PROCEEDINGS OF THE INTERNET ANALYSIS SEMINAR ON: RIESZ TRANSFORMS ASSOCIATED TO SCHRÖDINGER OPERATORS

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Overview of the Workshop

This workshop was part of the Internet Analysis Seminar that is the education component of the National Science Foundation – DMS # 0955432 held by Brett D. Wick. The Internet Analysis Seminar consists of three phases that run over the course of a standard academic year. Each year, a topic in complex analysis, function theory, harmonic analysis, or operator theory is chosen and an internet seminar will be developed with corresponding lectures. The course will introduce advanced graduate students and post-doctoral researchers to various topics in those areas and, in particular, their interaction.

This was a workshop that focused on the connections between Riesz transforms and Schrödinger operators. The canonical case of this arises when the operator is the standard Laplacian, and then leads to the basic material considered in standard harmonic analysis courses, such as the boundedness of the Riesz transforms on L^p, duality between H^1 and BMO and other related topics.

Changing the operator to $L = ?\operatorname{div}(A(x)f)$ where $A(x)$ is a matrix with complex bounded entries that satisfies an elliptic estimate the generalizations of the standard results encountered in harmonic analysis are obtained when considering the Riesz transforms associated to the square root of the operator $L^{-\frac{1}{2}}$. Topics covered will focus on the modifications in the techniques and proofs necessary to understand how the story changes when proving the boundedness of these Riesz transforms and the $H^1?BMO$ duality statements when no longer working with the Laplacian but instead working with more general elliptic operators.

The participants that presented, presented one of the following papers:

[1] Pascal Auscher, Thierry Coulhon, Xuan Thinh Duong, and Steve Hofmann, *Riesz transform on manifolds and heat kernel regularity*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 6, 911–957 (English, with English and French summaries). ↑

[2] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. (2) **156** (2002), no. 2, 633–654. ↑

[3] Pascal Auscher, Alan McIntosh, and Emmanuel Russ, *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal. **18** (2008), no. 1, 192–248. ↑

[4] Pascal Auscher and Emmanuel Russ, *Hardy spaces and divergence operators on strongly Lipschitz domains of* $\mathbb{R}^n$, J. Funct. Anal. **201** (2003), no. 1, 148–184. ↑

[5] Thierry Coulhon and Xuan Thinh Duong, *Riesz transforms for* $1 \leq p \leq 2$, Trans. Amer. Math. Soc. **351** (1999), no. 3, 1151–1169. ↑

[6] E. B. Davies, *Heat kernel bounds, conservation of probability and the Feller property*, J. Anal. Math. **58** (1992), 99–119. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. ↑

[7] Xuan Thinh Duong and Lixin Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. **18** (2005), no. 4, 943–973 (electronic). ↑

[8] Jacek Dziubański, *Atomic decomposition of* H^p *spaces associated with some Schrödinger operators*, Indiana Univ. Math. J. **47** (1998), no. 1, 75–98. ↑

[9] Steve Hofmann, Michael Lacey, and Alan McIntosh, *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Ann. of Math. (2) **156** (2002), no. 2, 623–631. ↑

[10] Steve Hofmann and José María Martell, *L^p bounds for Riesz transforms and square roots associated to second order elliptic operators*, Publ. Mat. **47** (2003), no. 2, 497–515. ↑
[11] Steve Hofmann and Svitlana Mayboroda, *Hardy and BMO spaces associated to divergence form elliptic operators*, Math. Ann. **344** (2009), no. 1, 37–116. ↑
[12] Jizheng Huang, *Hardy spaces associated to the Schrödinger operator on strongly Lipschitz domains of* $\mathbb{R}^d$, Math. Z. **266** (2010), no. 1, 141–168. ↑

They were then responsible to prepare two one-hour lectures based on the paper and an extended abstract based on the paper. This proceedings is the collection of the extended abstract prepared by each participant.

The following people participated in the workshop:

Dario Mena Arias	Georgia Institute of Technology
Tyler Bongers	Michigan State University
Cong Hoang	University of Alabama
Irina Holmes	Georgia Institute of Technology
Ishwari Kunwar	Georgia Institute of Technology
Bochen Liu	University of Rochestor
Shahaf Nitzan	Kent State University
Yumeng Ou	Brown University
Robert Rahm	Georgia Institute of Technology
Scott Spencer	Georgia Institute of Technology
Hagop Tossounian	Georgia Institute of Technology
Guillermo Rey	Michigan State University
Brett D. Wick	Georgia Institute of Technology

THE SOLUTION OF THE KATO PROBLEM FOR DIVERGENCE FORM ELLIPTIC OPERATORS WITH GAUSSIAN HEAT KERNEL BOUNDS

STEVE HOFMANN, MICHAEL LACEY AND ALAN MCINTOSH

presented by Tyler Bongers

Abstract. Given an $n \times n$ matrix A with complex-valued L^∞ coefficients and satisfying $\lambda I \leq A \leq \lambda^{-1} I$, it is possible to define a divergence form elliptic operator by $L = -\operatorname{div} A\nabla$; this operator has a square root $L^{1/2}$. A classical conjecture of Kato implies the bound $\|L^{1/2} f\|_{L^2(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^n)}$ with a constant independent of A. Under an additional assumption of pointwise bounds on the heat kernel of L, this estimate is proven.

1. Introduction and Main Results Presented

Fix a dimension n and a matrix $A = (a^{ij})_{1\leq i,j\leq n}$ so that each a^{ij} is a complex-valued L^∞ function defined on $\mathbb{R}^n$. Assume that A is elliptic (or accretive) in the sense that there exists $\lambda > 0$ with

$$\lambda|\xi|^2 \leq \operatorname{Re}\langle A\xi, \xi\rangle = \operatorname{Re}\sum_{i,j} a^{ij}\xi_j\bar{\xi}_i \leq \lambda^{-1}|\xi|^2$$

Using this, we define a divergence form operator

$$L = -\operatorname{div}\big(A(x)\nabla u\big)$$

interpreted in the weak sense as a sesquilinear form:

$$\int_{\mathbb{R}^n}\sum_{i,j} a^{ij}\partial_j f\overline{\partial g_i}$$

for f, g in the Sobolev space $W^{1,2}$. Due to the accretivity of A, it is possible to define a square root operator $L^{1/2}$ which is also accretive (see, e.g. [6]). This paper is concerned with the estimate

$$\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^n)} \qquad (1)$$

where C is a constant depending only on n, λ and $\Lambda = \|A\|_\infty$. This estimate is proven in [7] under the assumption that the heat kernel e^{-t^2L} satisfies some additional pointwise bounds in terms of Gaussian decay. Let us state exactly these bounds:

Definition Fix L, and let $W_{t^2}(x,y)$ denote the kernel of the operator e^{-t^2L}. We say that L satisfies property (G) if there exist positive constants α and β such that

- $|W_{t^2}(x,y)| \leq \beta t^{-n}\exp\left\{\frac{-|x-y|^2}{\beta t^2}\right\}$
- Whenever $|h| \leq t$ or $2|h| \leq |x-y|$,

$$|W_{t^2}(x+h,y) - W_{t^2}(x,y)| + |W_{t^2}(x,y+h)| - |W_{t^2}(x,y)| \leq$$
$$\leq \beta\frac{|h|^\alpha}{t^{\alpha+n}}\exp\left\{\frac{-|x-y|^2}{\beta t^2}\right\}$$

For example, the heat kernel of the Laplacian is given by

$$K_{t^2}(x,y) = \frac{1}{(4\pi)^{n/2}t^n} e^{-|x-y|^2/4t^2}$$

and satisfies property (G) in all dimensions; it is also known from [4] that this holds in dimension 2 always. The theorem proven here is

Theorem 1. *If L satisfies property G, then the square root estimate holds with constant depending on $\alpha, \beta, n, \lambda$ and Λ (called the allowable parameters).*

This theorem is generalized in [1] to remove the assumption of Gaussian decay.

2. Main Result 1

The proof of the main theorem will follow from a T(b) theorem of Auscher and Tchamitchian. Consider functions

$$\gamma_t(x) = e^{-t^2 L} t L \varphi = -e^{-t^2 L} t \operatorname{div} A$$

where $\varphi(x) = x$ throughout. Note that the equality follows from the fact that $\nabla \varphi = I_n$ is the $n \times n$ identity matrix: We interpret the gradient of an $\mathbb{R}^n$-valued function columnwise. First we bound γ_t.

Lemma 2. *If L satisfies property (G), then*

$$C_0 := \sup_{t>0} \left(\|\gamma_t(\cdot)\|_{L^\infty(\mathbb{R}^n)} + t^\alpha \sup_{h \neq 0} \| |h|^{-\alpha} (\gamma_t(\cdot + h) - \gamma_t(\cdot)) \|_{L^\infty(\mathbb{R}^n)} \right)^2 < \infty$$

and C_0 depends only on the allowable parameters.

