PROCEEDINGS OF THE INTERNET ANALYSIS SEMINAR ON THE UNCERTAINTY PRINCIPLE IN HARMONIC ANALYSIS

Contents

Overview of the Workshop

This workshop was part of the Internet Analysis Seminar that is the education component of the National Science Foundation – DMS # 0955432 held by Brett D. Wick. The Internet Analysis Seminar consists of three phases that run over the course of a standard academic year. Each year, a topic in complex analysis, function theory, harmonic analysis, or operator theory is chosen and an internet seminar will be developed with corresponding lectures. The course will introduce advanced graduate students and post-doctoral researchers to various topics in those areas and, in particular, their interaction.

This was a workshop that focused on the uncertainty principle in harmonic analysis. The famous Heisenberg Uncertainty Principle from Quantum Mechanics has a striking formulation in the language of Harmonic Analysis. It essentially says that a non-zero measure (distribution) and its Fourier transform cannot be simultaneously small. Throughout the years this broad statement raised a multitude of deep mathematical questions, each corresponding to a particular sense of "smallness." The majority of these questions were inspired and are closely connected to many fundamental problems in approximation theory, inverse spectral problems for differential operators and Krein's canonical systems, classical problems in the theory of stationary Gaussian Processes, signal processing, etc. Many of these problems remained open for more than half a century, and some of them were even considered completely intractable.

The participants that presented, presented one of the following papers:

[1] Anton Baranov, *Completeness and Riesz bases of reproducing kernels in model subspaces*, Int. Math. Res. Not. (2006), Art. ID 81530, 34. ↑

[2] S. V. Hruščëv, N. K. Nikol′skiĭ, and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, Complex analysis and spectral theory (Leningrad, 1979/1980), Lecture Notes in Math., vol. 864, Springer, Berlin, 1981, pp. 214–335. ↑

[3] N. Makarov and A. Poltoratski, *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle*, Perspectives in analysis, Math. Phys. Stud., vol. 27, Springer, Berlin, 2005, pp. 185–252. ↑

[4] Mishko Mitkovski and Alexei Poltoratski, *Pólya sequences, Toeplitz kernels and gap theorems*, Adv. Math. **224** (2010), no. 3, 1057–1070. ↑

[5] Joaquim Ortega-Cerdà and Kristian Seip, *Fourier frames*, Ann. of Math. (2) **155** (2002), no. 3, 789–806. ↑

[6] Alexei Poltoratski, *Spectral gaps for sets and measures*, Acta Math. **208** (2012), no. 1, 151–209. ↑

They were then responsible to prepare three one-hour lectures based on the paper and an extended abstract based on the paper. This proceeding is the collection of the extended abstract prepared by each participant.

The following people participated in the workshop:

Debendra Banjade	University of Alabama
Kelly Bickel	Georgia Institute of Technology
Anne Duffee	University of Alabama
Ishwari Kunwar	Georgia Institute of Technology
Michael Lacey	Georgia Institute of Technology
Jingguo Lai	Brown University
Alexei Poltoratski	Texas A&M University
James Murphy	University of Maryland
Robert Rahm	Georgia Institute of Technology
Rishika Rupam	Texas A&M University
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COMPLETENESS AND RIESZ BASES OF REPRODUCING KERNELS IN MODEL SUBSPACES

ANTON BARANOV

presented by Kelly Bickel

Abstract. This paper examines the behavior of systems of reproducing kernels in model subspaces $K_\Phi := H^2 \ominus \Phi H^2$, where Φ is inner and H^2 is the Hardy space. The first main result is a criterion of completeness for a system of reproducing kernels in K_Φ, phrased in terms of the argument of Φ. This result is used to show that the completeness property is stable under small perturbations. The author also derives a different completeness result, which relies on densities defined using a de-Branges-Clark basis of K_Φ. The second class of results examine conditions under which a system of reproducing kernels (indexed by real points) forms a Riesz basis in K_Φ.

1. Introduction and Main Results

1.1. **Introduction.** A function Φ on the upper half plane $\mathbb{C}^+$ is called *inner* if Φ is bounded, analytic, and $\lim_{y\to 0} |\Phi(x+iy)| = 1$ a.e. on $\mathbb{R}$. For each inner function, one can define the *model subspace associated to Φ:*

$$K_\Phi := H^2 \ominus \Phi H^2 = H^2 \cap \Phi \overline{H^2}.$$

Each K_Φ is a closed subspace of the Hardy space H^2 and is a reproducing kernel Hilbert space with reproducing kernel given by

$$k_w(z) := \frac{1}{2\pi i} \cdot \frac{1 - \overline{\Phi(w)}\Phi(z)}{\bar{w} - z}$$

for each $z, w \in \mathbb{C}^+$. Such model subspaces appear in many important areas of both function theory and operator theory and are arguably most well-known for their role in the Sz.-Nagy-Foias model of Hilbert space contractions. Given Φ inner and $\Lambda \subseteq \mathbb{C}^+ \cup \mathbb{R}$, it seems reasonable to ask

$$\text{When is the set } \mathcal{K}(\Lambda) := \{k_{\lambda_n}\} \text{ complete in } K_\Phi?$$

This question was originally motivated by interest in the completeness of systems of complex exponentials $\{\exp(2\pi i \lambda_n t)\}$ in $L^2([0,a])$ but is also interesting in its own right. The first section of this paper [1] answers the motivating question: when is $\mathcal{K}(\Lambda)$ complete in K_Φ? The second section restricts attention to meromorphic Φ and $\Lambda \subseteq \mathbb{R}$. It considers the question: When is $\mathcal{K}(\Lambda)$ a Riesz basis of K_Φ?

In this extended abstract, we first outline the paper's main results about completeness and Reisz bases of $\mathcal{K}(\Lambda)$ in K_Φ. The later two sections will address both the proofs of the main results and several applications.

1.2. **Completeness Results.** Baranov's first completeness result is phrased using the principle argument of Φ, denoted $\arg \Phi$, and the Hilbert transform. To state the theorem, let Π denote the Poisson measure on $\mathbb{R}$, i.e. $d\Pi(t) = \frac{dt}{1+t^2}$. Then for $g \in L^2(\Pi)$, the Hilbert transformed is defined as

$$\tilde{g}(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-t|>\epsilon} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) g(t) dt.$$

The following result characterizes completeness of $\mathcal{K}(\Lambda)$ for $\Lambda \subseteq \mathbb{C}^+$:

Theorem 1. *Let $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$. Then $\mathcal{K}(\Lambda)$ is* not complete *in K_Φ if and only if there are*

(1) *a nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$*
(2) *a measurable $\mathbb{Z}$-valued function k and a $\gamma \in \mathbb{R}$*

such that

$$\arg \Phi - \arg B_\Lambda = 2\widetilde{\log m} + 2\pi k + \gamma \quad a.e. \text{ on } \mathbb{R},$$

where B_Λ is the Blaschke product with zeros $\{\lambda_n\}$.

The paper separately considers the situation where some of the points defining reproducing kernels are real. To ensure such reproducing kernels exist, Φ is assumed to be analytic in a neighborhood of each such real point. In this situation, Baranov shows:

Theorem 2. *Let $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$ and $T = \{t_n\} \subseteq \mathbb{R}$ and let Φ be analytic in a neighborhood of each t_n. Then: $\mathcal{K}(\Lambda) \cup \mathcal{K}(T)$ is* not complete *in K_Φ if and only if there exist*

(1) *an inner function J with $\{J = 1\} = T$*
(2) *a nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$*
(3) *a measurable $\mathbb{Z}$-valued function k and a $\gamma \in \mathbb{R}$*

such that

$$\arg \Phi - \arg B_\Lambda - \arg J = 2\widetilde{\log m} + 2\pi k + \gamma \quad a.e. \text{ on } \mathbb{R},$$

where B_Λ is the Blaschke product with zeros $\{\lambda_n\}$.

These criteria can be used to show that the completeness property is stable under both small perturbations of Φ and small perturbations of Λ.

The last completeness result relies on a density criterion and follows from Theorem 2. It only applies to meromorphic inner functions Φ, which are always of the form

$$\exp(iaz) B_\Lambda(z)$$

where $a \geq 0$ and the zeros Λ of the Blaschke product B_Λ do not accumulate on $\mathbb{R}$. In this case, Φ extends analytically to $\mathbb{R}$ and $\Phi(t) = \exp(i\psi(t))$ for ψ C^∞ and increasing on $\mathbb{R}$.

Definition 3. To define the *density associated to* Φ, let Φ be a meromorphic inner function with increasing, smooth argument function ψ. Define the sequence $\{s_n\}$ by

$$\psi(s_n) = 2\pi n \qquad \forall\ n.$$

As long as $1 - \Phi \notin L^2(\mathbb{R})$, then the set of reproducing kernels $\{k_{s_n}\}$ is an orthogonal basis in K_Φ and is called *a de Branges-Clark basis.* Now let $T = \{t_n\}$ be another sequence in $\mathbb{R}$. Then the upper and lower densities D_+ and D_- of T with respect to Φ are defined as follows:

$$D_+(T, r) = \sup_n \#\{m : t_m \in [s_n, s_{n+r})\} \quad \text{and} \quad D_-(T, r) = \inf_n \#\{m : t_m \in [s_n, s_{n+r})\}$$

and

$$D_+(T) = \lim_{r\to\infty} \frac{D_+(T,r)}{r} \quad \text{and} \quad D_-(T) = \lim_{r\to\infty} \frac{D_-(T,r)}{r}.$$

Intuition suggests that if T is much denser than $\{s_n\}$, then $\mathcal{K}(T)$ is complete in K_Φ, and if T is much sparser than $\{s_n\}$, then $\mathcal{K}(T)$ is not complete. With additional restrictions on Φ and the points $\{s_n\}$, that is precisely Baranov's result:

Theorem 4. *Assume Φ is meromorphic, inner and $\Phi' \in L^\infty(\mathbb{R})$. Further, assume $\{s_n\}$ satisfy*

$$\sup_n \left| \sum_{k\neq n} \left(\frac{1}{s_n - s_k} + \frac{s_k}{1+s_k^2} \right) \right| < \infty.$$

Taking D_+, D_-, and T as before:

(1) *If $D_-(T) > 1$ then $\mathcal{K}(T)$ is* complete *in K_Φ.*
(2) *If $D_+(T) < 1$ then $\mathcal{K}(T)$ is* not complete *in K_Φ.*

1.3. **Riesz Bases Results.** A set of vectors $\{h_n\}$ is a *Riesz basis* for a Hilbert space H if $H = \overline{\text{Span}_n\, h_n}$ and there are constants $A, B > 0$ such that for every finite sum $h = \sum_n c_n h_n$,

$$A \sum_n |c_n|^2 \leq \left\| \sum_n c_n h_n \right\|_H^2 \leq B \sum_n |c_n|^2.$$

To obtain criteria for $\mathcal{K}(T)$ to be a Riesz basis in K_Φ, Baranov uses the connections between meromorphic inner functions Φ and Hermite-Biehler functions E, defined as follows:

Definition 5. A function E is in the *Hermite-Biehler class* HB if E is entire and

$$|E(z)| > |E(\bar{z})| \qquad \forall z \in \mathbb{C}^+.$$

For each Hermite-Biehler function, one can define the de Branges space $\mathcal{H}(E)$, which is the Hilbert space of entire functions F such that F/E and F^*/E are in H^2, where $F^*(z) := \overline{F(\bar{z})}$. The norm on $\mathcal{H}(E)$ is given by $\|F\|_E := \|F/E\|_2$, where $\|\cdot\|_2$ is the H^2 norm.

If $E \in HB$, then $\Phi := E^*/E$ is meromorphic inner and similarly, if Φ is meromorphic inner, then there is an HB function E such that $\Phi = E^*/E$. Moreover, as shown in [3], the map

$$F \mapsto F/E \quad \text{maps } \mathcal{H}(E) \text{ unitarily onto } K_\Phi.$$

The following result uses an associated de Branges space $\mathcal{H}(E)$ to characterize the Riesz bases of K_Φ:

Theorem 6. *Let $\Phi = E^*/E$ be be meromorphic inner with $E \in HB$ and let $T = \{t_n\} \subseteq \mathbb{R}$. Then $\mathcal{K}$ is a Riesz Basis for K_Φ if and only if there is a meromorphic inner $\Phi_1 = E_1^*/E_1$ such that*

(1) *$\mathcal{H}(E) = \mathcal{H}(E_1)$ as sets with equivalent norms*
(2) *$T = \{\Phi_1 = 1\}$ and $\Phi_1 - 1 \notin L^2(\mathbb{R})$*

Notice that (2) says $\mathcal{K}(T)$ is a de Branges-Clark basis for Φ_1. However, (1) is more mysterious and sufficient conditions for (1) are discussed later.

In the following two sections, we will highlight the main proof techniques for these results and discuss several applications.

2. Main Result 1: Completeness Criteria

2.1. **Proofs of Theorems 1 and 2.** The proofs of Theorems 1 and 2 rely on the following representations of arguments of inner and outer functions using Hilbert transforms:

Remark 7. Argument Representations. Assume $m \geq 0$, $m \in L^2(\mathbb{R})$, and $\log m \in L^1(\Pi)$. Then $f := \exp(\log m + i\widetilde{\log m})$ is an H^2 outer function and every H^2 outer function is of this form with $m = |f|$. Thus,

$$\arg f = \widetilde{\log |f|} + 2\pi k,$$

for a measurable $\mathbb{Z}$-valued function k. Similarly, if I is inner, then $1 - I$ is outer and since $I = -(1 - I)/(1 - \bar{I})$, we have

$$\arg I = 2\widetilde{\log |1 - I|} + 2\pi k + \pi,$$

for a measurable $\mathbb{Z}$-valued function k.