This is proven by using the analyticity of the semigroup to give pointwise bounds on the kernel of tLe^{-t^2L}. We now have a slightly modified T(b) theorem for square roots, which is a generalized version of Auscher and Tchamitchian's result in [2], using N functions instead of a single one. For notation, we let $|Q|$ be the measure of Q, $\ell(Q)$ its sidelength, and P_t an approximate identity given by $P_t(x) = t^{-n}\varphi(x/t)$ with $0 \le \varphi \le 1$, $\varphi \in C_0^\infty$ and $\int \varphi = 1$.

Theorem 3. *Suppose there is a finite index set J with cardinality N such that for each cube $Q \subseteq \mathbb{R}^n$, there is a family of functions $\{F_{Q,\nu}\}_{\nu \in J} : 5Q \to \mathbb{C}$ satisfying*

- *(i)* $\int_{5Q} |\nabla F_\nu|^2 \le C'|Q|$
- *(ii)* $\int_{5Q} |LF_\nu|^2 \le C'' \frac{|Q|}{\ell(Q)^2}$
- *(iii)*

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dtdx}{t} \le$$

$$\le C \sum_{\nu \in J} \left\{ C_0 + \sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x) P_t(\nabla F_\nu)(x)^2| \frac{dtdx}{t} \right\}$$

Then the square root estimate (1) *holds with constant on the order of* $C(1+\sqrt{N(C_0 + C' + C'')})$.

So in order to prove the main theorem, one needs to construct the family of functions for each Q.

Fix a cube, and let its side length be $\ell(Q)=\rho$; let $\epsilon>0$ be fixed later. We will choose ϵ to depend only on the allowable parameters, so this is possible. The Carleson box of a cube Q is $R_Q = Q\times[0,\ell(Q)]$. Define a mapping $F_Q : 5Q \to \mathbb{C}^n$ via

$$F_Q = e^{-\epsilon^2\rho^2 L}\varphi$$

If $\nu\in\mathbb{C}^n$ is a unit vector, define the $\mathbb{C}$-valued function

$$F_{Q,\nu} = F\cdot\overline{\nu}$$

It is easy to show that these functions satisfy properties (i) and (ii) above.
Given a fixed vector ν of modulus 1 in $\mathbb{C}^n$, define a "cone"

$$C_\nu = \{z\in\mathbb{C}^n : |z-\nu(z\cdot\overline{\nu})| < \epsilon|z\cdot\overline{\nu}|\}$$

We then have

Lemma 4. *For each $\epsilon>0$ and dimension n, there is an $N=N(\epsilon,n)<\infty$ so that exactly N such cones C_ν cover $\mathbb{C}^n$.*

The finite family of functions that we consider in the T(b) theorem will then be indexed by a set of vectors whose corresponding cones cover $\mathbb{C}^n$. All that remains is property (iii); first, note that it suffices to show the following estimate:

$$\sup_Q \frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |\gamma_t(x)1_{C_\nu}(\gamma_t(x))|^2\frac{dt}{t}dx \leq$$
$$CC_0 + C\sup_Q\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)}|\gamma_t(x)P_t(\nabla F_\nu)(x)|^2\frac{dt}{t}dx \qquad (2)$$

since we then sum over our finitely many vectors ν to prove the result. Furthermore, we can make even more of a reduction by well-known Carleson measure techniques (for example, following a minor modification of Lemma 3.3 in [3]):

Lemma 5. *Suppose there is a positive η with the following property: For each Q and ν, there is a collection $\{Q_k\}$ of nonoverlapping dyadic subcubes of Q and a decomposition $Q=E\cup B$ with $E\cap B=\emptyset$, $|E|\geq\eta|Q|$ and $B=\cup_k R_Q$ for which*

$$\iint_{E^*}|\gamma_t(x)1_{C_\nu}(\gamma_t(x))|^2\frac{dt}{t}dx \leq C\left\{C_0|Q| + \int_Q\int_0^{\ell(Q)}|\gamma_t(x)P_t(\nabla F_\nu)(x)|^2\frac{dt}{t}dx\right\}$$

where $E^* = R_Q\setminus\cup R_{Q_k}$. Then (2) holds.

We now proceed in proving this last estimate, which is done by a stopping time argument. First, notice that since φ is Lipschitz,

$$\|F-\varphi\|_\infty \leq C\epsilon\rho$$

This is proven by a direct computation with the kernel: Since $e^{-\epsilon^2\rho^2L}1=1$, we have

$$(F-\varphi)(x) = (e^{-\epsilon^2\rho^2L}-I)\varphi = \int_{\mathbb{R}^n} W_{(\epsilon\rho)^2}(x,y)(y-x)\,dy$$

The decay in the kernel then gives the result, after considering y in an $\epsilon\rho$ neighborhood of x, and y outside that neighborhood. Integrating this,

$$\frac{1}{|Q|}\left|\int_Q \nabla F - I_n\,dx\right| \leq C\epsilon$$

and so

$$\frac{1}{|Q|}\left|\int_Q \nu\cdot\nabla F_\nu - 1\,dx\right| \le C\epsilon \qquad (3)$$

We are now ready to define the stopping cubes.

- Type 1: Let S_1 be the collection of maximal dyadic subcubes $Q' \subseteq Q$ for which
$$\frac{1}{|Q'|}\operatorname{Re}\int_{Q'} \nu\cdot\nabla F_\nu\,dx \le \frac{3}{4}$$
- Type 2: Let S_2 be the collection of maximal dyadic subcubes $Q' \subseteq Q$ for which
$$\frac{1}{|Q'|}\int_{Q'} |\nabla F_\nu|\,dx > \frac{1}{8\epsilon}$$

We set $B_1 = \cup S_1$, $B_2 = \cup S_2$ and $B = B_1 \cup B_2$, $E = Q \setminus B$.
Thus a cube is added to the collection if the gradient has too large an average (which is quite rare because of the above lemma, which says that the average over all of Q is of constant order) or if the real part of the average gets too small. So there are now two points that remain to check: E needs to contain a proportion of Q, and the Carleson measure estimate must hold.
First, let us see why E is large: Taking real parts in (3), we see that

$$\begin{aligned}(1-C\epsilon)|Q| &\le \operatorname{Re}\int_Q \nu\cdot\nabla F_\nu\,dx \\ &= \operatorname{Re}\left(\int_E + \int_{B_1} + \int_{B\setminus B_1}\right)\end{aligned}$$

Note that

$$\operatorname{Re}\int_{B_1} = \operatorname{Re}\int_{Q'\in S_1}\int_{Q'} \nu\cdot\nabla F_\nu\,dx \le \frac{3}{4}\sum_{Q'\in S_1}|Q'| \le \frac{3}{4}|Q|$$

since the cubes are maximal, and hence nonoverlapping. The term involving $B \setminus B_1$ can be estimated away as $C\epsilon|Q|$ by using the weak $(2,2)$ bounds for the dyadic maximal function and property (ii). Finally, applying the Cauchy-Schwarz inequality to the term involving E and rearranging leads to

$$C\frac{|E|^{1/2}}{|Q|^{1/2}} \ge (1-C\epsilon) - \frac{3}{4} - C\epsilon$$

which suffices if ϵ is chosen small enough. Note that the constants here (and hence ϵ and η) depend only on the allowable parameters.
Finally, we must prove the Carleson measure estimate. Define a dyadic averaging operator:

$$A_t f(x) = \frac{1}{|Q(x,t)|}\int_{Q(x,t)} f(y)\,dy$$

where $Q(x,t)$ is the minimal dyadic subcube of Q containing x with sidelength at least t. Set

$$V(x,t) = A_t(\nabla F_\nu)(x)$$

We have the following lemma, which is proven by an easy geometric analysis of the cones:

Lemma 6. *If $z \in C_\nu$ then $|z\cdot V| \ge \frac{1}{2}|z|$.*

Given this lemma, the proof of the main theorem is essentially done: We have

$$\iint_{E^*} |\gamma_t(x) 1_{C_\nu}(\gamma_t(x))|^2 \frac{dt}{t} dx \le 4 \iint_{R_R} |\gamma_t(x) \cdot A_t(\nabla F_\nu)(x)|^2 \frac{dt}{t} dx$$

by using the fact that $z = \gamma_t(x) \in C_\nu$ whenever the left integrand is nonzero. It simply remains to replace A_t by P_t, but this follows from a standard orthogonality argument (e.g. [5]).

References

[1] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and Ph. Tchamitchian, The solution of the Kato square root problem for elliptic operators on $\mathbb{R}^n$, *Ann. of Math.* **156** (2002), 633-654.

[2] P. Auscher and Ph. Tchamitchian, Square Root Problem for Divergence Operators and Related Topics, *Asterisque* **249**, Soc. Math. Fiance, Paris, 1998.

[3] P. Auscher, S. Hofmann J. L. Lewis, and Ph. Tchamitchian, Extrapolation of Carleson measures and the analyticity of Kato's square root operators, *Acta Math.* **187** (2001), 161-190.

[4] P. Auscher, A. McIntosh, and Ph. Tchamtichian, Heat kernels of second order complex elliptic operators and applications, *J. Funct. Anal.* **152** (1998), 22-73.

[5] J.-L. Journé, Remarks on Kato's Square Root Problem, *Publ. Mat.* **35** (1991), 299-321.

[6] T. Kato, Fractional Powers of Dissipative Operators, *J. Math Soc. Japan* **13** (1961), 246-274.

[7] S. Hofmann, M. Lacey, A. McIntosh, The Solution of the Kato Problem for Divergence Form Elliptic Operators with Gaussian Heat Kernel Bounds, *Ann. of Math.* **156** (2002), 623-631.

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HARDY AND BMO SPACES ASSOCIATED TO DIVERGENCE FORM ELLIPTIC OPERATORS

STEVE HOFMANN AND SVITLANA MAYBORODA

presented by Cong Hoang

ABSTRACT. Consider a second order divergence form elliptic operator L with complex bounded measurable coefficients. In general, operators based on L, such as the Riesz transforms or square functions, need not be bounded in the classical Hardy, BMO and even some L^p spaces. In this paper, the two authors develop a new theory of Hardy and BMO spaces associated to L, which includes a molecular decomposition, maximal and square functions characterizations, duality of new Hardy and BMO spaces, and a John-Nirenberg inequality.

1. Introduction and Main Results Presented

Immense study of classical real-variable Hardy spaces in $\mathbb{R}^n$ began in 1960s with the fundamental paper of Stein and Weiss [27]. The theory of Hardy spaces is closely related to properties of harmonic functions and of the Laplacian. For instance, the classical Hardy space $H^1(\mathbb{R}^n)$ can be viewd as a collection of $f \in L^1(\mathbb{R}^n)$ such that the Riesz transform $\nabla\Delta^{-\frac{1}{2}}f$ belongs to $L^1(\mathbb{R}^n)$. One can also characterize the space via square functions and non-tangential maximal functions associated to semigroups generated by the Laplacian.

Let A be an $n \times n$ matrix with entries

$$a_{jk} \in L^\infty(\mathbb{R}^n), \quad j = 1, ..., n, \quad k = 1, ..., n$$

satisfying the ellipticity condition

$$\lambda|\xi|^2 \leqslant \mathcal{R}e(A\xi \cdot \bar{\xi}) \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \leqslant \Lambda|\xi||\zeta|, \quad \forall \xi, \zeta \in \mathbb{C}^n$$

for some positive constants λ and Λ. The second order divergence form operator is given by

$$Lf = -\text{div}(A\nabla f).$$

When the Laplacian is replaced by L, the classical Hardy space H^1 is not compatible with the corresponding operators associated to L any more. For example, the Riesz transform $\nabla L^{-\frac{1}{2}}$ is no longer bounded from classical H^1 to L^1. This fact comes from the results in [3, 5, 10], together with an interpolation argument. Therefore, this paper was written with two main goals: (1) to generalize the classical theory of Hardy spaces so that it is well-fit with the new operators that we are concerning about, and (2) to develop a corresponding BMO theory which includes an analogue of the $H^1 - BMO$ duality theorem and of the John-Nirenberg Lemma.

For any cube $Q \subset \mathbb{R}^n$, let

$$S_0(Q) = Q, \quad Q_i = 2^i Q \quad \text{and} \quad S_i(Q) = Q_i \setminus Q_{i-1}$$

where 2^iQ has the same center as Q but is twice as long in sidelength.

Let

$$p_L = \inf\{p \geqslant 1 : \sup_{t>0} \|e^{-tL}\|_{L^p\to L^p} < \infty\}$$

and

$$\widetilde{p}_L = \sup\{p \geqslant 1 : \sup_{t>0} \|e^{-tL}\|_{L^p\to L^p} < \infty\}.$$

Fix $p \in (p_L, \widetilde{p}_L)$, $\epsilon > 0$ and a natural number $M > \frac{n}{4}$. A function $m \in L^p(\mathbb{R}^n)$ is called a (p, ϵ, M)-molecule if there is a cube Q such that

$$(i) \quad \|m\|_{L^p(S_i(Q))} \leqslant 2^{-i(n-\frac{n}{p}+\epsilon)}|Q|^{\frac{1}{p}-1}, \quad i = 0, 1, 2, ...$$

and

$$(ii) \quad \|(l(Q)^{-2}L^{-1})^k m\|_{L^p(S_i(Q))} \leqslant 2^{-i(n-\frac{n}{p}+\epsilon)}|Q|^{\frac{1}{p}-1}, \quad i = 0, 1, 2, ...; k = 1, ..., M.$$

The Hardy space associated to L is defined to be

$$H^1_L(\mathbb{R}^n) = \left\{\sum_{j=0}^{\infty} \lambda_j m_j : \{\lambda_j\}_{j=0}^{\infty} \in \ell^1 \text{ and } m_j\text{'s are molecules}\right\}$$

with the norm given by

$$\|f\|_{H^1_L(\mathbb{R}^n)} = \inf\left\{\sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j m_j, \{\lambda_j\}_{j=0}^{\infty} \in \ell^1 \text{ and } m_j\text{'s are molecules}\right\}.$$

In this setting, we have

$$\nabla L^{-\frac{1}{2}} : H^1_L(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

as desired. However, it is still an open question (at least at the time this paper was written) if the "new" Riesz transform characterizes the space H^1_L.

Next, we consider the square and maximal operators associated to the heat semigroup generated by L

$$S_h f(x) = \left(\iint_{\Gamma(x)} \left|t^2 L e^{-t^2 L} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}$$

$$\mathcal{N}_h f(x) = \sup_{(y,t)\in\Gamma(x)} \left(\frac{1}{t^n}\int_{B(y,t)} \left|e^{-t^2 L} f(z)\right|^2 dz\right)^{\frac{1}{2}}$$

and the square and maximal operators associated to the Poisson semigroup generated by L

$$S_P f(x) = \left(\iint_{\Gamma(x)} \left|t\nabla e^{-t\sqrt{L}} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}$$

$$\mathcal{N}_P f(x) = \sup_{(y,t)\in\Gamma(x)} \left(\frac{1}{t^n}\int_{B(y,t)} \left|e^{-t\sqrt{L}} f(z)\right|^2 dz\right)^{\frac{1}{2}}$$

where $\Gamma(x) = \{(y, t) \in \mathbb{R}^n : |x - y| < t\}$.

We then define $H^1_{S_h}(\mathbb{R}^n)$ to be the completion of $\{f \in L^2(\mathbb{R}^n) : S_h f \in L^1(\mathbb{R}^n)\}$ with respect to the norm $\|f\|_{H^1_{S_h}(\mathbb{R}^n)} = \|S_h f\|_{L^1(\mathbb{R}^n)}$. The spaces $H^1_{\mathcal{N}_h}(\mathbb{R}^n)$, $H^1_{S_P}(\mathbb{R}^n)$ and $H^1_{\mathcal{N}_P}(\mathbb{R}^n)$ are defined analogously.

Theorem 1. *(Thm. 1.1) The spaces $H^1_L(\mathbb{R}^n)$, $H^1_{S_h}(\mathbb{R}^n)$, $H^1_{\mathcal{N}_h}(\mathbb{R}^n)$, $H^1_{S_P}(\mathbb{R}^n)$ and $H^1_{\mathcal{N}_P}(\mathbb{R}^n)$ coincide. Moreover,*

$$\|f\|_{H^1_L(\mathbb{R}^n)} \approx \|f\|_{H^1_{S_h}(\mathbb{R}^n)} \approx \|f\|_{H^1_{\mathcal{N}_h}(\mathbb{R}^n)} \approx \|f\|_{H^1_{S_P}(\mathbb{R}^n)} \approx \|f\|_{H^1_{\mathcal{N}_P}(\mathbb{R}^n)}$$

The second half of the paper is mainly dealing with the duality theorem. To do that, we need to introduce the adapted BMO space. We note here that the idea about adapted BMO space and duality has been previously introduced by Duong and Yan [15, 16], but they assume a point-wise boundedness for the heat kernel associated to L while the authors of this paper do not.

We define the adapted BMO space as follows. Fix $\epsilon > 0$ and a natural number $M > \frac{n}{4}$ we introduce the space

$$\mathbf{M}_0^{2,\epsilon,M} = \left\{\mu \in L^2(\mathbb{R}^n) : \|\mu\|_{\mathbf{M}_0^{2,\epsilon,M}} < \infty\right\}$$

where

$$\|\mu\|_{\mathbf{M}_0^{2,\epsilon,M}} = \sup_{j \geqslant 0} 2^{j(\frac{n}{2}+\epsilon)} \sum_{k=0}^{M} \|L^{-k}\mu\|_{L^2(S_j(Q_0))}$$

and Q_0 is the unit cube centered at origin. An element $f \in \cap_{\epsilon>0}(\mathbf{M}_0^{2,\epsilon,M})^* \equiv (\mathbf{M}_0^{2,M})^*$ is said to belong to $BMO^p_{L^*}$ if

$$\|f\|_{BMO^p_{L^*}} = \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q \left|\left(I - e^{-l(Q)^2 L^*}\right)^M f(x)\right|^p dx\right)^{\frac{1}{p}} < \infty.$$

By interchanging L and L^*, we have the definition for BMO^p_L. When $p = 2$, we simply write BMO_L for BMO^2_L and $\|f\|_{BMO_L}$ for $\|f\|_{BMO^2_L}$. We then have the following two theorems.