To illustrate the use of such representations, we summarize the proof of the forward implication of Theorem 1. The converse uses similar ideas and is omitted:

Proof: Assume $K(\Lambda)$ is not complete, so there is an $f \in K_\Phi$ that vanishes on Λ. We can assume $f = B_\Lambda g$, where g is outer. As $f \in K_\Phi$, this implies $\Phi\overline{B_\Lambda g} \in H^2$ and there is an inner function I so that $\Phi\overline{B_\Lambda g} = Ig$. Taking arguments and rearranging gives:

$$\arg \Phi - \arg B_\Lambda = 2 \arg g + \arg I + 2\pi k + \pi = 2\widetilde{\log m} + 2\pi k + \gamma,$$

where $m = |g||1 - I| \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$. $\square$

The proof of Theorem 2 is similar but requires the construction of an inner function J such that $\{J = 1\} = T$.

Remark 8. Constructing the Inner Function. Define a Poisson-finite, positive measure ν supported on T. If δ_x denotes the Dirac measure at the point x, just select:

$$\nu = \sum_n \nu_n \delta_{t_n} \quad \text{where} \quad \nu_n > 0 \quad \text{and} \quad \sum_n \frac{\nu_n}{1 + t_n^2} < \infty.$$

Then construct a meromorphic Herglotz function G using ν as follows

$$G(z) = \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\nu = \sum_n \nu_n \left(\frac{1}{t_n - z} - \frac{t_n}{1 + t_n^2} \right)$$

and the function $J = (G - i)/(G + i)$ is meromorphic inner with $\{J = 1\} = T$. The only additional technicality in the proof of Theorem 2 is that the $\{\nu_n\}$ must be sufficiently small so that, if $f \in K_\Phi$ is the function vanishing on $\Lambda \cup T$, then $f/(1 - J) \in H^2$. This construction appears in [5].

2.2. **Application I: Proving Theorem 4.** Theorems 1 and 2 are particularly useful because previous results by Baranov, Havin, and Mashreghi [2, 4] show that every *mainly increasing function* can be written as $2\widetilde{\log m} + 2\pi k$, where $m \geq 0$, $m \in L^2(\mathbb{R})$, and $\log m \in L^1(\Pi)$ and k is a $\mathbb{Z}$-valued measurable function.

Definition 9. If $f \in C^1(\mathbb{R}$, then f is *mainly increasing* if there is an increasing sequence $\{d_n\} \subseteq \mathbb{R}$ such that $\lim_{n \to \infty} |d_n| = \infty$ and

(a) $f(d_{n+1}) - f(d_n) \approx 1 \qquad \forall\ n.$
(b) There is a constant C such that

$$\sup_{s,t\in(d_n,d_{n+1})} |f(s)-f(t)| \le C \text{ and } \sup_{s,t\in(d_n,d_{n+1})} |f'(s)-f'(t)| \le C \quad \forall\ n.$$

To see the usefulness of mainly increasing functions, consider this (very) briefly sketch of the proof of Theorem 4 :

Proof: First, prove part (2). Use the density assumption to construct an inner function J with $\{J=1\}=T$ such that $\arg\Phi-\arg J$ is mainly increasing. Then Theorem 2 implies that $\mathcal{K}(\Lambda)\cup\mathcal{K}(T)$ is not complete in K_Φ. To prove part (1), similarly construct an inner function with $\arg J-\arg\Phi$ mainly increasing, so it is of the form $2\widetilde{\log m}+2\pi k$. Simultaneously, assume that $\mathcal{K}(\Lambda)\cup\mathcal{K}(T)$ is not complete and use Theorem 2. This results in a contradiction. $\square$

2.3. **Application:Perturbations of** Λ**.** Theorem 2 also has implications about the stability of these completeness properties. For example, Baranov proves the following corollary:

Corollary 10. *Let $\Lambda=\{\lambda_n\}\subseteq\mathbb{C}^+$ and $M=\{\mu_n\}\subseteq\mathbb{C}^+$. Assume $\mathcal{K}(M)$ is complete in K_Φ. If for some choice of arguments ψ_Λ of B_Λ and ψ_M of B_M, $(\psi_\Lambda-\psi_M)\in L^1(\Pi)$ and*

$$(\psi_\Lambda-\psi_M)^\sim \in L^\infty(\mathbb{R})$$

then $\mathcal{K}(\Lambda)$ is also complete in K_Φ.

This result follows quickly from Theorem 2:
Proof: Assume $\mathcal{K}(\Lambda)$ is not complete, so $\arg\Phi-\psi_\Lambda = 2\widetilde{\log m}+2\pi k+\gamma$. The assumption $(\psi_\Lambda-\psi_M)^\sim = u\in L^\infty(\mathbb{R})$ implies

$$\psi_\Lambda-\psi_M = -\tilde{u} = 2\widetilde{\log m_1},$$

for $m_1=e^{-u/2}$. Since $u\in L^\infty(\mathbb{R})$, then $m_1\in L^\infty(\mathbb{R})$ and clearly $\log m_1\in L^1(\Pi)$. So

$$\arg\Phi-\psi_M = \arg\Phi-\psi_\Lambda+(\psi_\Lambda-\psi_M) = 2\widetilde{\log m_1 m}+2\pi k+\gamma,$$

which means $\mathcal{K}(M)$ is not complete in K_Φ, a contradiction. $\square$

3. Main Result 2: Riesz Basis Criteria

3.1. **Proof of Theorem 6.** The proof of Theorem 6 relies on the following condition for a system of reproducing kernels $\mathcal{K}(T)$ to be a Riesz basis in K_Φ, which can be found in [6]:

Remark 11. The system $\mathcal{K}(T)$ is a Riesz basis for K_Φ if and only if for each $\{c_n\}$ satisfying

$$\sum_n \frac{|c_n|^2}{\|k_{t_n}\|_2^2} < \infty\ , \text{ there is a unique function } f\in K_\Phi \text{ such that } f(t_n)=c_n$$

and each $\|f\|_2^2 \approx \sum |c_n|^2/\|k_{t_n}\|_2^2$. Such a set T is called *a complete interpolating set for K_Φ.* If $\Phi=E^*/E$, where E is Hermite-Biehler, an equivalent condition is: T is a *a complete interpolating set for $\mathcal{H}(E)$*, which means that for each $\{d_n\}$ satisfying

$$\sum_n \frac{|d_n|^2}{|E(t_n)|^2\|k_{t_n}\|_2^2} < \infty\ , \text{ there is a unique function } F\in\mathcal{H}(E) \text{ with } F(t_n)=d_n$$

and each $\|F\|_E^2 \approx \sum |d_n|^2/|E(t_n)|^2\|k_{t_n}\|_2^2$.

The ($\Leftarrow$) direction of Theorem 6's proof is almost immediate:

Proof: Assumption (2) coupled with the above remark implies that T is a complete interpolating set for $\mathcal{H}(E_1)$. Assumption (1) implies not only that $\mathcal{H}(E_1) = \mathcal{H}(E)$ and $\|F\|_E \approx \|F\|_{E_1}$ but also gives

$$|E(t_n)|^2 \|k_{t_n}\|_2^2 \approx |E_1(t_n)|^2 \|k^1_{t_n}\|_2^2 \quad \forall\ n,$$

where k^1_z denotes the reproducing kernel of K_{Φ_1}. Working through the definitions, it follows that T is a complete interpolating set for $\mathcal{H}(E)$, so $\mathcal{K}(T)$ is a Riesz basis for K_Φ. □

The ($\Rightarrow$) direction of the proof uses the following result of Ortega-Cerda and Seip [7]:

Theorem 12. *Let $E \in HB$ and let $T = \{t_n\} \subseteq \mathbb{R}$. If T is a sampling set for K_Φ, then there exist entire functions E_1 and E_2 where $E_1 \in HB$ and $E_2 \in HB$ or is constant such that*

(1) *$\mathcal{H}(E) = \mathcal{H}(E_1)$ with norm equivalence.*
(2) *If $\Phi_1 = E_1^*/E_1$ and $\Phi_2 = E_2^*/E_2$, then $\{\Phi_1\Phi_2 = 1\} = T$.*
(3) *$(1 - \Phi_1\Phi_2) \notin L^2(\mathbb{R})$*

Given that, here is a brief proof sketch:

Proof: Assume T is a complete interpolating set for K_Φ; this implies T is a sampling set of K_Φ, so there are E_1 and E_2 as in Theorem 12. To obtain the desired result, Baranov shows E_2 is constant. This relies on factoring:

$$\mathcal{H}(E_1E_2) = E_2\mathcal{H}(E_1) \oplus E_1^*\mathcal{H}(E_2).$$

Facts (2) and (3) about E_1, E_2 imply that T is a complete interpolating set for $\mathcal{H}(E_1E_2)$. Assumption (1) implies that T is also a complete interpolating set for $E_2\mathcal{H}(E_1)$. Combining the two results, one can show that $\mathcal{H}(E_1E_2) = E_2\mathcal{H}(E_1)$, and so E_2 must be constant. □

3.2. **Using Theorem 6.** To demonstrate the usability of Theorem 6, in the last section of the paper, Baranov examines when inner functions Φ and Φ_1 can be factored using HB functions E, E_1 so that $\mathcal{H}(E) = \mathcal{H}(E_1)$ with equivalent norms. One sufficient condition for $\mathcal{H}(E) = \mathcal{H}(E_1)$ with equivalent norms is for $|E(z)| \approx |E_1(z)|$ for all $z \in \overline{\mathbb{C}^+}$. Baranov finds necessary and sufficient conditions for this in the following theorem:

Theorem 13. *Let Φ, Φ_1 be meromorphic inner with increasing arguments ψ,ψ_1. Assume $(\psi - \psi_1) \in L^1(\Pi)$. Then there are $E_1, E \in HB$ such that $\Phi = E^*/E$, $\Phi_1 = E_1^*/E_1$ and*

$$|E(z)| \approx |E_1(z)| \ \ \forall z \in \overline{\mathbb{C}^+} \quad \textit{if and only if} \quad (\psi - \psi_1)^\sim \in L^\infty(\mathbb{R}).$$

Proof: The idea is to construct an entire function S that is real on $\mathbb{R}$ such that

$$S = \frac{E_1}{E} w$$

for an outer function w with $w, w^{-1} \in H^\infty$ and continuous on $\overline{\mathbb{C}^+}$. Then $E_2 := SE$ satisfies $\Phi = E_2^*/E_2$ and $E_1(z) \approx E_2(z)$ on $\overline{\mathbb{C}^+}$. The condition $(\psi - \psi_1)^\sim \in L^\infty(\mathbb{R})$ allows Baranov to define w by

$$w(t) := \exp(-\tilde{f}(t) + if(t)) \text{ on } \mathbb{R}$$

and extend w to an outer function on $\overline{\mathbb{C}^+}$ with the desired properties. □

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PÓLYA SEQUENCES, TOEPLITZ KERNELS AND GAP THEOREMS

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presented by Anne Duffee

Abstract. We summarize the paper on Pólya sequences by Poltoraski and Minkovski [5]. A sequence $\Lambda = \{\lambda_n\}$ is a Pólya sequence if any entire function of zero exponential type bounded on Λ is constant. We seek to fully characterize these Pólya sets and show that this is equivalent to the Beurling gap problem on Fourier transform. We achieve this via techniques using Toeplitz kernels and de Branges spaces of entire functions.

1. Introduction and Main Results Presented

1.1. Introduction.

1.1.1. *Definitions.* An entire function F is said to have *exponential type zero* if

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} = 0.$$

A sequence is *separated* if it satisfies $|\lambda_n - \lambda_m| \geq \delta > 0, \ \ (n \neq m)$. A separated real sequence $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ is *Pólya* if any entire function of exponential type zero that is bounded on Λ is constant. It is our goal in this paper to fully characterize Pólya sets.

1.1.2. *Examples.*

(i) $\Lambda = \mathbb{Z}$ is a Pólya sequence.
(ii) $\lambda_n := n + n/\log(|n| + 2),\ n \in \mathbb{Z}$ is a Pólya sequence.
(iii) $\lambda_n = n^2$ is the zero set of the zero type function $F(z) := \cos\sqrt{2\pi z}\cos\sqrt{-2\pi z}$ and thus is not a Pólya sequence.

1.1.3. *Background.* From Levinson [3], we have that if $|\lambda_n - n| \leq p(n)$, where $p(t)$ satisfies $\int \frac{p(t)}{1+t^2}\log|\frac{t}{p(t)}|dt < \infty$ and some smoothness conditions, then $\Lambda = \{\lambda_n\}$ is a Pólya sequence. For each such $p(t)$ that also satisfies $\int_{-\infty}^{\infty} \frac{p(t)}{1+t^2}\,dt = \infty$, there exists $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ that is not Pólya. How do we close the gap between our sufficient condition and the counterexamples?

From de Branges [1], we have that Λ is Pólya if $\int_{-\infty}^{\infty} \frac{p(t)}{1+t^2}\,dt < \infty$. However, there exists a Pólya sequence, Λ when $\int_{-\infty}^{\infty} \frac{p(t)}{1+t^2}\,dt = \infty$. Thus we have yet to fully characterize the sets of Pólya sequences.

However, de Branges' work [1] contains the following necessary condition: if Λ is Pólya, then the complement of Λ, $\{I_n\}$, is short, i.e.:

$$\sum_n \frac{|I_n|^2}{1 + \mathrm{dist}^2(I_n, 0)} < \infty.$$

But this condition is not sufficient—as we saw in the examples above, $\lambda_n = n^2$ is the zero set of the zero type function $F(z) := \cos\sqrt{2\pi z}\cos\sqrt{-2\pi z}$, which is non constant on $\mathbb{C}_+$ but bounded on $\Lambda = \{\lambda_n\}$, and the complement of this Λ is short.
What this paper shows, then, is that a separated real sequence Λ is not Pólya iff there exists a long sequence of intervals $\{I_n\}$ such that

$$\frac{\#(\Lambda \cap I_n)}{|I_n|} \to 0,$$

where a sequence of intervals is long if it is not short.

1.1.4. *Beurling-Malliavin densities.* We will define the interior and exterior densities in terms of Toeplitz kernels and the exponential function $S^a(z) = e^{iaz}$:

The *interior BM (Beurling-Malliavin) density* is defined as

$$D_*(\Lambda) = \frac{1}{2\pi}\sup\{a : N[\bar{\Theta}S^a] = 0\},$$

where $\Theta(z)$ denotes some/any meromorphic inner function with $\{\Theta = 1\} = \Lambda$. The *exterior BM density* is defined as

$$D^*(\Lambda) = \frac{1}{2\pi}\sup\{a : N[\bar{S}^a\Theta] = 0\},$$

where $\Theta(z)$ denotes some/any meromorphic inner function with $\{\Theta = 1\} = \Lambda$. An equivalent definition of the interior BM density follows: let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\gamma(\mp\infty) = \pm\infty$. We say that γ is *almost decreasing* if the family of intervals $\mathcal{BM}(\gamma)$ is short, where $\mathcal{BM}(\gamma)$ is the collection of the connected components of the open set

$$\left\{x \in \mathbb{R} : \ \gamma(x) \neq \max_{t\in[x,+\infty)} \gamma(t)\right\}.$$

1.1.5. *Background Theorems.*

Theorem 1 ([2])**.** *Let μ be a positive measure on $\mathbb{R}$ satisfying $\int d\mu(t)/(1+t^2) < \infty$. Then there exists a short de Branges space B_E contained isometrically in $L^2(\mu)$, with de Branges function $E(z)$ being of Cartwright class and having no real zeros. Moreover, if there exists such a space B_E with $E(z)$ of positive exponential type, then there also exists such a space B_E that is contained properly in $L^2(\mu)$.*

Theorem 2 ([4, Section 4.2])**.** *Suppose that $\Theta(z)$ is a meromorphic inner function with the derivative of $\arg\Theta(t)$ bounded on $\mathbb{R}$. Then for any meromorphic inner function $J(z)$, we have*

$$N^+[\bar{\Theta}J] \neq 0 \quad \Rightarrow \quad \forall\epsilon > 0, \quad N[\bar{S}^\epsilon\bar{\Theta}J] \neq 0.$$

Theorem 3 ([4, Section 4.3])**.** *Suppose $\gamma'(t) > -\text{const}$.*

(i) If γ is not almost decreasing, then for every $\epsilon > 0$, $N^+[S^\epsilon e^{i\gamma}] = 0$.
(ii) If γ is almost decreasing, then for every $\epsilon > 0$, $N^+[\bar{S}^\epsilon e^{i\gamma}] \neq 0$.

As noted in [4], part (i) of Theorem 3 holds without the assumption $\gamma'(t) > -\text{const}$.

1.2. **Main Results Presented.**

Theorem 4. *Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty} \subset \mathbb{R}$ be a separated sequence of real numbers. The following are equivalent:*

(i) $\Lambda = \{\lambda_n\}$ *is a Pólya sequence.*
(ii) There exists a non-zero measure μ of finite total variation, supported on Λ, such that the Fourier transform of μ vanishes on an interval of positive length.
(iii) The interior Beurling-Malliavin density of Λ, $D_*(\Lambda)$, is positive.
(iv) There exists a meromorphic inner function $\Theta(z)$ with $\{\Theta = 1\} = \{\lambda_n\}$ such that $N[\bar{\Theta}S^{2c}] \neq 0$, for some $c > 0$.

Corollary. *Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ be a separated sequence of real numbers. Then Λ is a Pólya sequence if and only if for every long sequence of intervals $\{I_n\}$ the sequence $\frac{\#(\Lambda \cap I_n)}{|I_n|}$ is not a null sequence, i.e., $\frac{\#(\Lambda \cap I_n)}{|I_n|} \nrightarrow 0$.*

With regard to the gap problem we will prove the following result. Let M denote the set of all complex measures of finite total variation on $\mathbb{R}$. For $\mu \in M$ its Fourier transform $\hat{\mu}(x)$ is

$$\hat{\mu}(x) = \int e^{ixt} d\mu(t).$$

If X is a closed subset of the real line denote by $\mathcal{G}X)$ the gap characteristic of X:

$$G(X) := \sup\{a \mid \exists\ \mu \in M,\ \mu \not\equiv 0,\ \operatorname{supp} \mu \subset X, \text{ such that } \hat{\mu} = 0 \text{ on } [0, a]\}.$$

Theorem 5. *The following are true:*

(i) For any separated sequence $\Lambda \subset \mathbb{R}$, $G(\Lambda) \geq 2\pi D_*(\Lambda)$.
(ii) For any closed set $X \subset \mathbb{R}$, $G(X) \leq 2\pi D_*(X)$.

Corollary. *For separated sequences $\Lambda \subset \mathbb{R}$, $G(\Lambda) = 2\pi D_*(Ł)$.*

Corollary. *Let X be a closed subset of the real line. If there exists a long sequence of intervals $\{I_n\}$ such that*

$$\frac{\#(X \cap I_n)}{|I_n|} \to 0$$

then any measure μ of finite total variation supported on X, whose Fourier transform vanishes on an interval of positive length, is trivial.

The formula for $G(X)$ for a general closed set X is more involved, and thus in the proof, we will consider only a closed set X equal to a separated real sequence Λ. Another immediate consequence of Theorem 2 is the following extension of Beurling's gap theorem:
If $X \subset \mathbb{R}$ is closed, define

$$T(X) = \sup\{a \mid \exists \text{ meromorphic inner } \Theta(z) \text{ with } \{\Theta = 1\} \subset X \text{ and } N[\bar{\Theta}S^a] \neq 0\}.$$

Theorem 6. *For any closed $X \subset \mathbb{R}$,*

$$T(X) = G(X).$$

2. Main Results

2.1. **Technical Lemmas.**

Lemma 7. *Let* $\Theta(z)$ *be a meromorphic inner function with* $1-\Theta(t)\notin L^2(\mathbb{R})$ *and let* σ *be the corresponding Clark measure. If* $N[\bar{\Theta}S^{2a}]\neq 0$ *for some* $a>0$*, then for any* $\epsilon>0$ *there exists* $h\in L^2(\sigma)$ *such that*

$$\lim_{y\to\pm\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t)=0$$

for every $x\in(-a+\epsilon,a-\epsilon)$ *and the measure* $hd\sigma$ *has finite total variation.*

Proof. The idea of the proof is as follows: If $N[\bar{\Theta}S^{2a}]\neq 0$ for some $a>0$ then there exists some $f\in K_\Theta$ such that $f/S^{2a-2\epsilon}\in N[\bar{\Theta}S^{2a-2\epsilon}]$ and thus is in $N[\bar{\Theta}S^{2a}]$. We can then find our measure $hd\sigma$ from the Clark representation of f with $h\in L^2(\sigma)$ and of finite total variation, and the limit follows from $f/S^{a-\epsilon}\in N[\bar{\Theta}S^{a-\epsilon}]$. □

Lemma 8. *Let* μ *be a measure with finite total variation. Then the Fourier transform of* μ *vanishes on* $[-a,a]$ *if and only if*

$$\lim_{y\to\pm\infty} e^{xy}\int \frac{d\mu(t)}{t-iy}=0,$$

for every $x\in[-a,a]$.

2.2. **Proofs of Main Results.**

We will present the proofs of the main theorems in reverse order.

Proof of Theorem 6. The inequality $T(X)\leq G(X)$ follows from the lemmas. To prove the opposite inequality, let $G(X)=a$. Then for any $\epsilon>0$ there exists a non-zero complex measure of total variation no greater than 1 supported on X whose Fourier transform vanishes on $[0,a-\epsilon]$. Consider the set of all such measures. Since this set is closed, convex and contains non-zero elements, by the Krein-Milman theorem it has an extreme point, a non-zero measure ν. We can show that the extremality of ν implies that it is supported on a discrete subset of X. Let $\Theta(z)$ be the meromorphic inner function whose Clark measure is $|\nu|$. Then $\{\Theta=1\}\subset X$. It is left to notice that the function

$$f(z)=\frac{1-\Theta(z)}{2\pi i}\int\frac{d\nu(t)}{t-z}$$

belongs to K_Θ and is divisible by $S^{a-\epsilon}$ (as follows, for instance, from the proof of Lemma 8). Hence $f/S^{a-\epsilon}\in N[\bar{\Theta}S^{a-\epsilon}]\neq 0$. □

Proof of Theorem 5. By Theorem 6 it is enough to prove that $T(\Lambda)=2\pi D_*(\Lambda)$.

(i) Suppose that $D_*(\Lambda)=a/2\pi$. By the definition given of D_* the function

$$\phi(x)=-2\pi n_\Lambda(x)+(a-\epsilon)x$$

is almost decreasing for any $\epsilon>0$. Consider a meromorphic inner function Θ with $\{\Theta=1\}=\Lambda$ and bounded derivative on $\mathbb{R}$. Then $\arg\bar{\Theta}S^{a-\epsilon}$ differs from ϕ by a bounded function. Hence $\arg\bar{\Theta}S^{a-2\epsilon}$ is almost decreasing. By Theorem 2 and Theorem 3, $N[\bar{\Theta}S^{a-3\epsilon}]\neq 0$.

(ii) Again we will prove that $T(X) \leq 2\pi D_*(X)$. If $T(X) = a$ then for any $\epsilon > 0$ there exists a meromorphic inner $\Theta(z)$ such that $\Gamma := \{\Theta = 1\} \subset X$ and $N[\bar{\Theta}S^{a-\epsilon}] \neq 0$. By Theorem 2 (and remark after it) this means that $\arg \bar{\Theta}S^{a-2\epsilon}$ is almost decreasing. Hence

$$-2\pi n_\Gamma(x) + (a - 3\epsilon)x$$

is almost decreasing. Since ϵ is arbitrary, $D_*(X) \geq a/2\pi$. □

Proof of Theorem 4. $(ii) \Leftrightarrow (iii)$ follows from Theorem 5 and $(ii) \Leftrightarrow (iv)$ from Theorem 6.

$(i) \Rightarrow (iii)$ Assume (iii) is not true, i.e. for every meromorphic inner function $\Theta(z)$ with $\{\Theta = 1\} = \Lambda$, $N[\bar{\Theta}S^{2c}] = 0$ for every $c > 0$. We construct a non-constant zero type entire function which is bounded on Λ, which will mean that Λ is not a Pólya set. Define a measure μ to be the counting measure of Λ. Then by Theorem 1 there exists a short de Branges space B_E contained isometrically in $L^2(\mu)$. If $E(z)$ has type zero then all functions in B_E have type zero. If the type of E is positive, then by Theorem 1 we can assume that B_E is contained properly in $L^2(\mu)$.

Suppose that $F(z) \in B_E$ has positive type, and assume that $F(iy)$ grows exponentially in y as $y \to \infty$. Since $B_E \neq L^2(\mu)$, there exists $g \in L^2(\mu)$ with $\bar{g} \perp B_E$. Then
for any $w \in \mathbb{C}$. Thus the function

$$G(z) := \frac{1 - \Theta(z)}{2\pi i} \int \frac{1}{t - z} g(t) d\mu(t)$$

can be represented as $G(z) = S^c(z)h(z)$ for some nonzero $h(z) \in H^2(\mathbb{C}_+)$ and $c > 0$, and belongs to K_Θ, where $\Theta(z)$ is the inner function corresponding to the measure μ. Hence $h \in N[\bar{\Theta}S^c]$ and we have a contradiction.

Therefore any $F(z) \in B_E$ has zero type. It is left to notice that

$$|F(\lambda_n)| \leq \sqrt{\sum_m |F(\lambda_m)|^2} = \|F\|_{L^2(\mu)} < \infty,$$

which means that $F(z)$ is bounded on Λ.

$(ii) \Rightarrow (i)$ from de Branges' proof [1]. □

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UNCONDITIONAL BASES OF EXPONENTIALS AND OF REPRODUCING KERNELS

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ABSTRACT. We give several concise characterizations on when a given family of exponentials and reproducing kernels form an unconditional bases. The method using to analyze the problem is of independent interest.

1. INTRODUCTION

A well-known fact in Fourier analysis is: the family $\{e^{inx}\}_{n\in\mathbb{Z}}$ forms a complete orthogonal system in $L^2(0, 2\pi)$. So for any $f \in L^2(0, 2\pi)$, we can write

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \text{ in } L^2(0, 2\pi)$$

and this sum converges unconditionally.
A natural generalization would be: given an interval $I \subset \mathbb{R}$ and a family of complex frequencies $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$, can we make good sense of

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n x}, \text{ for every } f \in L^2(I) \text{ in terms of } \Lambda?$$

Our enemy here is: the family $\mathcal{E}_\Lambda = \{e^{i\lambda x}\}_{n\in\mathbb{Z}}$ might NOT be complete or orthogonal.
A first progress on this type of problem is

Theorem 1. *(Paley-Wiener, 1934) The system $\mathcal{E}_\Lambda = \{e^{i\lambda x}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0, 2\pi)$ if $\lambda_n \in \mathbb{R}$ and $\sup_n |n - \lambda_n| < \pi^{-2}$.*

This result has been repeatly revised and generalized. The most equisite formulation is

Theorem 2. *(Ingham-Kadec 1/4-Theorem) Let $\delta > 0$. Every family $\mathcal{E}_\Lambda = \{e^{i\lambda x}\}_{n\in\mathbb{Z}}$ satisfying $\lambda_n \in \mathbb{R}$ and $\sup_n |\lambda_n - n| = \delta$ forms a Riesz basis in $L^2(0, 2\pi)$ iff $\delta < 1/4$.*

This theorem will be a consequence of our main results. We should mention the definition of Riesz basis and unconditional bases.