Theorem 2. *(Thm. 1.2) For all $p \in (p_L, \widetilde{p}_L)$, the spaces BMO^p_L coincide.*

Theorem 3. *(Thm 1.3) $(H^1_L)^* = BMO_{L^*}$.*

Due to time limited, we cover following topics in the presentation.

1/ Notation and preliminaries
2/ Sublinear operators in Hardy spaces
3/ Characterization by the square operator associated to the heat semigroup
4/ BMO_L: duality

2. Sublinear Operator in Hardy Spaces

Theorem 4. *(Thm. 3.2) Let $p \in (p_L, 2]$. Assume that $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded sublinear operator, and that there exist $M \in \mathbb{N}, M > \frac{n}{4}$ such that for all closed sets $E, F \subset \mathbb{R}^n$ with $dist(E,F) > 0$ and all $f \in L^p(\mathbb{R}^n)$ supported in E*

$$\left\|T(I - e^{-tL})^M f\right\|_{L^p(F)} \leqslant C\left(\frac{t}{dist(E,F)^2}\right)^M \|f\|_{L^P(E)}, \quad \forall t > 0$$

$$\left\|T(tLe^{-tL})^M f\right\|_{L^p(F)} \leqslant C\left(\frac{t}{dist(E,F)^2}\right)^M \|f\|_{L^P(E)}, \quad \forall t > 0$$

Then,

$$T : H^1_L(\mathbb{R}^n) \to L^1(\mathbb{R}^n).$$

Remark: The result of Theorem 3.2 also holds for $p \in (2, \widetilde{p}_L)$, but one needs to take $M > \frac{1}{2}\left(n - \frac{n}{p}\right)$.

For $f \in L^2(\mathbb{R}^n)$, we define

$$g_h f(x) = \left(\int_0^\infty \left|t^2 Le^{-t^2 L} f(x)\right| \frac{dt}{t}\right)^{\frac{1}{2}}.$$

Then g_h and the Riesz transform are two of the examples for such operators T in the above theorem.

3. Charaterization by the Square Operator S_h

Theorem 5. *(Thm. 4.1) $H^1_L = H^1_{S_h}$ and $\|f\|_{H^1_L} \approx \|f\|_{H^1_{S_h}}$.*

Remark: Since the definition of the space $H^1_{S_h}$ does not depend on the parameters p, ϵ and M, from the proof of this theorem, we will see that no matter what p, ϵ and M we start with, we are going to have the same space H^1_L.

4. BMO_{L^*} as the Duality of H^1_L

The duality is the result of the follwing two theorems.

Theorem 6. *(Thm. 8.2) Let $f \in BMO_{L^*}(\mathbb{R}^n)$. Then the linear functional given by*

$$l(g) = \langle f, g\rangle$$

which is initially defined on the dense subspace of finite linear combinations of $(2, \epsilon, M)$-molecules, via pairing $\boldsymbol{M}_0^{2,\epsilon,M}$ with its dual, has a unique extention to $H^1_L(\mathbb{R}^n)$. Moreover,

$$\|l\| \leqslant C\|f\|_{BMO_{L^*}(\mathbb{R}^n)}.$$

Theorem 7. *(Thm. 8.6) If $l \in (H^1_L)^*$, then formally $l \in BMO_{L^*}(\mathbb{R}^n)$. Furthermore, for all $g \in H^1_L(\mathbb{R}^n)$ which can be represented as a finite linear combination of $(2, \epsilon, M)$-molecules, we have*

$$l(g) = \langle l, g\rangle$$

where the later pairing is in the sense of $\boldsymbol{M}_0^{2,\epsilon,M}$ with its dual, and

$$\|l\|_{BMO_{L^*}(\mathbb{R}^n)} \leqslant C\|l\|.$$

References

[1] Albrecht, D., Duong, X., McIntosh, A., *Operator theory and harmonic analysis. Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995)*, Proc. Centre Math. Appl. Austral. Nat. Univ. **34** (1996), 77-136.

[2] Auscher, P.: Personal communication

[3] Auscher, P., *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates*, Memoirs of the Amer. Math. Soc. **871** (2007), 186.

[4] Auscher, P.,Coulhon,T.,Tchamitchian, Ph., *Absence de principe dumaximum pour certaines equations paraboliques complexes*, Coll. Math. **171** (1996), 87-95.

[5] Auscher, P.,Hofmann, S., Lacey, M.,McIntosh, A., Tchamitchian, Ph., *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*, Ann. Math. (2) **156**(2) (2002), 633-654.

[6] Auscher, P., Hofmann, S., Lewis, J.L., Tchamitchian, Ph., *AExtrapolation of Carleson measures and the analyticity of Katos square-root operators.*, Acta Math. **187**(2) (2001), 161-190.

[7] Auscher, P., McIntosh, A., Russ, E.: preprint

[8] Auscher, P., Russ, E., *Hardy spaces and divergence operators on strongly Lipschitz domains of $\mathbb{R}^n$*, J. Funct. Anal. **201**(1) (2003), 148-184.

[9] Blunck, S., Kunstmann, P., *Weak-type (p, p) estimates for Riesz transforms*, Math. Z. **247**(1) (2004), 137-148.

[10] Blunck, S., Kunstmann, P., *CaldernZygmund theory for non-integral operators and the H^∞ functional calculus*, Rev. Mat. Iberoamericana **19**(3) (2003), 919-942.

[11] Coifman, R.R., *A real variable characterization of H^p*, Studia Math. **51** (1974), 269-274.

[12] Coifman, R.R., Meyer, Y., Stein, E.M., *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. **62**(2) (1985), 304-335.

[13] Coifman, R.R.,Weiss, G., *Extensions of Hardy spaces and their use in analysis*, Bull. Am. Math. Soc. **83**(4) (1977), 569-645.

[14] Davies, E.B., *Limits on L^p regularity of selfadjoint elliptic operators*, J. Differ. Equ. **135**(1) (1997), 83-102.

[15] Duong, X.T., Yan, L.X., *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Am. Math. Soc. **18** (2005), 943-973.

[16] Duong, X.T., Yan, L.X., *New function spaces of BMO type, the JohnNirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58**(10) (2005), 1375-1420.

[17] Fefferman, C., Stein, E.M., *H^p spaces of several variables*, Acta Math. **129**(3-4) (1972), 137-193.

[18] Garcia-Cuerva, J., Rubio de Francia, J.L., *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies, vol. 116. North-Holland Publishing Co., Amsterdam (1985).

[19] Hofmann, S., Martell, J.M., *L^p bounds for Riesz transforms and square roots associated to second order elliptic operators*, Publ. Math. **47**(2) (2003), 497-515.

[20] John, F., Nirenberg, L., *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415-426.

[21] Kenig, C.E., Pipher, J., *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math. **113**(3) (1993), 447-509.

[22] Latter, R.H., *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. **62**(1) (1978), 93-101.

[23] Mazya, V.G., Nazarov, S.A., Plamenevski, B.A., *Absence of a De Giorgi-type theorem for strongly elliptic equations with complex coefficients. Boundary value problems of mathematical physics and related questions in the theory of functions*, 14. Zap. Nauchn. Sem. Leningrad.Otdel. Mat. Inst. Steklov. (LOMI) **115** (1982), 156-168, 309.

[24] Stein, E.M., *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, NJ (1970).

[25] Stein, E.M., *Harmonic analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ (1993).

[26] Stein, E.M., *Maximal functions. II. Homogeneous curves*, Proc. Natl. Acad. Sci. U.S.A. **72**(7) (1976), 2176-2177.

[27] Stein, E.M., Weiss, G., *On the theory of harmonic functions of several variables. I. The theory of H^p-spaces*, Acta Math. **103** (1960), 25-62.

[28] Taibleson, M., Weiss, G., *The molecular characterization of certain Hardy spaces. Representation theorems for Hardy spaces*, Soc. Math. France, Paris **77** (1980), 67-149.

[29] Wilson, J.M., *On the atomic decomposition for Hardy spaces*, Pac. J. Math. **116**(1) (1985), 201-207.

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DUALITY OF HARDY AND BMO SPACES ASSOCIATED WITH OPERATORS WITH HEAT KERNEL BOUNDS

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presented by Irina Holmes

Abstract. This paper is part of an extensive body of work concerning generalizations of the classical Hardy and BMO spaces to Hardy and BMO spaces adapted to certain linear operators L with bounded holomorphic calculus. This began with the introduction of such a generalized Hardy space H^1_L by Auscher, Duong and McIntosh, and was further continued by Duong and Yan, who introduced the spaces BMO_L and established some of their important properties (such as a John-Nirenberg inequality). The paper being considered here bridges the gap between the two and, in its first main result, establishes the fundamental duality relationship between Hardy and BMO spaces in this more general context - specifically, it is proved that the dual space of H^1_L is BMO_{L^*}, where L^* denotes the adjoint of L. Moreover, the second main result of this paper proves a characterization of the BMO_L spaces in terms of Carleson measures - another fundamental and deeply impactful feature of BMO spaces.