Definition 1. *A family $\{\psi_n\}_{n\in\mathbb{Z}}$ of non-zero vectors in a Hilbert space $\mathcal{H}$ is called an unconditional basis in $\mathcal{H}$ if*

1. *the family $\{\psi_n\}_{n\in\mathbb{Z}}$ spans the space $\mathcal{H}$;*
2. $||\sum_n a_n\psi_n||^2 \asymp \sum_n |a_n|^2||\psi_n||^2$.

If, in addition, $||\psi_n|| \asymp 1$, then the family $\{\psi_n\}_{n\in\mathbb{Z}}$ is called a Riesz basis.

2. Functional Model

2.1. H^p spaces and Nevanlinna spaces. Let us work on the upper-half plane $\mathbb{C}_+$. The H^p spaces and Nevanlinna spaces consists of analytic functions on $\mathbb{C}_+$ such that

$$H^p(\mathbb{C}_+) : ||f||^p = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p dx < \infty, 0 < p < \infty.$$

$$H^\infty(\mathbb{C}_+) : ||f|| = \sup_{z \in \mathbb{C}_+} |f(z)| < \infty.$$

$$N : \sup_{y>0} \int_{\mathbb{R}} \log^+ |f(x+iy)| dx < \infty, \log^+ |f(z)| = \max\{\log |f(z)|, 0\}.$$

Note that the first two are norms on their corresponding spaces, but not the third one.

Theorem 3. *(Fatou) $H^p(\mathbb{C}_+) = \{f \in L^p(\mathbb{R}) : \widehat{f}$ is supported on $[0, \infty)\}, p \geq 1$, and we can recover by $f(x+iy) = f * P_y(x)$.*

So we sometimes identify $H^p, p \geq 1$ as a closed subspace of L^p.

Theorem 4. *(Riesz-Nevanlinna factorization) If $f \in N, f \not\equiv 0$, then $f(z) = cBf_eS_1/S_2$, where $|c| = 1$, B is the Blaschke product, S_1, S_2 are singular inner functions and f_e is outer function. If $f \in H^p, f \not\equiv 0$, then $f(z) = cBf_eS$.*

Definition 2. *A Blaschke produce B with zeros $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is an infinite product*

$$B(z) = \prod_{n \in \mathbb{Z}} \epsilon_n \frac{z - \lambda_n}{z - \overline{\lambda_n}} = \prod_{n \in \mathbb{Z}} b_n(z),$$

where signs ϵ_n, $|\epsilon_n| = 1$ make each factor $b_n(z)$ nonnegative at $z = i$.

A well-known Blaschke condition: $\sum_{n \in \mathbb{Z}} \frac{\Im \lambda_n}{|\lambda_n + i|^2} < \infty$ is necessary and sufficient for the Blaschke product to converge.

Definition 3. *A singular inner function S is of the form*

$$S(z) = \exp\left\{-\frac{1}{\pi i} \int_{\widehat{\mathbb{R}}} \frac{tz+1}{t-z} d\mu(t)\right\},$$

where μ is a non-negative finite measure on $\widehat{\mathbb{R}} = \mathbb{R} \bigcup \{\infty\}$ and is singular with respect to the Lebesgue measure on $\mathbb{R}$.

Example. If taking $\mu = \pi a \delta_\infty$, then $S(z) = e^{iaz}$ is a singular inner function.

Definition 4. *An outer function f_e of a function f is defined by*

$$f_e(z) = \exp\left\{\frac{1}{\pi i} \int_{\mathbb{R}} \frac{tz+1}{t-z} \frac{\log |f(t)|}{t^2+1} dt\right\}.$$

Lemma 5. (1) *$|B| = |S| = 1$ a.e. on $\mathbb{R}$.*
(2) *f is an outer function iff $\log |f(z)| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\Im(z)}{|z-t|^2} \log |f(t)| dt$.*

There is a parallel theory on the unit disc $\mathbb{D}$, which we skip. Their relation is the isometry of $L^p(\mathbb{T})$ onto $L^p(\mathbb{R})$

$$U_p f(x) = \frac{1}{\pi^{\frac{1}{p}}} \frac{1}{(x+i)^{\frac{2}{p}}} f\left(\frac{x-i}{x+i}\right), x \in \mathbb{R}.$$

Our main results are stated on the upper-half plane $\mathbb{C}_+$, but sometimes it is easier to work on $\mathbb{D}$.
In this note, we mainly focus on the space $H^2(\mathbb{C}_+)$ or $H^2(\mathbb{D})$, which are reproducing kernel Hilbert spaces (RKHS). Some results concerning H^p for $p \neq 2$ can be found in Chapter 2, Section 6 of the original paper.

2.2. **Two Theorems of Paley-Wiener.**

Theorem 6. *If $f \in L^2(0,\infty)$, then $F(z) = \int_0^\infty f(t)e^{itz}dt$ satisfies $F(z) \in H^2(\mathbb{C}_+)$. Conversely, if $F \in H^2(\mathbb{C}_+)$, then there exists $f \in L^2(0,\infty)$ such that $F(z) = \int_0^\infty f(t)e^{itz}dt$.*

Theorem 7. *Let A and C be two positive constants. If $f \in L^2(-A,A)$, then $F(z) = \int_{-A}^{A} f(t)e^{itz}dt$ is an entire function of exponential type A, i.e. $|F(z)| \leq Ce^{A|z|}$, and $\int_{\mathbb{R}} |F(x)|^2dx < \infty$. Conversely, if F is an entire function of exponential type A and $\int_{\mathbb{R}} |F(x)|^2dx < \infty$., then there exists $f \in L^2(-A,A)$ such that $F(z) = \int_{-A}^{A} f(t)e^{itz}dt$.*

2.3. **Functional Model.** Now we are in a position to carefully formulate our problem. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}_+$ and $\sup_n |\Im\lambda_n| < \infty$ be a fixed subset and $a > 0$. Let B be the Blaschke product for the sequence Λ if it satisfies the Blaschke condition and identically zero otherwise. We want to find condtions on Λ such that the family $\mathcal{E}_\Lambda = \{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0,a)$.
We first note that $||e^{i\lambda_n x}|| \asymp 1$, so in this case, Riesz basis means exactly unconditional basis. Note also $\{e^{i\lambda_n x}\}$ is an unconditional basis iff $\{e^{-i\overline{\lambda_n}x}\}$ is an unconditional basis. Let $\Lambda^* = \{-\overline{\lambda_n}\}_{n\in\mathbb{Z}}$. The inverse Fourier transform gives $\mathcal{F}^*(e^{-i\bar{\lambda}x}\chi_{[0,\infty)})(z) = \frac{i}{z-\bar{\lambda}}$. Note that $\{k(z,\lambda) = \frac{i}{z-\bar{\lambda}}\}$ are exactly the reproducing kernels of $H^2(\mathbb{C}_+)$ and since B is the Blaschke product for Λ, we would have $span\{k(z,\lambda_n)\}_{n\in\mathbb{Z}} = K_B = H^2(\mathbb{C}_+) \ominus BH^2(\mathbb{C}_+)$. Hence, $\mathcal{F}^*$ maps $span\{e^{-i\overline{\lambda_n}x}\chi_{[0,\infty)}\}_{n\in\mathbb{Z}}$ onto the subspace $K_B = H^2(\mathbb{C}_+) \ominus BH^2(\mathbb{C}_+)$.
This says $\{e^{-i\overline{\lambda_n}x}\}$ is an uncondition basis of $L^2(0,\infty)$ iff $\{k(z,\lambda_n)\}$ is an unconditional basis.
Let $\theta^a(z) = e^{iaz}$, we can deduce $\mathcal{F}^*L^2(0,a) = \mathcal{F}^*L^2(\mathbb{R}_+) \ominus \mathcal{F}^*L^2(a,\infty) = H^2 \ominus \theta^a H^2 = K_{\theta^a}$.
So $\{e^{-i\overline{\lambda_n}x}\}$ is an uncondition basis of $L^2(0,a)$ iff $\{P_{K_{\theta^a}}k(z,\lambda_n)\}$ is an unconditional basis.
At this stage, it is natural to consider the following general problem:
Let θ be any inner function and let B be a Blaschke product with the sequence Λ. Let $P_\theta = P_{K_\theta}$ be the orthogonal projection onto K_θ. Then the function $k_\theta(z,\lambda) = P_\theta k(z,\lambda)$ is the reproducing kernel for K_θ, because for $f \in K_\theta$,

$$\langle f(z), k_\theta(z,\lambda)\rangle = \langle f(z), P_\theta k(z,\lambda)\rangle = \langle f(z), k(z,\lambda)\rangle = f(\lambda).$$

So $k_\theta(z,\lambda) = i\frac{1-\overline{\theta(\lambda)}\theta(z)}{z-\bar{\lambda}}$. In the case of $\mathbb{D}$, $k(z,\lambda) = \frac{1}{1-\bar{\lambda}z}$ and $k_\theta(z,\lambda) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar{\lambda}z}$.
General problem of unconditional bases for reproducing kernel:
What is to be assumed about the pair (θ,Λ) for the family $\{k_\theta(z,\lambda)\}_{\lambda\in\Lambda}$ to be an unconditional basis in K_θ?

3. Towards a solution (I)

3.1. **Carleson's Interpolation Theorem and Carleson Embedding Theorem.**

Theorem 8. *(Carlenson's Interpolation Theorem) Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}_+$ be a sequence, the following are equivalent:*

(1) *The sequence is an interpolating sequence: every interpolation problem*

$$f(\lambda_n)(\Im\lambda_n)^{\frac{1}{2}} = a_n, n \in \mathbb{Z} \text{ with } \{a_n\} \in l^2$$

has solution $f \in H^2(\mathbb{C}_+)$.

(2) Λ *satisfies the well-known Carleson condition:*

$$\inf_n \prod_{n \neq k} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \overline{\lambda_k}} \right| = \delta > 0. \ (C)$$

(3) Λ *is a rare set, i.e. there exists* $\varepsilon > 0$, *such that*

$$D(\lambda_n, \varepsilon\Im\lambda_n) \bigcap D(\lambda_k, \varepsilon\Im\lambda_k) = \emptyset, \ n \neq k, \ (R)$$

and the measure $\mu = \sum_{n\in\mathbb{Z}} \Im\lambda_n \delta_{\lambda_n}$ *is a Carleson measure.*

On the unit disc $\mathbb{D}$, the conditions are slightly different:
Carleson condition: $\inf_n \prod_{k\neq n} \left| \frac{\lambda_n-\lambda_k}{1-\overline{\lambda_k}\lambda_n} \right| = \delta > 0. \ (C)$
Rarity condition: $D(\lambda_n, \varepsilon(1-|\lambda_n|)) \bigcap D(\lambda_k, \varepsilon(1-|\lambda_k|)) = \emptyset, \ n \neq k. \ (R)$
Carleson measure: $\mu = \sum_{n\in\mathbb{Z}} (1-|\lambda_n|)\delta_{\lambda_n}$.
It is not hard to check that condition (R) is equivalent to $\inf_{n\neq k} \left| \frac{\lambda_n-\lambda_k}{1-\overline{\lambda_k}\lambda_n} \right| \geq \delta > 0$. To understand the Carleson measure condition, we need the Carleson Embedding Theorem.

Theorem 9. *(Carleson Embedding Theorem on the unit disc* $\mathbb{D}$*) Let* μ *be a positive measure on* $\mathbb{D}$*. The following are equivalent:*

(1) μ *is a Carleson measure.*

(2)

$$\sup_{f\in H^2(\mathcal{D}), ||f||\leq 1} \int_{\mathbb{D}} |f|^2 d\mu < \infty.$$

(3)

$$\sup_{\lambda\in supp(\mu)} \int_{\mathbb{D}} |k(z,\lambda)|^2 d\mu(z) < \infty \text{ for all normalized reproducing kernel } k(z,\lambda) = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda}z}.$$

Using statement (3) of Carleson Embedding Theorem, $\mu = \sum_{n\in\mathbb{Z}} (1-|\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure on $\mathbb{D}$ iff $\sup_n \sum_k \frac{(1-|\lambda_n|^2)(1-|\lambda_k|^2)}{|1-\overline{\lambda_k}\lambda_n|^2} < \infty$.

3.2. **A first reduction of the problem.** We first mention a nice characterization due to N.K.Nikol'skii and B.S.Pavlov, which will be appealed to.

Theorem 10. *(Nikol'skii-Pavlov) Let* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}_+$*. The following are equivalent:*

(1) *The family* $\{k(z,\lambda_n)\}_{n\in\mathbb{Z}}$ *forms an unconditional basis in its own span in* $H^2(\mathbb{C}_+)$.
(2) *The family* $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ *forms an unconditional basis in its* $L^2(\mathbb{R}_+)$*-span.*
(3) $\Lambda \in (C)$.

And it comes our main results:

Theorem 11. *Let* θ *be an inner function and* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}_+$*. If the family* $\{k_\theta(z,\lambda_n)\}_{n\in\mathbb{Z}}$ *is an unconditional basis in* K_θ*, then* $\Lambda \in (C)$.

Theorem 12. *Let* θ *be an inner function and* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}_+$*. Moreover, assume that* $\sup_n |\theta(\lambda_n)| < 1$*. The following are equivalent:*

(1) $\{k_\theta(z,\lambda)\}_{n\in\mathbb{Z}}$ *forms an unconditional basis in* K_θ.
(2) $\Lambda \in (C)$ *and* $P_\theta|K_B$ *maps isomorphically the space* K_B *onto* K_θ, *where* B *is the Blaschke product for the sequence* Λ.

We need to understand the meaning of $\sup_n |\theta(\lambda_n)| < 1$. Since

$$||(z-\overline{\lambda})^{-1}||^2_{H^2} = \langle k(z,\lambda), k(z,\lambda)\rangle = k(\lambda,\lambda) = \frac{i}{\lambda-\overline{\lambda}} = \frac{1}{2\Im\lambda},$$

$$||P_\theta(z-\overline{\lambda})^{-1}||^2_{H^2} = k_\theta(\lambda,\lambda) = i\frac{1-|\theta(\lambda)|^2}{\lambda-\overline{\lambda}} = \frac{1-|\theta(\lambda)|^2}{2\Im\lambda},$$

thus $\sup_n |\theta(\lambda_n)| < 1$ is equivalent to $||k(z,\lambda_n)||_{H^2} \asymp ||k_\theta(z,\lambda_n)||_{H^2}$. There is yet a theorem without the condition $\sup_n |\theta(\lambda_n)| < 1$ formulated in Chapter 2, Section 4 of the original paper.
The above theorem, in particular, implies

Theorem 13. *Let* θ *be an inner function,* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}_+$ *and* $a > 0$. *The following are equivalent:*

(1) $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ *is an unconditional basis in* $L^2(0,a)$.
(2) $\Lambda \in (C)$ *and the restriction* $f \to f\chi_{(0,a)}$ *maps isomorphically the space* $span_{L^2(\mathbb{R})}\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ *onto* $L^2(0,a)$.

4. Towards a solution (II)

We need to undersand when $P_\theta|K_B$ maps isomorphically the space K_B onto K_θ? To do this, we have to do more preparations.

4.1. **A lemma from functional analysis.**

Lemma 14. *Let* M *and* N *be two closed subspaces of a Hilbert space* H.
$P_M|N$ *is an isomorphism onto its image (of left invertible) iff* $||P_N|M^\perp|| < 1$.
$P_M|N$ *is an isomorphism onto* M *(or invertible) iff both* $||P_N|M^\perp|| < 1$ *and* $||P_M|N^\perp|| < 1$.

Proof. For the first assertion, we just need to remember:
An operator $T : X \to Y$ defined on two Banach spaces satisfies $ker(T) = \{\mathbf{0}\}$ and $Range(T)$ is closed iff there exists $c > 0$ such that $||Tx|| \geq c||x||$ for all $x \in X$.
Thus, $P_M|N$ is an isomorphism onto its image iff $||P_Mx|| \geq c||x||, x \in N$. Note that $P_M|N + P_{M^\perp}|N = Id|N$, so we have $||P_{M^\perp}|N|| < 1$. Finally, because $(P_{M^\perp}|N)^* = P_N|M^\perp$, we conclude $||P_N|M^\perp|| < 1$.
For the second assertion, since $(P_M|N)^* = P_N|M$, so $P_M|N$ is an isomorphism onto M iff both $P_M|N$ and $P_N|M$ are left invertible. The first assertion then implies the second. □

4.2. **Hankel operators, Toeplitz operators and Nehari Theorem.**

Definition 5. *Let* $\varphi \in L^\infty(\mathbb{R})$. *The Toeplitz operator with symbol* φ *is the operator* T_φ *on* $H^2(\mathbb{C}_+)$ *defined by* $T_\varphi f = \mathbb{P}_+(\varphi f)$ *for* $f \in H^2(\mathbb{C}_+)$. *The Hankel operator with the same symbol* φ *is the defined by* $H_\varphi f = \mathbb{P}_-(\varphi f)$ *for* $f \in H^2(\mathbb{C}_+)$.

Hence, we have the simple relation $\varphi f = H_\varphi f + T_\varphi f$ for $f \in H^2(\mathbb{C}_+)$.

Theorem 15. *(Nehari) If* $\varphi \in L^\infty(\mathbb{R})$, *then* $||H_\varphi|| = dist(\varphi, H^\infty)$.

A simple corollary of Nehari Theorem is

Corollary 16. *Let $\varphi \in L^\infty(\mathbb{R})$ and $|\varphi| = 1$ a.e., then*
T_φ is an isomorphism onto its image iff $||H_\varphi|| = dist(\varphi, H^\infty) < 1$,
T_φ is an isomorphism onto $H^2(\mathbb{C}_+)$ iff $||H_\varphi|| = dist(\varphi, H^\infty) < 1$, $|H_{\overline{\varphi}}|| = dist(\overline{\varphi}, H^\infty) < 1$.

Proof. Note that the relation $\varphi f = H_\varphi f + T_\varphi f$ gives $||f||^2 = ||H_\varphi f||^2 + ||T_\varphi f||^2$ for $f \in H^2(\mathbb{C}_+)$. Hence, $||H_\varphi|| < 1$ iff $||T_\varphi f|| \geq c||f||, f \in H^2(\mathbb{C}_+)$ for some $c > 0$, which is equivalent to T_φ being an isomorphism onto its image. Finally, using Nehari Theorem, the first assertion is proved. The second assertion follows. □

4.3. **When is $P_\theta|B$ invertible?** First **Lemma 14** tells us $P_\theta|B$ maps isomorphically the space K_B onto K_θ iff $||P_B|K_\theta^\perp|| < 1$ and $||P_\theta|K_B^\perp|| < 1$. To better understant these, we need the following lemma.

Lemma 17. *Let φ be an inner function, then $P_\varphi = \varphi\mathbb{P}_-\overline{\varphi}$.*

Proof. Write $P' = \varphi\mathbb{P}_-\overline{\varphi}$, then P' is idemponent and normal.
Idemponence: $P'P' = \varphi\mathbb{P}_-\overline{\varphi}\varphi\mathbb{P}_-\overline{\varphi} = \varphi\mathbb{P}_-\overline{\varphi} = P'$.
Self-ajoint operator: $\langle P'f, g\rangle = \langle \varphi\mathbb{P}_-\overline{\varphi}f, g\rangle = \langle f, \varphi\mathbb{P}_-\overline{\varphi}g\rangle = \langle f, P'g\rangle$.
These two imply P' is a projection. Moreover, $P'f = 0$ iff $f \in \theta H^2$. So $P' = P_\theta$. □

Having this lemma, we can conclude: $P_B|K_\theta^\perp = P_B|\theta H^2 = B\mathbb{P}_-\overline{B}|\theta H^2 = BH_{\overline{B}\theta}\overline{\theta}|\theta H^2$.
So $||P_B|K_\theta^\perp|| = ||H_{\overline{B}\theta}||$. Similarly, $||P_\theta|K_B^\perp|| = ||H_{B\overline{\theta}}||$.
Hence, by **Corollary 16**, $P_\theta|B$ maps isomorphically the space K_B onto K_θ iff $||H_{\overline{B}\theta}|| = dist(\overline{B}\theta, H^\infty) < 1$ and $||H_{B\overline{\theta}}|| = dist(B\overline{\theta}, H^\infty) < 1$ iff both $T_{\overline{B}\theta}$ and $T_{B\overline{\theta}}$ are invertible.

5. Towards a solution (III)

Up to now, we have obtained a few non-trivial characterizations. To go further, we need to consider a more general situation. Again, we begin with some preparations.

5.1. **Hilbert transform, Helson-Szegö condition and (A_2)-condition.** For $v \in L^\infty(\mathbb{R})$, we define its Hilbert transform $\widetilde{v}$ by

$$\widetilde{v}(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) v(t)dt,$$

here, we need the term $\frac{t}{1+t^2}$ to remove the sigularity at infinity.
The Schwartz formula

$$V(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) v(t)dt$$

recovers the function V by its real part v only, provided $V \in H^\infty$ and $\Im V(i) = 0$.
The function $\widetilde{v}(x)$ is the boundary value of $\Im V(z)$.

Definition 6. *A non-negative function w is called a function satisfying Helson-Szegö condition ($w \in (HS)$) if there are functions u, v in $L^\infty(\mathbb{R})$ such that*

$$||v||_{L^\infty} < \frac{\pi}{2} \text{ and } w = \exp(u + \widetilde{v}).$$

Definition 7. *A unimodular function φ on $\mathbb{R}$ is called a Helson-Szegö function if there are a unimodular constant λ and an outer function h satisfying*

$$\varphi = \lambda \frac{\overline{h}}{h} \text{ and } |h|^2 \in (HS).$$

Another equivalent form of the Helson-Szegö condition has been obtained by B.Muckenhoupt, R.A. Hunt and R.L.Wheeden.

Theorem 18. *(Muckenhoupt-Hunt-Wheeden) The (HS)-condition is equivalent to the Mukenhoupt (A_2)-condition*

$$\sup_{I\in\mathcal{I}} \frac{1}{|I|}\int_I w dx \frac{1}{|I|}\int_I w^{-1}dx < \infty, \ where\ \mathcal{I}\ is\ the\ family\ of\ all\ intervals\ on\ \mathbb{R}.$$

We now state our main theorem.

Theorem 19. *Let φ be a unimodular function. The following are equivalent.*

(1) *The Toeplitz operator T_φ is invertible.*
(2) *$dist_{L^\infty}(\varphi, H^\infty) < 1$ and $dist_{L^\infty}(\overline{\varphi}, H^\infty) < 1$.*
(3) *There exists an outer function f, $f \in H^\infty$, satisfying $||\varphi - f||_{L^\infty} < 1$.*
(4) *There exists a branch of the argument α of φ, $\varphi(x) = e^{i\alpha(x)}$, such that*

$$dist_{L^\infty}(\alpha, \widetilde{L^\infty} + \mathbb{C}) = \inf\{||\alpha - \widetilde{v} - c||_{L^\infty} : v \in L^\infty(\mathbb{R}), c \in \mathbb{C}\} < \frac{\pi}{2}.$$

(5) *φ is a Helson-Szegö function.*

The equivalence of (1) and (2) is exactly **Corollary 16** from the last Chapter.
To obatin a list of invertibility tests for $P_\theta|B$, it suffices to put $\varphi = \overline{B}\theta$ in the condition of the theorem.

6. Towards a solution (IV)

6.1. **A particular interesting characterization.** We want to further refine assertion (4) of **Theorem 19**.
Before doing that, let us mention a simple but important remark. The following isomorphisms in $L^2(0, a)$ preserves the exponentials:

$$f(x) \to e^{i\alpha x} f(x),\ f(x) \to f(a - x),\ f(x) \to \overline{f(x)}.$$

Any of these isomorphisms preserves the property of being an unconditional basis. So we always can move a frequency set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ from $\mathbb{C}_+$ to $\mathbb{C}_\delta = \{z \in \mathbb{C} : \Im z > \delta\}, \delta > 0$.
Now, let $\Lambda \subset \mathbb{C}_\delta, \delta > 0$ and let B be a Blaschke product with zero set Λ. For $x \in \mathbb{R}$, we have

$$\frac{d}{dx} arg B(x) = \Im(\frac{d}{dx}\log B(x)) = \Im(\sum_{n\in\mathbb{Z}} \frac{d}{dx}\log\left(\frac{x-\lambda_n}{x-\overline{\lambda_n}}\right)) = 2\sum_{n\in\mathbb{Z}} \frac{\Im\lambda_n}{|x-\lambda_n|^2},$$

thus it is easy to see that the function α_Λ defined by

$$\alpha_\Lambda(x) = 2\int_0^x \sum_{n\in\mathbb{Z}} \frac{\Im\lambda_n}{|t-\lambda_n|^2} dt - ax, x \in \mathbb{R},$$

is a continuous branch of argument, up to an additive constant, of the unimodular function $B\overline{\theta^a}$ on $\mathbb{R}$. Now, we refine assertion (4) of **Theorem 19** using this specific function α_Λ.

Theorem 20. *Let $\Lambda \subset \mathbb{C}_\delta, \delta > 0$. Then the family $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ forms an unconditional basis in $L^2(0, a)$ iff $\Lambda \in (C)$ and $dist_{L^\infty}(\alpha_\Lambda, \widetilde{L^\infty} + \mathbb{C}) < \frac{\pi}{2}$.*

The sufficency of this theorem is a consequence of **Theorem 13** and **Theorem 19**.
As a corollary, we can easily prove Ingham-Kadec 1/4-Theorem stated in the introduction.

6.2. **Entire functions of exponential type.** An important tool used to prove **Theorem 20** is the properties of entire functions of exponential type.
Let $F(z)$ be an entire function of exponential type A, i.e. for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|F(z)| \leq C_\varepsilon e^{(A+\varepsilon)|z|}$.

Definition 8. *The 2π-periodic indicator function of $F(z)$ is defined by*

$$h_F(\varphi) = \overline{\lim_{r\to\infty}} \frac{\log|F(re^{i\varphi})|}{r}, \ \varphi \in \mathbb{R}.$$

Definition 9. *Let $K \subset \mathbb{C}$ be a convex compact set. The supporting function $k(\varphi)$ of the set K is $k(\varphi) = \sup\{\Re(\zeta e^{-i\varphi}) : \zeta \in K\}$.*

A classical result on entire functions of exponential type asserts that: $h_F(\varphi)$ is a supporting function of a convex compact set G_F, i.e. $h_F(\varphi) = \sup\{\Re(\zeta e^{-i\varphi}) : \zeta \in G_F\}$. The convex compact set G_F is called the *indicator diagram* of F.
Write $F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$. The function $f(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}$ is called the *Borel transform* of $F(z)$.
The smallest convex compact set containing all singularities of $f(z)$, denoted by G_F^*, is called the *conjugate diagram* of $F(z)$. We denote by $k_F(\varphi)$ the supporting function of this set G_F^*.