1. Introduction

The spaces H^1 and BMO have long been crucial tools in harmonic analysis, especially since the seminal work of Fefferman and Stein [4], where the $H^1 - BMO$ duality was proved. Among many avenues opened by this work, one point of interest in the literature has been to generalize these results to Hardy and BMO spaces associated with a certain type of linear operator. In particular, one can think of the classical H^1 and BMO spaces as being defined in terms of the Laplacian operator Δ on $\mathbb{R}^n$, or in terms of $\sqrt{\Delta}$. The question was, can one expand these definitions using other operators besides the Laplacian, and obtain more general H^1 and BMO spaces with similar properties to the classical ones?
These generalized Hardy spaces H^1_L, associated to a certain kind of linear operator L, were introduced in [1], followed by the introduction of the appropriately defined corresponding BMO_L spaces in [2]. The paper presented here is an important milestone, as it makes the connection between the two, and proves the duality between H^1_L and BMO_{L^*}, as well as a characterization of BMO_L in terms of Carleson measures.

1.1. Assumptions on The Operator L.

For $0 < \omega < \pi$, denote by S_ω the closed sector

$$S_\omega := \{z \in \mathbb{C} : |\arg(z) \leq \omega|\} \cup \{0\},$$

in the complex plane, and denote its interior by S^0_ω. In what follows, L is a linear operator of **type** ω on $L^2(\mathbb{R}^n)$, for some $\omega < \pi/2$. This means that L is a closed operator whose spectrum $\sigma(L) \subset S_\omega$, and such that, for every $\nu > \omega$, there is $c_\nu > 0$ such that

$$\|(L - \lambda\mathcal{I})^{-1}\| \leq c_\nu |\lambda|^{-1}, \forall \lambda \notin S_\nu.$$

In this case, L generates a holomorphic semigroup e^{-zL}, $0 < |\mathrm{Arg}(z)| < \pi/2 - \omega$. Now, there are two main assumptions on L, which can be summarized as:

(a). The semigroup generated by L is represented by a kernel which satisfies a Poisson bound of order $m > 0$;
(b). L has a bounded holomorphic calculus on $L^2(\mathbb{R}^n)$.

We explain these in more detail below. Assumption (a) says that the semigroup e^{-zL} is represented by a kernel $p_z(x, y)$ which satisfies:

$$|p_z(x, y)| \leq c_\theta h_{|z|}(x, y),$$

for all $x, y \in \mathbb{R}^n$, $|\mathrm{Arg}(z)| < \pi/2 - \theta$, and all $\theta > \omega$. Here

$$h_t(x, y) := t^{-n/m} s\left(\frac{|x - y|}{t^{1/m}}\right), \tag{1}$$

for some positive, bounded, decreasing function s that satisfies

$$\lim_{r \to \infty} r^{n+\epsilon} s(r) = 0, \tag{2}$$

for some $\epsilon > 0$. We define also

$$\Theta(L) := \sup\{\epsilon > 0 : (2) \text{ holds}\}.$$

Remark that, if $L = \Delta$ (the Laplacian on $\mathbb{R}^n$), $\Theta(\Delta) = \infty$ and if $L = \sqrt{\Delta}$, then $\Theta(\sqrt{\Delta}) = 1$.
To briefly explain holomorphic functional calculi, we look at two important subspaces of $\mathcal{H}(S^0_\nu)$, the space of holomorphic functions on S^0_ν, for $\nu > \omega$:

- $\mathcal{H}_\infty(S^0_\nu)$ - the space of bounded holomorphic functions b on S^0_ν;
- $\Psi(S^0_\nu)$ - the space of all $\psi \in \mathcal{H}(S^0_\nu)$ for which there exists $s > 0$ such that

$$|\psi(z)| \leq c|z|^s(1 + |z|^{2s})^{-1}.$$

Then for $\psi \in \Psi(S^0_\nu)$, define $\psi(L)$ as

$$\psi(L) := \frac{1}{2\pi i}\int_\Gamma (L - \lambda\mathcal{I})^{-1}\psi(\lambda)\, d\lambda, \tag{3}$$

where Γ is the contour $\Gamma = \{\xi = re^{\pm i\theta} : r \geq 0\}$, parametrized clockwise around S_ω, and $\theta \in (\omega, \nu)$. The integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$, so $\psi(L) : L^2 \to L^2$ is a bounded linear operator on L^2, and, by Cauchy's theorem, the definition of $\psi(L)$ is independent of the choice of θ. Now, we say that L has a **bounded holomorphic calculus** on L^2 if, for every $\nu > \omega$, there is $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$ and $\|b(L)\| \leq c_{\nu,2}\|b\|_\infty$ for all $b \in \mathcal{H}_\infty(S^0_\nu)$.
We now look at the class of functions that the operators P_t act upon. A function $f \in L^2_{loc}(\mathbb{R}^n)$ is said to be of **β-type**, for some $\beta > 0$, provided that

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1 + |x|^{n+\beta}}\right)^{1/2} \leq c < \infty. \tag{4}$$

We denote by $\mathcal{M}_\beta$ the collection of all functions of β-type. This is a Banach space under the norm

$$\|f\|_{\mathcal{M}_\beta} := \inf\{c \geq 0 : (4) \text{ holds}\}.$$

Finally, let $\mathcal{M} := \mathcal{M}_{\Theta(L)}$, if $\Theta(L) < \infty$, and

$$\mathcal{M} := \bigcup_{\beta \in (0,\infty)} \mathcal{M}_\beta,$$

otherwise.

2. First Main Result: H^1_L – BMO_{L^*} Duality

Suppose now that L is an operator satisfying the assumptions in Section 1.1, and define for any $(x,t) \in \mathbb{R}^{n+1}_+$ and $f \in \mathcal{M}$:

$$P_t f(x) := e^{-tL} f(x) = \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy,$$

and

$$Q_t f(x) := tLe^{-tL} f(x) = \int_{\mathbb{R}^n} -t \left(\frac{d}{dt} p_t(x,y)\right) f(y)\, dy.$$

Given $f \in L^1(\mathbb{R}^n)$, the *area integral function* $S_L(f)$, associated to the operator L, is defined as

$$S_L(f)(x) := \left(\int_{\Gamma(x)} |Q_{t^m} f(y)|^2 \frac{dy\, dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ is the standard cone with vertex x. Then we say $f \in L^1$ belongs to the *Hardy space* H^1_L associated with L, provided that $S_L(f) \in L^1$, and define its H^1_L norm by

$$\|f\|_{H^1_L} := \|S_L(f)\|_{L^1}.$$

The main properties of these spaces were studied in [1]. Remark that if we let $L = \Delta$ or $L = \sqrt{\Delta}$, the spaces H^1_Δ and $H^1_{\sqrt{\Delta}}$ coincide with the classical H^1 space, and the norms are equivalent.

Now for the corresponding BMO space, we say that $f \in \mathcal{M}$ belongs to BMO_L, the space of functions of *bounded mean osciallation* associated with L, provided that

$$\|f\|_{BMO_L} := \sup_B \frac{1}{|B|} \int_B |f(x) - P_{r_B^m} f(x)|\, dx < \infty,$$

where the supremum is over all balls B in $\mathbb{R}^n$ and r_B denotes the radius of B. In [2], the authors establish some important properties of these generalized BMO spaces, chief among them a John-Nirenberg inequality for BMO_L.

The first main result of the paper presented here is the duality between H^1_L and BMO_{L^*}.

Theorem 1. *Suppose L satisfies the assumptions in Section 1.1. Then the dual space of H^1_L is BMO_{L^*}, where L^* denotes the adjoint operator of L, in the sense that, for every $f \in BMO_{L^*}$, the linear functional:*

$$l(g) := \int_{\mathbb{R}^n} f(x) g(x)\, dx, \tag{5}$$

initially defined on the dense subspace $H^1_L \cap L^2$, has a unique extension to H^1_L. Conversely, every $l \in (H^1_L)^*$ can be expressed as in (5) for a unique $f \in BMO_{L^*}$ and, in this case, $\|f\|_{BMO_{L^*}} \leq c\|l\|$.

Remark that, in the result above, the authors only impose a Poisson upper bound on the kernel p_t, but impose no regularity assumptions on the space variables x and y. Moreover, they do not impose the conservation property $p_t(1) = 1$, $t > 0$ on the semigroup either. Both of these freedoms on the kernel make the results in this paper applicable to a large class of operators L.

3. Second Main Result: BMO_L and Carleson Measures

Besides the $H^1 - BMO$ duality, the second crucial aspect of BMO spaces established by Fefferman and Stein in [4] is their profound connection to Carleson measures. Recall that a Carleson measure μ on $\mathbb{R}^{n+1}_+$ is a measure for which there exists a constant c such that for every ball B on $\mathbb{R}^n$:

$$\mu(\widehat{B}) \leq c|B|,$$

where $\widehat{B}$ denotes the tent over B. The smallest c that satisfies the inequality above is the norm of μ, denoted $|||\mu|||_c$. In the classical case, BMO spaces are characterized by Carleson measures in the sense that for every $f \in BMO$:

$$\mu_f(x,t) := \left| t\frac{\partial}{\partial t} e^{-t\sqrt{\Delta}} f(x) \right|^2 \frac{dx\, dt}{t}$$

is a Carleson measure on $\mathbb{R}^{n+1}_+$.