Theorem 21. *(Pólya) For every entire function of exponential type $F(z)$, the relation $h_F(\varphi) = k_F(-\varphi)$ holds and hence $G_F^* = \{\overline{\zeta} : \zeta \in G_F\}$.*

We denote by $\mathcal{M}_a = \{$all entire functions F of exponential type with $G_F^* = [0, ia]$ and $|F|^2|\mathbb{R} \in (HS)$ or equivalently $(A_2)\}$.
The most important intermediate step in proving **Theorem 20** is:

Theorem 22. *Let $\Lambda \subset \mathbb{C}_\delta, \delta > 0$ be a Blaschke set, let B denote the corresponding Blaschke product and let $\theta^a = \exp(iaz), a > 0$. The following are equivalent:*

(1) *There exists a function of the class $\mathcal{M}_a$ with simple zeros whose zere set is Λ.*
(2) *The restriction $B\overline{\theta^a}|\mathbb{R}$ is a Helson-Szegö function, i.e. there exists a unimodular constant c and an outer function h such that $|h|^2|\mathbb{R} \in (HS)$ and $Bh = c\overline{h}\theta^a$ a.e. on $\mathbb{R}$.*

Theorem 2, **Theorem 11**, **Theorem 12**, **Theorem 19**, **Theorem 20** and **Theorem 22** will be carefully explained during the workshop.

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FOURIER FRAMES

JOACHIM ORTEGA-CERDÀ AND KRISTIAN SEIP

Presented by James Murphy

Abstract. We give an extended overview of Fourier frames. Our focus shall be on the 2002 paper of Ortega-Cerdà and Seip entitled "On Fourier Frames." We give background in the form of major definitions and early results in the theory, and briefly review de Branges' theory of Hilbert spaces of entire functions. We then move to the two major results of "On Fourier Frames": characterizations of sampling sequences for the Paley-Wiener space.

Contents

1. Introduction and background on de Branges spaces

Recalling that $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(-\pi,\pi)$, we generalize to the case when a family of complex exponentials spans $L^2(-\pi,\pi)$, though perhaps with redundancy.

Definition 1.1. A family of complex exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$, with $\Lambda=\{\lambda_k\}_{k\in\mathbb{Z}}\subset\mathbb{R}$, is a *Fourier frame* if there exist $0<A\le B<\infty$ such that $\forall f\in L^2(-\pi,\pi)$:

$$A\int_{-\pi}^{\pi}|f(x)|^2dx \le \sum_{k\in\mathbb{Z}}\left|\int_{-\pi}^{\pi}f(x)e^{-i\lambda_k x}dx\right|^2 \le B\int_{-\pi}^{\pi}|f(x)|^2dx. \tag{1.2}$$

The statement (1.2) is a particular instance of the "frame condition," which has been generalized to a variety of settings. Of particular relevance to the notion of Fourier frame is the applicability of the Paley-Wiener theorem. There are many formulations of this classical result, but we use the following version [7] to motivate an alternate characterization of Fourier frames.

Theorem 1.3 (Paley-Wiener)**.** *Let $\sigma>0$ be constant. Then a function $F(x)$ is of the form:*

$$F(x)=\int_{-\sigma}^{\sigma}f(\xi)e^{i\xi x}d\xi \text{ for some } f\in L^2(-\sigma,\sigma)$$

if and only if $F(x)\in L^2(\mathbb{R})$ and F can be extended to an entire function of exponential-type at most σ, meaning F extends to an entire function $\tilde{F}$ such that $\exists\ C>0$ with the property that $|\tilde{F}(z)|\le Ce^{\sigma|z|}$ everywhere.

We call the space of entire functions of exponential type at most π, whose restriction to $\mathbb{R} \subset \mathbb{C}$ is square-integrable, the Paley-Wiener space, denoted PW.

Definition 1.4. A sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is *sampling* for PW if there exist $0 < A \leq B < \infty$ such that $\forall f \in PW$:

$$A\int_{\mathbb{R}} |f(x)|^2 dx \leq \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^2 \leq B\int_{\mathbb{R}} |f(x)|^2 dx.$$

The Paley-Wiener theorem can be used together with the Plancherel theorem to show that $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is sampling for PW if and only if $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a Fourier frame. Hence, we will study sampling sequences in order to understand Fourier frames. Indeed, the two main results of "On Fourier Frames" are characterizations of sampling sequences for PW; we must keep in mind that such characterizations also describe Fourier frames, our initial objects of study.

We now introduce several properties of sequences $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, which will be used to prove the two main results characterizing sampling sequences.

Definition 1.5. $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a *complete interpolating sequence* if the interpolation problem $f(\lambda_k) = a_k, k \in \mathbb{Z}$ has a unique solution $f \in PW$ for all $l^2(\mathbb{C})$ data $\{a_k\}_{k\in\mathbb{Z}}$.

We note that an alternate characterization of Λ being a complete interpolating sequence is that Λ is sampling, but $\Lambda \setminus \{\lambda_j\}$ is not, for any $j \in \mathbb{Z}$. In this sense, complete interpolating sequences can be considered "minimal" sampling sequences.

Definition 1.6. Consider $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ where $\lambda_k \leq \lambda_{k+1}$, $\forall k \in \mathbb{Z}$. Such a sequence is *separated* if $q := \inf_{k\in\mathbb{Z}}(\lambda_{k+1} - \lambda_k) > 0$; q is the *separation constant*. For a separated sequence, define the associated *distribution function* n_Λ as:

$$n_\Lambda(0) = 0, \quad \forall a < b,\ n_\Lambda(b) - n_\Lambda(a) = |\Lambda \cap (a,b)|.$$

Theorem 1.7 (Landau's Inequality)**.** *If Λ is a separated sampling sequence for PW, then there exist constants A, B such that for all $a < b$:*

$$n_\Lambda(b) - n_\Lambda(a) \geq b - a - A\log^+(b-a) - B.$$

Landau's inequality provides a necessary condition for sampling. This sophisticated result can be considered alongside a more elementary inequality, which gives a sufficient condition:

$$n_\Lambda(b) - n_\Lambda(a) \geq (1+\epsilon)(b-a) - C,\ C, \epsilon > 0 \text{ independent of } a < b \Longrightarrow \Lambda \text{ is sampling.}$$

Indeed, the example of $\Lambda = \{k + \log^+|k|\}_{k\in\mathbb{Z}}$ optimizes Landau's inequality. Of paramount importance to the study of Fourier frames is the notion of Beurling density.

Definition 1.8. For a separated sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ with associated distribution n_Λ, the *lower Beurling uniform density* is

$$D^-(\Lambda) := \lim_{R\to\infty} \frac{\min_{x\in\mathbb{R}}(n_\Lambda(x+R) - n_\Lambda(x))}{R}.$$

It can be shown that lower uniform Beurling density gives an *almost* total characterization of sampling sequences, and hence of Fourier frames as well. Indeed, if $D^-(\Lambda) > 1$, then Λ is necessarily sampling. Conversely, if $D^-(\Lambda) < 1$, then Λ is necessarily not sampling [3]. The critical case is when $D^-(\Lambda) = 1$. The two main theorems of "On Fourier Frames" give characterizations of Fourier frames that apply to the critical case $D^-(\Lambda) = 1$.

One of the crucial tools in our analysis will be de Branges' theory of Hilbert spaces of entire functions. This theory combines aspects of analytic function theory and functional analysis, and was deployed successfully by Louis de Branges to prove the Bieberbach conjecture. We present a few basic ideas from this theory.

Definition 1.9. A *de Branges space* is a Hilbert space H of entire functions with the following three properties:

(1) If $f \in H$, ζ is non-real such that $f(\zeta) = 0$, then $g \in H$, where $g(z) := \dfrac{f(x)(z-\bar{\zeta})}{z-\zeta}$. Moreover, $\|f\|_H = \|g\|_H$.
(2) For every ζ non-real, the linear functional on H given by $\zeta \mapsto f(\zeta)$ is continuous.
(3) If $f \in H$, then $f^* \in H$, where $f^*(z) := \overline{f(\bar{z})}$.

A prime example of a de Branges space is PW intself. Also of importance for us is the so-called *Hermite-Biehler* space $\overline{HB}$. This is the space of entire functions f without roots in the upper half plane $\mathbb{H}$ and such that $|f(z)| \geq |f(\bar{z})|$ whenever $\Im(z) > 0$. A crucial construction involving $\overline{HB}$ involves generating Hilbert spaces from elements of $\overline{HB}$. Explicitly, given $E \in \overline{HB}$, we associate a Hilbert space of entire functions

$$H(E) := \left\{ f \text{ entire} \;\middle|\; \frac{f(z)}{E(z)}, \frac{f^*(z)}{E(z)} \in H^2(\mathbb{H}) \right\},\ \|f\|^2_{H(E)} = \|f\|^2_E := \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2} dt.$$

It is worth noting that such spaces $H(E)$ actually constitute *all* de Branges spaces [2].

Theorem 1.10 (Characterization of de Branges spaces)**.** *A Hilbert space of entire functions satisfying the three criterion of a de Branges space is isometrically isomorphic to $H(E)$, some $E \in \overline{HB}$.*

In order to bring the rich theory of de Branges spaces to bear on the problem of characterizing sampling sequences of PW, we shall state a generalized Plancherel theorem. We first require a couple of developmental results.

Theorem 1.11 (Reproducing Kernel for $H(E)$)**.** *Let $E \in \overline{HB}$. For each fixed $\zeta \in \mathbb{C}$, the function*

$$K_E(\zeta, z) := \frac{i}{2} \frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{\pi(z-\bar{\zeta})}$$

as a function of z is in $H(E)$. Moreover, K_E is a reproducing kernel for $H(E)$:

$$\forall f \in H(E),\ \langle f, K_E(\zeta, \cdot)\rangle_E = \int_{-\infty}^{\infty} \frac{f(t)\overline{K_E(\zeta,t)}}{|E(t)|^2} dt = f(\zeta).$$

Proposition 1.12. *Given $E \in \overline{HB}$, we may, for $x \in \mathbb{R}$, write $E(x) = |E(x)|e^{-i\phi(x)}$, where $\phi(x) \in \mathcal{C}(\mathbb{R})$ is such that $E(x)e^{i\phi(x)} \in \mathbb{R},\ \forall x \in \mathbb{R}$.*

We shall call such ϕ a *phase function*. It captures in some sense the "angular behavior" of the analytic function E.

Theorem 1.13 (Generalized Plancherel)**.** *Let $H(E)$ be a de Branges space, ϕ the phase function associated to E. Suppose $\alpha \in \mathbb{R}$ and let $\Gamma := \{\gamma_k\}$ be the sequence of real numbers such that $\phi(\gamma_k) = \alpha + k\pi, k \in \mathbb{Z}$. Then if $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$, the family of normalized reproducing kernels*

$$\left\{ \frac{K_E(\gamma_k, z)}{\|K_E(\gamma_k, \cdot)\|_E} \right\}_{k\in\mathbb{Z}}$$

is an orthonormal basis for $H(E)$. In particular, $\|f\|_E^2 = \sum_{k\in\mathbb{Z}} \frac{\pi |f(\gamma_k)|^2}{\phi'(\gamma_k)|E(\gamma_k)|^2}$.

Note that $e^{i\alpha}E - e^{-i\alpha}E^* \in H(E)$ for at most a single $\alpha \in [0, \pi)$, making the above result quite applicable.

2. Main Result 1: Complex Analysis and Sampling

The relevance of the $\overline{HB}$ space can be seen in the following characterization of sampling sequences.

Theorem 2.1 (Main Result 1)**.** $\Lambda \subset \mathbb{R}$ *is sampling for PW if and only if there exist $E, F \in \overline{HB}$ such that $H(E) = PW$ and Λ is the zero sequence of $EF + E^*F^*$.*

We can interpret the function F as follows. Using theorem 1.10 and the generalized Plancherel theorem, it can be shown that if Λ is a complete interpolating sequence, there exists $E \in \overline{HB}$ such that $H(E) = PW$ and Λ is the zero sequence of $E + E^*$. In this sense, we may view F as accounting for the "redundancy in Λ." In particular, Seip [6] has shown that if $D^-(\Lambda) > 1$, then $\Lambda = \Lambda' \setminus (\Lambda \setminus \Lambda')$ with Λ' a complete interpolating sequence. Then the hypothesis of Theorem 2.1 are met if we pick

$$F(z) := \prod_{\lambda_k \in (\Lambda\setminus\Lambda')} \left(1 - \frac{z}{\lambda_k}\right) e^{\frac{z}{\lambda_k}}.$$

In this case, F accounts for the part of Λ that doesn't contribute to the complete interpolating sequence, that is to say, the redundancy of Λ.

As an interesting corollary, a separated sampling sequence is "everywhere denser" than some complete interpolating sequence.

Corollary 2.2. *If Λ is a separated sampling sequence for PW, then there exists a complete interpolating sequence $\Gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ such that for every $k \in \mathbb{Z}$, there is at least one $\lambda \in \Lambda$ with the property that $\gamma_k \leq \lambda \leq \gamma_{k+1}$.*

3. Main Result 2: Approximation of Subharmonic Functions and Sampling

In this section, we consider $\psi \in \mathcal{C}^1(\mathbb{R})$ non-decreasing such that $\psi(\infty) - \psi(-\infty) = \infty$ and $\psi'(x) = o(1)$ as $|x| \to \infty$. Such a ψ generates a sequence $\Lambda(\psi) = \{\lambda_k\}_{k\in\mathbb{Z}}$ given by $\lambda_k = k - \psi(\lambda_k)$. Alternatively, setting $\psi(0) = 0$, this means that $n_\Lambda(t) = [t + \psi(t)]$.

Such $\Lambda(\psi)$ are relevant to the study of Fourier frames because none of them contain a complete interpolating sequence as a subsequence [6]. In particular, $D^-(\Lambda) \leq 1$. We shall give a characterization of the functions ψ such that $\Lambda(\psi)$ is sampling for PW. More precisely, we shall examine the critical growth rate of ψ for which $\Lambda(\psi)$ is sampling for PW. Loosely, $\Lambda(\psi)$ is sampling for PW if and only if $\psi(x)$ grows at least as fast as $\log^+(x)$ as $|x| \to \infty$.

Our analysis hinges on understanding the *potential function*:

Definition 3.1. For ψ as above, the *potential* of ψ is defined in the principle value sense by:

$$U_\psi(z) := \int_{-\infty}^{\infty} \left[\log\left|1 - \frac{z}{t}\right| + \Re\left(\frac{z}{t}\right)\right] d\psi(t).$$

Note that since ψ' is assumed non-decreasing, U_ψ is subharmonic. We now relate sampling to approximating U_ψ.

Corollary 3.2. *$\Lambda(\psi)$ is sampling for PW if there exists $f \in \overline{HB}$ such that:*

(1) $\phi_f(x) = o(1)$ as $|x| \to \infty$.

(2) $|U_\psi(z) - \log|f(z)|| \lesssim 1$ for $\Im(z) \geq 0$.

Definition 3.3. Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Let $\{t_n\}_{n=0}^{\infty}$ be such that $t_0 = 0$ and $\psi(t_n) = n$, $\forall n \geq 1$. Set $d_n := t_n - t_{n-1}$. We say ψ induces a *logarithmically regular partition* if $d_n \simeq d_{n+1}$ and

$$\sup_{x>0} \sum_{\frac{x}{2}<t_n<2x} \frac{d_n^2}{(x-t_n)^2 + d_n^2} < \infty.$$

Theorem 3.4 (Main Result 2)**.** *Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:*

(1) If $\psi'(x) = \frac{1}{O(x)}$ when $x \to \infty$ and ψ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.

(2) If $\psi'(x) = o(\frac{1}{x})$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.

The proof of main result 2 involves showing U_ψ can be approximated as in corollary 3.2. The proof method is rather intricate, and is derived from results advanced by Lyubarskii and Malinnikova [4].

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MEROMORPHIC INNER FUNCTIONS, TOEPLITZ KERNELS AND THE UNCERTAINTY PRINCIPLE

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presented by Rishika Rupam

Abstract. The aim of this paper is to describe certain problems like the completeness problem of some families of functions and problems in the spectral theory of selfadjoint differential operators in the setting of meromorphic inner functions and Toeplitz kernels. The injectivity of the Toeplitz operator is seen to play a crucial role in interpreting these problems. The paper concludes with an illustration of the effect of the Buerling Malliavin theory on the Toeplitz kernels.

1. Introduction

An inner function θ on $\mathbf{C}_+$ is a bounded analytic function on $\mathbf{C}_+$ such that $|\theta| = 1$ a.e. on $\mathbb{R}$. A *meromorphic* inner function on $\mathbf{C}_+$ is an inner function with a meromorphic continuation to the lower half plane. These functions can be characterized by the canonical factorization, due to Riesz and Smirnov,

$$\theta = B_\Lambda e^{iaz}, \tag{1}$$

where $a \geq 0$ and Λ is the discrete set in $\mathbf{C}_+$ satisfying the Blaschke condition

$$\sum_{\lambda\in\Lambda} \frac{\Im\lambda}{1+|\lambda|^2} < \infty. \tag{2}$$

A Blaschke term, B_λ can be easily seen to have the property that $B = e^{i\theta}$ on $\mathbb{R}$, where θ is a real analytic and increasing function. This property can be extended to finite Blashcke products and thence to infinite Blaschke products and meromorphic inner functions.
A meromorphic Herglotz function, m is a meromorphic function such that $\Im m > 0$ in $\mathbf{C}_+$, $m(\bar{z}) = \overline{m(z)}$. Meromorphic herglotz functions are in one-to-one correspondence with meromorphic inner functions using the relations,

$$\theta = \frac{m-i}{m+i}, \qquad m = i\frac{1+\theta}{1-\theta}. \tag{3}$$

Let Θ be an inner function. The $\mathcal{H}^2$-model space of Θ given by

$$K_\Theta \equiv K[\Theta] = \mathcal{H}^2 \ominus \Theta\mathcal{H}^2 = \mathcal{H}^2 \cap \Theta\overline{\mathcal{H}^2}, \tag{4}$$

is a Hilbert space with reproducing kernel

$$k_\lambda^\theta = \frac{1}{2\pi i}\frac{1-\overline{\Theta(\lambda)}\Theta(z)}{\bar{\lambda}-z}, \qquad \lambda \in \mathbf{C}_+. \tag{5}$$

Consider the Smirnov-Nevanlinna class $\mathcal{N}^+ = \mathcal{N}^+(\mathbf{C}_+)=\{g/h : g,h \in \mathcal{H}^\infty$ and h is outer$\}$ and the Hardy spaces $\mathcal{H}^p = \mathcal{N}^+ \cap L^p(\mathbb{R})$. One can consider model spaces in these classes, described as

$$K_\Theta^+ = \{F \in \mathcal{N}^+ \cap \mathcal{C}^\omega(\mathbb{R}) : \Theta\overline{F} \in \mathcal{N}^+\}, \tag{6}$$

and

$$K_\Theta^p = K_\Theta^+ \cap L^p(\mathbb{R}). \tag{7}$$

Consider the second order Schrödinger operator $u \to -\ddot{u} + qu$ on an interval (a, b). Assume that $q \in L^1_{loc}(a, b)$. Let us fix a boundary condition β at b. The Weyl Titchmarsh m-function of $(q; b, \beta)$ evaluated at a is defined by

$$m(\lambda) = m^a_{b,\beta}(\lambda) = \frac{\dot{u}_\lambda(a)}{u_\lambda(a)}, \qquad \lambda \in \mathbf{C} \tag{8}$$

where $u_\lambda(.)$ is a non-trivial solution of the Schrödinger equation satisfying the boundary condition β at b. It is known that m is a Herglotz function [3]. Thus, one can determine the corresponding inner funtion $\Theta^a_{b,\beta}$, which is called the *Weyl* inner function of q. Similarly, if we consider a non trivial solution satisfying a boundary condition α at a and let u_λ be a non trivial solution, satisfying α, then we can define the m-function of $(q; a, \alpha)$ evaluated at b: $m^b_{a,\alpha}(\lambda) = -\frac{\dot{u}_\lambda(b)}{u_\lambda(b)}$ and thus define the corresponding Weyl inner function $\Theta^b_{a,\alpha}$.
Let $L = (q, \alpha, \beta)$ be the Schrödinger operator with self adjoint boundary condition α and β at a and b respectively. We denote by $\sigma(L)$, the spectrum of L.
Let U be a function in $L^\infty(\mathbb{R})$. We may consider this function to be a mutiplication operator on $\mathcal{H}^2$, when projected onto $\mathcal{H}^2$, giving us what is called the Toeplitz operator. Formally, the operator $T_U : \mathcal{H}^2 \to \mathcal{H}^2$ is defined as

$$T_U(f) = P|_{\mathcal{H}^2}(Uf). \tag{9}$$

U is denoted as the symbol of the operator. We only consider unitary symbols, i.e., $U = e^{i\gamma}$, where $\gamma : \mathbb{R} \to \mathbb{R}$. $N[U] = \ker T_U$. As in the case of model spaces, we will consider the Toeplitz kernels in the Smirnov-Nevanlinna class and the Hardy spaces.

$$N^+[U] = \{F \in \mathcal{N}^+ \cap L^1_{loc}(\mathbb{R}) : \overline{UF} \in \mathcal{N}^+\} \tag{10}$$

and

$$N^p[U] = N^+[U] \cap L^p(\mathbb{R}), \qquad (0 < p \le \infty). \tag{11}$$

In particular, if Θ is a meromorphic inner function, then $N^+[\overline{\Theta}] = K^+_\Theta$ and $N^p[\overline{\Theta}] = K^p_\Theta$.

We now describe some problems and their reformulation using the tools we just defined.
Suppose $a < c < b$ and let $q_- = q|_{(a,c)}$ and $q_+ = q|_{(c,b)}$. The triplet $(q_-, \alpha, \sigma(L))$ is said to determine L if for any other Schrödinger operator $\tilde{L} = (\tilde{q}, \tilde{\alpha}, \tilde{\beta})$ such that $q_- = \tilde{q_-}$, $\alpha = \tilde{\alpha}$ and $\sigma(L) = \sigma(\tilde{L})$ must be the same as L, i.e, must have $q_+ = \tilde{q_+}$ and $\beta = \tilde{\beta}$. This problem has been long studied and significant work has been done on this by Hochstadt and Libermann, Gesztezy, Simon and Del Rio [2] to name a few. We now study this problem from a complex analytic point of view.
Given two meromorphic inner functions Φ and Ψ, let $\Theta = \Phi\Psi$. Let $\sigma(\Theta)$ denote the point spectrum of Θ, i.e., $\sigma(\Theta) = \{x \in \mathbb{R} : \Theta(x) = 1\}$. The data $[\Psi, \sigma(\Theta)]$ is said to determine Θ (or Φ) if for any other meromorphic inner function $\tilde{\Theta}$ $(\tilde{\Phi})$,

$$\tilde{\Theta} = \Psi\tilde{\Phi}, \sigma(\tilde{\Theta}) = \sigma(\Theta) \Rightarrow \Theta = \tilde{\Theta}. \tag{12}$$

Let $\Theta_- = \Theta^c_{a,\alpha}$ and $\Theta_+ = \Theta^c_{b,\beta}$ be the Weyl inner functions corresponding to q_- and q_+ respectively, evaluated at the point c. We have the following lemma

Lemma 1.

$$\sigma(L) = \sigma(\Theta_-\Theta_+) \tag{13}$$

which leads to the following sufficient condition for determining potential,

Corollary 1. *(q_-, a, α) determine L if $(\Theta_-, \sigma(\Theta_-\Theta_+))$ determine Θ_+.*

We recall the completeness problem for exponentials. Given a set of separated points $\Lambda \subset \mathbb{R}$, is the set $\{e^{i\lambda} : \lambda \in \Lambda\}$ complete in $L^2(a,b)$, i.e., is $\text{span}\{e^{i\lambda} : \lambda \in \Lambda\} = L^2(a,b)$? We can ask the same and related questions with different families of functions. In particular, for each $\lambda \in \mathbf{C}$, consider the solution u_λ to the Schrödinger equation that satisfies boundary condition β at b. Let $\Lambda \subset \mathbf{C}$. The question is if $\{u_\lambda : \lambda \in \Lambda\}$ is complete in $L^2(a,b)$. Let $\Lambda_+ = \Lambda \cap (\mathbf{C}_+ \cup \mathbb{R})$ and $\Lambda_- = \Lambda \cap \mathbf{C}_-$. The following lemma translates the problem above into the language of model spaces.

Lemma 2. *The family $\{u_\lambda\}_{\lambda\in\Lambda}$ is complete in $L^2(a,b)$ if and only if the family $\{k_{\bar\lambda}\}_{\lambda\in\Lambda_-} \cap \{k^*_\lambda\}_{\lambda\in\Lambda_+}$ is complete in K_Θ.*

Another useful way to see these problems is to use the notion of uniqueness sets. We say that $\Lambda \subset \mathbf{C}$ is a uniqueness set of K_Θ if there is no non trivial $F \in K_\Theta$ such that $F = 0$ on Λ. It is not difficult to see that a necessary and sufficient condition for uniqueness is the triviality of the Toeplitz kernel with symbol $\bar\Theta B_\Lambda$, i.e. $f \in K_\Theta$ is zero on Λ if and only if $g = \bar{B_\Lambda} f \in N[\overline{\Theta} B_\Lambda]$.

2. Main Results presented

The following basic criterion for the (non) triviality of the Toeplitz kernel sets the tone for further results.

Proposition 1. *Let $\gamma \in C^\omega(\mathbb{R})$. Then $N^+[e^{i\gamma}] \neq 0$ iff γ has a representation*

$$\gamma = -\alpha + \tilde h,$$

where $\alpha \in C^\omega(\mathbb{R})$ is an increasing function and $h \in L^1_\Pi$.

A similar result in the case of the Hardy space $\mathcal{H}^p$ is the following

Proposition 2. *Let $U = e^{i\gamma}$ with $\gamma \in C^\omega(\mathbb{R})$. Then $N^p[U] \neq 0$ iff*

$$U = \bar\Phi \frac{\bar H}{H},$$

where H is an outer function in $\mathcal{H}^p \cap C^\omega(\mathbb{R})$, $H \neq 0$ on $\mathbb{R}$, and Φ is a meromorphic inner function. Alternatively, $N^p[U] \neq 0$ iff

$$\gamma = -\phi + \tilde h, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2}(\mathbb{R}), \tag{14}$$

where ϕ is the argument of some meromorphic inner function.

The above results are the basis for most of the applications stated in the paper.

Lemma 3. $\Lambda \subset \mathbf{C}_+$ *is not a uniqueness set of* K^p_Θ *iff the function*

$$\gamma = \arg B_\Lambda - \arg \Theta \tag{15}$$

has a representation

$$\gamma = -\phi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2},$$

where ϕ *is the argument of a meromorphic inner function.*

We return to our questions about the Schrödinger operator. The following results are applications of the above propositions.