The paper presented here proves a characterization of BMO_L spaces in terms of Carleson measures:

Theorem 2. *Suppose L satisfies the assumptions in Section 1.1 and let $f \in BMO_L$. Then $f \in \mathcal{M}$ and*

$$\mu_f(x,t) := |Q_{t^m}(\mathcal{I} - P_{t^m})f(x)|^2 \frac{dx\, dt}{t} \tag{6}$$

is a Carleson measure, with $|||\mu_f||| \equiv \|f\|^2_{BMO_L}$. Conversely, if $f \in \mathcal{M}$ and μ_f is a Carleson measure, then $f \in BMO_L$.

References

[1] P. Auscher, X. T. Duong, and A. McIntosh, *Boundedness of Banach space valued singular integral operators and Hardy spaces*, Unpublished manuscript (2004).

[2] Xuan Thinh Duong and Lixin Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1375–1420.

[3] ———, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. **18** (2005), no. 4, 943–973 (electronic).

[4] Fefferman and Stein, *H^p spaces of several variables*, Acta Mathematica **129** (1972), no. 1, 137–193.

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HEAT KERNEL BOUNDS, CONSERVATION OF PROBABILITY AND THE FELLER PROPERTY

E. B. DAVIES

presented by Bochen Liu

ABSTRACT. Let M be a compelete Riemannian manifold and L be the weighted Lapace-Beltrami operator on M. By generalizing a lemma from Gaffney [5], the author proves, if for some $p \in M, \alpha > 0, \beta > 0$,

$$|B(p,r)| \leq \alpha e^{\beta r^2},\ \forall r > 0,$$

the operator e^{Lt} conserves probability. The author also gives a pointwise upper bound on heat kernels without any assumption on Ricci curvature. Using this pointwise upper bound of heat kernels, a new condition for the Feller property is given. The application to second order elliptic operators on $\mathbb{R}^N$ is also discussed.

1. Introduction and Main Results Presented

Let M be a complete Riemannian manifold. Denote $W_b^{1,\infty}$ as the space of continuous functions f of bounded support whose weak first derivatives lie in L^∞. Let $\overline{\nabla}$ be the L^2 operator closure of ∇ initially defined on $W_b^{1,\infty}$. Then one can define the weighted Lapace-Beltrami operator L by

$$Lf = \sigma^{-2}\overline{\nabla} \cdot (\sigma^2 \overline{\nabla} f), \tag{1}$$

where $f \in L^2$ and σ is a positive weight with $\sigma^{\pm 1} \in L^\infty_{loc}$. Note that L associates the quadratic form

$$Q(f) = \int_M |\nabla f|^2 \sigma^2 \, dx.$$

It is known (see e.g. [3],[4]) that e^{Lt} is a positivity preserving contraction semi-group on L^p for $1 \leq p \leq \infty$ which is strongly continous for $1 \leq p < \infty$. The first result of this paper is a generalization of a lemma due to Gaffney [5], who only treated the case $\sigma = 1$.

Lemma 1. *Let $\phi = e^{\alpha\psi}$ where $\alpha \in \mathbb{R}$ and ψ is a bounded function such that $|\nabla\psi| \leq 1$. If $0 \leq f \in L^2$ and $u_t = e^{Lt} f$, then*

$$||\phi u_t||_2 \leq e^{\alpha^2 t}||\phi f||_2$$

and

$$I_t \equiv \int_0^t ||\phi \overline{\nabla} u_s||_2^2 \, ds \leq 2e^{2\alpha^2 t}||f\phi||_2^2$$

for all $t > 0$.

There are three main results in this paper.

1.1. **Conservation of probability.** We say e^{Lt} conserves probability if

$$e^{Lt}1 = 1 \tag{2}$$

for all $t > 0$. A lot of different conditions on the Ricci curvature or volume growth of M which imply this property are known (see, for example, [1] [3] [5] [6], etc.). In this paper, the author gives a condition which does not depend on the Ricci curvature.

Let $B(p, r)$ be the ball in M with center p and radius r, denote

$$|E| = \int_E \sigma^2(x)\, dx = ||\chi_E||_2^2.$$

Combining Lemma 1 and a method similar to proof in [6], the author proves

Theorem 2. *If M is a complete Riemannian manifold and for some $p \in M, \alpha, \beta > 0$,*

$$|B(p, r)| \le \alpha e^{\beta r^2}$$

for all $r > 0$, the semigroup e^{Lt} on $L^1(M, \sigma^2 dx)$ conserves probability.

1.2. **Heat kernel bounds.** More surprisingly, Lemma 1 also yields a general L^2 upper bound on heat kernels, which has been proved by wave equation techniques [2].

Theorem 3. *If E and F are two Borel subsets of M with $|E|, |F| < \infty$, then*

$$0 \le \int \left(e^{Lt}\chi_E\right) \cdot \chi_F\, dx \le |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} e^{-\frac{d(E,F)}{4t}}$$

for all $t > 0$.

To pass from L^2 estimates to pointwise bounds, we need the following assumptions on local geometry.

We say that M, σ has weak bounded geometry if there are positive constants c_1 and r and a diffeomorphism

$$\Phi_x : \{y : ||y|| < r\} \subset \mathbb{R}^N \to U_x \subset M$$

for all $x \in M$, with $\Phi_x(0) = x$ and the following properties. If $y \in U_x$ then

$$c_1^{-1} \le \frac{\sigma(y)}{\sigma(x)} \le c_1.$$

Considering $\{y : ||y|| < r\}$ as a coordinate patch, the metric satisfies

$$c_1^{-1} \le g_{ij} \le c_1.$$

Under the above condition if we define the distorted balls $\tilde{B}(x, \rho)$ by

$$\tilde{B}(x, \rho) = \Phi_x\{y : ||y|| < \rho\}$$

then there exists a constant $c_2 = c_2(c_1, N)$ such that

$$c_2^{-1}\sigma(x)^2\rho^N \le |\tilde{B}(x, \rho)| \le c_2\sigma(x)^2\rho^N$$

and $z \in \tilde{B}(x, \rho)$ implies $d(x, z) \le c_2\rho$.

Combining Theorem 3 and Moser's parabolic Harnack inequality [7], the author obtains a upper bound on heat kernel $K(t, x, y)$ of L without any assumption on the Ricci curvature.

Theorem 4. *Suppose that M, σ has weak bounded geometry in the sense above. Then*

$$0 \le K(t,x,y) \le c_4 \sigma(x)^{-1} \sigma(y)^{-1} \max\left\{ \left(\frac{2}{r}\right)^N, t^{-\frac{N}{2}}, \left(\frac{d}{t}\right)^N \right\} e^{-\frac{d^2}{4t}}$$

for all $x, y \in M$ and $t > 0$, where $d = d(x,y)$ and $c_4 = c_4(c_1, N)$.

1.3. **Feller property.** Denote C_0 as the space of continuous functions on M which vanish at infinity. We say e^{Lt} has Feller property if

$$e^{Lt} C_0 \subset C_0$$

for all $t > 0$. By applying the upper bound on the heat kernel in Theorem 4, the author gives a new condition for the Feller property which does not depend upon lower bounds on the Ricci curvature, but rather upon the behaviour at infinity of a modified injectivity radius.

We suppose that $c > 0$ and that for each $x \in M$ there exists $r(x) > 0$, called the bounded geometry radius at x, with the following property. The ball $B(x, r(x))$ can be mapped diffeomorphically onto the coordinate patch U in $\mathbb{R}^N$ given by

$$U = \{y \in \mathbb{R}^N : ||y|| < r(x)\}$$

in such a way that for any $y, z \in U$ one has

$$c^{-1} \le g_{ij}(z) \le c,$$

$$c^{-1} \le \frac{\sigma(x)}{\sigma(y)} \le c.$$

If $\sigma(x)$ and the metric depend continuously upon $x \in M$, then such an $r(x) > 0$ always exists and one should take it as large as possible.

The new condition for Feller property is the following.

Theorem 5. *Let M be a complete Riemannian manifold. If there exists $p \in M$ and positive constants c_1, c_2 such that the bounded geometry radius $r(x)$ satisfies*

$$\sigma(x)^2 \min\{r(x), 1\}^N \ge c_1 e^{-c_2 d(x,p)^2} \tag{3}$$

for all $x \in M$, then e^{Lt} has the Feller property.