Proposition 3. *If* $N^\infty[\bar{\Phi}\Psi] \neq \{0\}$*, then the data* $[\Psi, \sigma(\Theta)]$ *does not determine* Θ*.*

We write $N^p_\pi[U]$ for $N^+[U] \cap L^p_\pi$. Below is a partial converse to the above proposition.

Proposition 4. *If* $N^p_\Pi[\bar{\Phi}\Psi] = \{0\}$ *for some* $p < 1$*, then* $[\Psi, \sigma(\Theta)]$ *determine* Θ*.*

As an example we see have the following

Corollary 2. *Suppose* $\Theta = \Psi^2$ *and* $\infty \notin \sigma(\Theta)$*. Then the set of solutions is exactly one-dimensional:* $\tilde{\Theta}$ *satisfies* $\Psi|\tilde{\Theta}$*,* $\sigma(\tilde{\Theta}) = \sigma(\Theta)$ *iff*

$$\exists r \in (-1, 1), \qquad \tilde{\Theta} = \Psi \frac{r + \Psi}{1 + r\Psi}. \tag{16}$$

One can show this even without the assumption that $\infty \notin \sigma(\Theta)$. Thus, if b is the single Blaschke term $\frac{z-i}{z+i}$, then $\sigma(b^2) = \{0, \infty\}$and thus b^2 is not determined by $[b, \sigma(b^2)]$.
These problems can also be described using the tools of defining sets. A set $\Lambda \subset \mathbb{R}$ is said to be defining for a meromorphic inner function Φ, if for any other function $\tilde{\Phi}$ such that $\Phi = \tilde{\Phi}$ as well as $\arg \Phi = \arg \tilde{\Phi}$ on Λ forces $\Phi = \tilde{\Phi}$. Let $\Lambda = \sigma(\Theta)$, then it is not difficult to see that $(\Psi, \sigma(\Theta))$ determines Θ if and only if Λ is defining for Φ. A useful characterization in this case is the following

Proposition 5. Λ *is not defining for* Φ *if there is a non-constant function* $G \in K^\infty_\Phi$ *such that*

$$G = \overline{G} \quad \text{on } \Lambda. \tag{17}$$

We conclude with the consequences of the Beurling Malliavin theory on Toeplitz kernels. In the words of the authors, a metric criterion for the (non)-triviality of the Toeplitz kernel $N^+[e^{i\gamma}]$ up to a gap of $S^{\pm\epsilon} = e^{i\pm\epsilon z}$ is given. As before, $U = e^{i\gamma}$. Suppose $\gamma(\mp\infty) = \pm\infty$ is continuous. The family $\mathcal{BM}(\gamma)$ is defined as the collection of the components of the open set $\{\gamma^* \neq \gamma\}$, where $\gamma^*(x) = \max\limits_{[x,\infty]} \gamma$.
For an interval $l = [a, b] \subset \mathbb{R}_+$ or $\subset \mathbb{R}_-$ we write $|l|$ for the Euclidian length, and $\delta(l)$ for the distance from the origin. A family of finite disjoint intervals $\{l\}$ is called *long* if

$$\sum_{\delta(l) \geq 1} \frac{|l|^2}{\delta(l)^2} = \infty.$$

Otherwise, we call the family *short*.

Theorem 4. *Suppose* $\gamma' > -\text{const}$*.*

(1) *If* $\gamma(\mp\infty) \neq \pm\infty$*, or if* $\gamma(\mp\infty) = \pm\infty$ *but the family* $\mathcal{BM}(\gamma)$ *is long, then* $\forall\epsilon > 0, \quad N^+[S^\epsilon U] = 0$*.*

(2) *If* $\gamma(\mp\infty) = \pm\infty$ *and* $\mathcal{B}M(\gamma)$ *is short, then*

$$\forall\epsilon > 0, \quad N^+[\bar{S}^\epsilon U] \neq 0.$$

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SPECTRAL GAPS FOR SETS AND MEASURES

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presented by Prabath Silva

Abstract. This paper solves the Gap Problem by obtaining a formula for the supremum of spectral gaps of measures supported on a closed subset X of the real line. The proof uses the formulation of the Gap Problem and Beurling-Malliavin theory in terms of the injectvity of Toeplitz kernels.

1. Introduction and Main Result

For a finite complex measure μ on $\mathbb{R}$, the Fourier transform of the measure is defined by

$$\hat{\mu}(z) = \int_{\mathbb{R}} e^{-izt} d\mu(t).$$

Note that $\hat{\mu}$ is defined for $z \in \mathbb{R}$. Under appropriate additional conditions on μ, $\hat{\mu}$ will be defined for all $z \in \mathbb{C}$.
The uncertainty principle says that the supports of μ and $\hat{\mu}$ cannot both be small at the same time. Various qualitative statements about this phenomenon were known to mathematicians for a long time. In this paper we obtain a quantitative result about the supports of a measure and its Fourier transform.
We define the spectral gap of a measure μ.

Definition 1 (Spectral Gap of a measure). *For a finite complex measure on $\mathbb{R}$, define its spectral gap by*

$$G_\mu = \sup\{a : \exists f \in L^1(|\mu|), \widehat{f\mu} = 0 \text{ on } (0,a)\}.$$

Next, for a closed subset $X \subset \mathbb{R}$, define the gap characteristic of X.

Definition 2 (Gap Characteristic of a set). *For a closed subset $X \subset \mathbb{R}$, we define the gap characteristic of X by*

$$G_X = \sup\{a : \exists \mu \neq 0, \operatorname{supp} \mu \subset X, \hat{\mu} = 0 \text{ on } [0,a]\}.$$

Proposition 1. [1] $G_\mu = G_{\operatorname{supp} \mu}$.

The Gap Problem asks us to formulate G_X in terms of X, that is without using the Fourier transform. This paper solves the Gap Problem by obtaining a formula, !!!, for G_X.
The Gap Problem has connections to the following completeness problem of exponentials. These connections were well known to experts in the field, but formulating both the Gap Problem and the completeness problem using Toeplitz kernels [2] was a key step towards the solution to the Gap Problem; see Section 3.
For a set $\Lambda = \{\lambda_n\} \subset \mathbb{C}$, consider the complex exponential functions with frequencies from Λ,

$$E_\Lambda = \{e^{i2\pi\lambda_n x}\}.$$

The completeness radius of Λ is defined by

$$R(\Lambda) = \sup\{a : E_\Lambda \text{ is a complete set in } L^2(0,a)\}.$$

Beurling and Malliavin in their celebrated work in the 1960's, later called Beurling-Malliavin theory [4], [5] found a formula for the radius of completeness.
A set of disjoint intervals $\{I_n\}_{n\in\mathbb{Z}}$ is called a long sequence if

$$\sum \frac{|I_n|^2}{1+\text{dist}^2(0,I_n)} = \infty.$$

If the sum is finite, it is called a short sequence.
One can reduce the case $\Lambda \subset \mathbb{C}$ to the case with real frequencies, so it is enough to consider $\Lambda \subset \mathbb{R}$.

Definition 3. *For a subset $\Lambda \subset \mathbb{R}$, define its BM density by*

$$d_{BM}(\Lambda) = \sup\{d | \exists \text{ long sequence } \{I_n\}, \forall n\ |\Lambda \cap I_n| \geq d|I_n|\}.$$

Theorem 1. *For $\Lambda \subset \mathbb{R}$, we have $R(\Lambda) = d_{BM}(\Lambda)$.*

Now we are ready to formulate the main theorem of the paper. First we define the notion of a short partition.

Definition 4 (Short partition)**.** *A collection of intervals I_n is caled a short partition of intervals if*
(1) $I_n = (a_n, a_{n+1}]$ for some collection of real numbers a_i such that $\cdots < a_{-1} < a_0 = 0 < a_1 < \cdots$.
(2) $\{I_n\}$ is a short sequence of intervals.
(3) $|I_n| \to \infty$ as $|n| \to \infty$.

Now for a closed set $X \subset \mathbb{R}$ we define the metric C_X.

Definition 5. *Let $\Lambda = \{\lambda_n\} \subset \mathbb{R}$. We say $C_\Lambda \geq a$ if there exists a short partition $\{I_n\}$ that satisfies the following conditions:*
(1) Density condition

$$\Delta_n = |\Lambda \cap I_n| \geq a|I_n|.$$

(2) Energy condition

$$\sum_n \frac{\Delta_n^2 \log|I_n| - E_n}{1+dist^2(0,I_n)} < \infty,$$

where

$$E_n = \sum_{\lambda_k,\lambda_l \in I_n, \lambda_k \neq \lambda_l} \log|\lambda_k - \lambda_l|.$$

For a closed subset $X \subset \mathbb{R}$ we define

$$C_X = \sup\{a : \exists \Lambda \subset X, C_\Lambda \geq a\}.$$

The following theorem gives the solution to the long standing Gap Problem, and a quantitative realization of the uncertainty principle.

Theorem 2. [1] *For a closed set $X \subset \mathbb{R}$, we have*

$$G_X = 2\pi C_X.$$

2. Background

The first result on the completeness problem of exponentials was the following Payley-Weiner Theorem.

Theorem 3. *[Paley and Wiener, 1934]*

$$R(\Lambda) \geq \bar{D}(\Lambda) = \limsup \frac{|\Lambda \cap (0,x)|}{x}.$$

The proof of this theorem follows from the following connection to the distribution of zeros of analytic functions of exponential type.
Since $L^2(0,a)$ is a Hilbert space, we have that E_Λ is not a complete set when we have a non-zero $f \in L^2(0,a)$ which is orthogonal to E_Λ. This reduces the problem to the distribution of zeroes of analytic functions that arise from the Fourier transforms of functions from $L^2(0,a)$.

Theorem 4. *[Paley and Wiener] For $f \in L^2(-a,a)$, we have*

$$|\hat{f}(z)| \leq e^{2\pi a|z|}.$$

Also any function that satisfies the above decay estimate arises from a Fourier transform of a function from $L^2(-a,a)$.

Now one can use these decay estimates, for example Jensen's formula, to obtain estimates on the distribution of zeros of the Fourier transform, but this is same as the set Λ.
This started a hunt for a $D(\Lambda)$, and it was finally obtained by Beurling-Malliavin by proving Theorem 1.
For the Gap Problem we are interested in the spectral gap of measures. The following lemma can be viewed as an analogue for Theorem 4.

Lemma 1. *Let μ be a complex measure with finite total variation. Then $\hat{\mu} = 0$ on $[-a,a]$ if and only if*

$$\lim_{y \to \pm\infty} e^{ay} \int \frac{d\mu(t)}{t - iy} = 0.$$

For a complex measure μ with finite total variation, define the Cauchy integral of μ by

$$K\mu(z) = \int \frac{d\mu(t)}{t - z}.$$

Thus Lemma 1 gives a characterization of measures μ with spectral gap using the decay properties of $K\mu$ along the y-axis. The following qualitative theorem can be proved using the connection of G_μ to $K\mu$; see [9].

Theorem 5. [8] *Let μ be a finite measure on $\mathbb{R}$ with $\mu = 0$ on an interval. Suppose there exists a sequence of disjoint intervals I_n such that*

$$\sum \frac{|I_n|}{1 + dist^2(0, I_n)} \min\{|I_n|, -\log|\mu|(I_n)\} = \infty.$$

Then $\mu \equiv 0$.

This theorem is powerful enough to derive some known results on the Gap Problem, like Beurling's Gap Theorem [6] and de Branges's Theorem 63 [7].

3. Toeplitz Approach

As mentioned in the introduction, the key to the recent advances, [2], [3], [8] in the circle of problems related to the uncertainty principle comes from the formulation of these problems in terms of the injectivity of Toeplitz kernels.

Definition 6 (Toeplitz kernels)**.** *For $U \in L^\infty\mathbb{R}$, define the Toeplitz kernel $N[U]$ by*

$$N[U] = \{F \in H_2 : T_U(F) = P_+(FU) = 0\},$$

where P_+ denotes the projection from L^2 to H^2.
Define the Toeplitz kernel in the Smironv class $\mathcal{N}^+$ by

$$N^+[U] = \{F \in \mathcal{N}^+ \cap L^1_{loc}(\mathbb{R}) : \bar{U}\bar{F} \in \mathcal{N}^+\}.$$

For $0 < p \leq \infty$, define the Toeplitz kernels in H_p by

$$N^p[U] = N^+[U] \cap L^p(\mathbb{R}).$$

The inner function is a bounded analytic function in the upper half plane with boundary values with absolute value one. The following theorem connects G_X to the Toeplitz kernels.

Theorem 6 ([2])**.** *For a closed set $X \subset \mathbb{R}$ we have*

$$G_X = T_X,$$

where

$$T_X = \sup\{a : N[\bar{\theta} e^{iaz}] \neq 0 \textit{ for some meromorphic inner function } \theta \textit{ with } \{\theta = 1\} \subset X\}.$$

The following theorem formulates BM density in terms of the injectivity of Toeplitz kernels.

Theorem 7. [3] *We have*

$$d_{BM}(\Lambda) = \sup\{a : N[e^{-2\pi a z}\theta] = 0\},$$

where θ is a some/any meromorphic inner function with $\{\theta = 1\} = \Lambda$. Also by the Toeplitz formulation of the BM multiplier theorem, one can replace the above kernel N with N^+ or N^p for $0 < p \leq \infty$.

The following lemma from [1] gives a nice application of these Toeplitz formulations. This allow one to use different results in a unified setting.

Lemma 2. *Let $X \subset \mathbb{R}$ and let Λ be a discrete sequence. Then*

$$G_{X \sup \Lambda} \leq G_X + 2\pi d_{BM}(\Lambda).$$

The proof of Theorem 2 is far from being soft, even with the BM theorem in hand. Proof of the theorem uses intricate constructions. The density condition is used to control the argument of inner functions, and the energy condition helps in showing certain functions are in the Dirichlet class.

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