2. Applications to second order elliptic operators on $\mathbb{R}^N$

On $M = \mathbb{R}^N$, let $D_{ij}(x)$ be a real symmetric matrix which has bounded distortion in the sense that

$$D_{ij}(x) = B(x) C_{ij}(x)$$

where $\det(C_{ij}) = 1$ and

$$0 < c^{-1} \le C_{ij}(x) \le c < \infty$$

for all $x \in \mathbb{R}^N$. We also assume $A(x)^{\pm 1}, B(x)^{\pm 1} \in L^\infty_{loc}$. Then H, which associates the quadratic form

$$Q(f) = \int_{\mathbb{R}^N} \sum_{ij} D_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \, d^N x,$$

formally defined by

$$Hf = -A^{-1} \sum_{i,j} \frac{\partial}{\partial x_i} \left(D_{ij} \frac{\partial f}{\partial x_j} \right),$$

is a non-negative self-adjoint operator on $L^2(\mathbb{R}^N, Adx)$.

Using Theorem 2, the author proves the following.

Theorem 6. *If*

$$A(x) \geq c_5 |x|^{2\alpha-2} B(x)$$

for large enough $|x|$, where $c_5 > 0$ and $\alpha > 0$ and

$$\int_{S^{N-1}} A(r\omega)\, dS(\omega) \leq \delta e^{\gamma r^\beta}$$

for all $r > 0$, where β, γ, δ are positive constants and $\beta \leq 2\alpha$, then the semigroup e^{Lt} on $L^1(\mathbb{R}^N, Adx)$ conserves probability.

We now specialize even further to the case where $M = \mathbb{R}^N$ and $\sigma^{\pm 1} \in L^\infty_{loc}$ are sufficiently smooth. Define $H' = ULU^{-1}$ on $L^2(\mathbb{R}, d^N x)$ where L is the same as (1) and the unitary operator U from $L^2(\mathbb{R}^N, \sigma^2 d^N x)$ to $L^2(\mathbb{R}, d^N x)$ is defined by $Uf = \sigma f$. We call H' a "singular Schrödinger operator" since it is of form

$$H' f = -\Delta f + V f$$

with $V = \frac{\Delta \sigma}{\sigma}$. Notice the heat kernel K' of H' is $\sigma(x)\sigma(y)K(t, x, y)$, where K is the heat kernel of L. Therefore, Theorem 4 implies

Theorem 7. *Suppose that there are positive constants c_1 and r such that the positive weight σ on $\mathbb{R}^N$ satisfies*

$$c_1^{-1} \leq \frac{\sigma(x)}{\sigma(y)} \leq c_1$$

whenever $|x - y| \leq r$. Then the heat kernel K' of the "singular Schrödinger operator" H' satisfies

$$0 \leq K'(t, x, y) \leq c_4 \max \left\{ \left(\frac{2}{r} \right)^N, t^{-\frac{N}{2}}, \left(\frac{|x-y|}{t} \right)^N \right\} e^{-\frac{|x-y|^2}{4t}}$$

for all $x, y \in \mathbb{R}^N$ and $t > 0$, where $c_4 = c_4(c_1, N)$.

We now study the Feller property for e^{Lt} for particular weights. If $\sigma(x) \geq c^{-1} e^{-cx^2}$ for some $c \geq 1$, then (3) holds with $p = 1$ and $r(x) = c^{-1}(1 + |x|)^{-c}$. Therefore by Theorem 5, it follows that

Theorem 8. *Suppose that there exists a constant $c \geq 1$ such that*

$$\sigma(x) \geq c^{-1} e^{-cx^2}$$

for all $x \in \mathbb{R}^N$, and

$$c^{-1} \leq \frac{\sigma(x)}{\sigma(y)} \leq c$$

for all $x, y \in \mathbb{R}^N$ such that

$$|x - y| \leq c^{-1}(1 + |x|)^{-c}.$$

Then the semigroup e^{Lt} *on* $L^2(\mathbb{R}^N, \sigma^2 dx)$ *has the Feller property.*

References

[1] R. Azencott, *Behaviour of diffusion semigroups at infinity*, Bull. Soc. Math. Fr. 102 (1974), 193-240.
[2] J. Cheeger, M. Gromov and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifold.* J. Differ. Geom. 17 (1982), 15-53.
[3] E. B. Davies, *Heat Kernels and Spectral Theory.* Cambridge University Press, 1989.
[4] M. Fukushima, Dirichlet Forms and Markov Processes, North-Holland, Amsterdam, 1980.
[5] M. P. Gaffney, *The conservation property of the heat equation on Riemannian manifolds*, Comm. Pure Appl. Math. 12 (1959), 1-11.
[6] A. A. Grigor'yan, *On stochastically complete Riemannian manifolds.* Soviet Math. Doklady 34 (1987), 310-313.
[7] J. Moser, *On a pointwise estimate for parabolic differential equations.* Commun. Pure Appl. Math. 24 (1971), 727-740.

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HARDY SPACES ASSOCIATED TO THE SCHRÖDINGER OPERATOR ON STRONGLY LIPSCHITZ DOMAINS OF $\mathbb{R}^d$

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presented by Darío Mena

Abstract. Let $L=-\Delta+V$ denote a Schrödinger operator on $\mathbb{R}^d$, where $V(x)$ is a nonnegative polynomial and Δ is the Laplacian on $\mathbb{R}^d$, and let Ω be a strongly Lipschitz domain of $\mathbb{R}^d$. Denote by $\{T_t^L\}_{t>0}$ the semigroup of linear operators generated by $-L$, and by $K_t^L(x,y)$ their kernels. We define the maximal-type function

$$\mathcal{M}f(x)=\sup_{t>0}\left|T_t^L f(x)\right|$$

We say that a Schwartz function f is in $H_L^p(\mathbb{R}^d)$ if and only if

$$\|f\|_{H_L^p}=\|\mathcal{M}f\|_{L^p}<\infty.$$

In this paper, Hardy spaces associated to the operator L are studied with relation to maximal functions and area integral functions.

1. Introduction and Main Results Presented

1.1. **Definition and preliminaries.** Let Ω be a strongly Lipschitz domain, that is, a domain in $\mathbb{R}^d$ whose boundary is covered by a finite number of parts of Lipschitz graphs and at most one of them is unbounded.
Let $L=-\Delta+V$ denote a Schrödinger operator on $\mathbb{R}^d$, where $V(x)$ is a nonnegative polynomial $V(x)=\sum_{\beta\leq\alpha}a_\beta x^\beta$.
Denote by $\{T_t^L\}_{t>0}$ the semigroup of linear operators generated by $-L$, and by $K_t^L(x,y)$ their kernels. We define the maximal-type function

$$\mathcal{M}f(x)=\sup_{t>0}\left|T_t^L f(x)\right|$$

Definition 1. *We say that a Schwartz function f is in $H_L^p(\mathbb{R}^d)$ if and only if*

$$\|f\|_{H_L^p}=\|\mathcal{M}f\|_{L^p}<\infty.$$

Consider the auxiliary function

$$\rho(x)=\rho(x,V)=\left(\sum_{\beta\leq\alpha}|\partial^\beta V(x)|^{1/(|\beta|+2)}\right)^{-1},$$

and define the "sharp" function

$$f_V^\sharp(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)-f(B,V)|\,dy,$$

where

$$f(B,V)=\begin{cases}\dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} f(y)\,dy, & \text{if } r<\rho(x),\\ 0, & \text{if } r\geq \rho(x).\end{cases}$$

Definition 2 ($BMO_L(\mathbb{R}^d)$)**.** *A locally integrable function f is in $BMO_L(\mathbb{R}^d)$ if and only if $f_V^\sharp \in L^\infty(\mathbb{R}^d)$, and we set $\|f\|_{BMO_L}=\|f_V^\sharp\|_{L^\infty}$.*

This space is the dual of $H^1_L(\mathbb{R}^d)$ [4].
For $0<p\leq 1$, $1\leq q\leq\infty$, we say that $a\in H^p_L(\mathbb{R}^d)$ is an $H^{p,q}_L$-atom associate to a ball $B(x_0,r)$ if it satisfies the following properties

(1) $\operatorname{supp} a\subset B(x_0,r)$.
(2) $\|a\|\leq |B(x_0,r)|^{\frac{1}{q}-\frac{1}{p}}$.
(3) If $r<\rho(x_0)$, then $\int a(x)x^\gamma\,dx=0$ for $|\gamma|\leq d\left(\frac{1}{p}-1\right)$

Define the atomic quasi-norm of a function $f\in H^p_L(\mathbb{R}^d)$ by

$$\|f\|_{L-atom,q}=\inf\left\{\left(\sum |c_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all possible representations of f as $f=\sum c_j a_j$, where the a_j's are $H^{p,q}_L$-atoms. This quasi-norm is equivalent to the H^p_L-norm [3].

Definition 3. *Let $\frac{d}{d+1}<p\leq 1$, we say that $f\in H^p_{L,r}(\Omega)$ if it is the restriction to Ω of a function $F\in H^p_L(\mathbb{R}^d)$ and the quasi-norm is defined by*

$$\|f\|_{H^p_{L,r}}=\inf_{\substack{F\in H^p_L(\mathbb{R}^d)\\ F\restriction\Omega=f}}\|F\|_{H^p_L(\mathbb{R}^d)}$$

Let $\{P^L_t\}$ represent the semigroup of linear operators associated to $-\sqrt{L}$ (and their kernels). For a locally integrable function f such that $|y|^{-d-1}f(y)$ is integrable, define, for $x\in\Omega$

$$f^*_L(x)=\sup_{\substack{y\in\Omega,t>0\\ |x-y|<t}}\left|P^L_t f(y)\right|$$

Definition 4. *We say that $f\in H^p_{\max,L}(\Omega)$, for $\frac{d}{d+1}<p\leq 1$, if and only if $f^*_L\in L^p(\Omega)$ and define*

$$\|f\|_{H^p_{\max,L}(\Omega)}=\|f^*_L\|_{L^p(\Omega)}$$

If we set $\overline{\nabla}u=(\nabla u,\partial_t u)$, and the cone $\Gamma(x)=\{(y,t)\in\Omega\times(0,\infty):|x-y|<t\}$, then we can define the following area integral function associated to L.

$$S_L f(x)=\left(\int_{\Gamma(x)}\left|t\overline{\nabla}P^L_t f(y)\right|^2\frac{dy\,dt}{t^{d+1}}\right)^{1/2},$$

and also the area integral restricted to time derivative

$$s_L f(x)=\left(\int_{\Gamma(x)}\left|t\partial_t P^L_t f(y)\right|^2\frac{dy\,dt}{t^{d+1}}\right)^{1/2},$$

Definition 5 (Campanato space associated to L). *Let $0 \leq \alpha < 1$, a locally integrable function g on $\mathbb{R}^d$ belongs to $\Lambda^L_\alpha(\mathbb{R}^d)$ if and only if*

$$\|g\|_{\Lambda^L_\alpha(\mathbb{R}^d)} = \sup_{B\subset\mathbb{R}^d}\left\{\frac{1}{|B|^{\alpha/d}}\left(\int_B |g-g(B,V)|^2\frac{dx}{|B|}\right)^{1/2}\right\} < \infty.$$

Definition 6. *Let D^L_t denote the operator $t\partial_t P^L_t$. We say that $f \in \left(\Lambda^L_\alpha(\mathbb{R}^d)\right)^*$ vanishes at infinity in a generalized sense, if it satisfies*

$$\lim_{A\to\infty}\int_A^\infty \left\langle f,(D^L_t)^2 g\right\rangle \frac{dt}{t} = 0,$$

for every $g \in \Lambda^L_\alpha(\mathbb{R}^d)$.

With all the necessary terminology defined, the main result of the paper can be stated

Theorem 1. *Let $\frac{d}{d+1} < p \leq 1$, Ω be a domain with Ω^c unbounded, and f be a locally integrable function on Ω, that vanishes at infinity in a generalized sense. Then, if the operator L satisfies the Dirichlet boundary condition (DBC) on Ω, we have*

$$\|f\|_{H^p_{L,r}(\Omega)} \simeq \|s_L f\|_{L^p(\Omega)} \simeq \|S_L f\|_{L^p(\Omega)} \simeq \|f\|_{H^p_{\max,L}(\Omega)}.$$

2. Main result

2.1. **Dual and predual of H^p_L.** The proof of the following result can be found in [7]

Proposition 1. *Let $0 < p \leq 1$, then the dual of $H^p_L(\mathbb{R}^d)$ is $\Lambda^L_{d(1/p-1)}(\mathbb{R}^d)$.*

Definition 7. *Let $f \in \Lambda^L_\alpha(\mathbb{R}^d)$, we say that f is in $\lambda^L_\alpha(\mathbb{R}^d)$ if $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where*

$$\gamma_1(f) = \lim_{r\to 0}\sup_{\substack{B\subset\mathbb{R}^d\\ r_B<r}}\frac{1}{|B|^{\alpha/d+1/2}}\left(\int_B |f-f(B,V)|^2\frac{dx}{|B|}\right)^{1/2},$$

$$\gamma_2(f) = \lim_{r\to \infty}\sup_{\substack{B\subset\mathbb{R}^d\\ r_B\geq r}}\frac{1}{|B|^{\alpha/d+1/2}}\left(\int_B |f-f(B,V)|^2\frac{dx}{|B|}\right)^{1/2},$$

$$\gamma_3(f) = \lim_{r\to \infty}\sup_{B\subset B(0,r)^c}\frac{1}{|B|^{\alpha/d+1/2}}\left(\int_B |f-f(B,V)|^2\frac{dx}{|B|}\right)^{1/2}.$$

The dual of the space $\lambda^L_{d(1/p-1)}(\mathbb{R}^d)$ is the completion of $H^p_L(\mathbb{R}^d)$ as stated in the following theorem whose proof can be found in [8].

Theorem 2. *Let $\frac{d}{d+1} < p \leq 1$ and $\alpha = d(1/p-1)$, then*

(a) *Suppose that $\in H^p_L(\mathbb{R}^d)$. Then $\mathcal{L}_f = \displaystyle\int_{\mathbb{R}^d} f(x)g(x)\,dx$ defined (initially) for $g \in L^2_{loc}(\mathbb{R}^d)$ can be extended to a bounded linear functional on $\lambda^L_\alpha(\mathbb{R}^d)$ and satisfies*

$$\|\mathcal{L}_f\| \lesssim \|f\|_{H^p_L(\mathbb{R}^d)}.$$

(b) *Conversely, every bounded linear functional $\mathcal{L}$ on $\lambda^L_\alpha(\mathbb{R}^d)$ can be realized as $\mathcal{L} = \mathcal{L}_f$ with $f \in H^p_L(\mathbb{R}^d)$ and*

$$\|f\|_{H^p_L(\mathbb{R}^d)} \lesssim \|\mathcal{L}\|.$$

2.2. **Proof of the main result.** The proof of the main result is divided in several parts, similar to [1].

2.2.1. *From Hardy spaces to maximal Hardy spaces.* The lemma is the following

Lemma 1. *Let $\frac{d}{d+1} < p \leq 1$ and f a locally integrable function on Ω. Then, under DBC one has*

$$\|f\|_{H^p_{\max,L}(\Omega)} \lesssim \|f\|_{H^p_{L,r}(\Omega)}$$

The proof of this lemma is relatively easy, and it relies on the fact that if $f \in H^p_{L,r}(\Omega)$, then it can be extended to $\mathbb{R}^d$ to a function that admits an atomic decomposition. Therefore the estimates are reduced to those of an atom.
The computations of the estimate for an atom, use some estimates on the Poisson kernels $\{P^L_t\}_{t>0}$, L^2-bounds for the Hardy-Littlewood maximal function, the Dirichlet boundary condition, and some other geometric considerations.

2.2.2. *From maximal functions to area integral functions.*

Lemma 2. *Let $\frac{d}{d+1} < p \leq 1$ and f a locally integrable function on Ω. Then, under DBC one has*

$$\|S_L f\|_{L^p(\Omega)} \lesssim \|f\|_{H^p_{\max,r}(\Omega)}$$

The proof of this lemma, depends strongly on the Caccioppoli inequality, whose proof can be found in [9]: if $Q((x_0,t_0),r) = \{(y,t) : \max\{|y-x_0|, |t-t_0|\} < r\}$, then we have

$$\int_{Q((x_0,t_0),r)\cap(\Omega\times\mathbb{R}^+)} |\overline{\nabla} u|^2 + V(x)|u|^2 \, dx\, dt \lesssim \frac{1}{r^2} \int_{Q((x_0,t_0),2r)\cap(\Omega\times\mathbb{R}^+)} |u(x,t)|^2 \, dx\, dt.$$

If $\Gamma^{\epsilon,R}_\alpha(x) = \{(y,t) \in \Omega \times (\epsilon, R) : |y-x| < \alpha t\}$, we define the averaging operator

$$\widetilde{S}^{\epsilon,R}_\alpha f(x) = \left(\int_1^2 \int_{\Gamma^{a\epsilon,aR}_{\alpha/a}(x)} t^{1-d} \left|\overline{\nabla} P^L_t f(y)\right|^2 \, dy\, dt\, da \right).$$

Then, an important "good λ" inequality required for the proof is the following.

Lemma 3. *There exists $C > 0$, such that, for $0 < \gamma < 1$, $\lambda > 0$, $0 < \epsilon < R < \infty$ and $f \in H^p_{\max,L} \cap L^2(\Omega)$, we have*

$$\left|\left\{x \in \Omega : \widetilde{S}^{\epsilon,R}_{1/20} f(x) > 2\lambda, f^*_L(x) \leq \gamma\lambda\right\}\right| \leq C\gamma^2 \left|\left\{x \in \Omega : \widetilde{S}^{\epsilon,R}_{1/2} f(x) > \lambda\right\}\right|.$$

These two results, combined with Besicovitch and Whitney covering lemmas, provide the desired result.

2.2.3. *From area integral functions to Hardy spaces.*

Lemma 4. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^d$ with Ω^c unbounded. Then, under DBC and f vanishes at infinity in a generalized sense, we have*

$$\|f\|_{H^p_{L,r}(\Omega} \lesssim \|s_L f\|_{L^p(\Omega)}.$$

The proof is based on an atomic decomposition for tent spaces, and uses the duality mentioned above for the space $\lambda^L_\alpha(\mathbb{R}^d)$.
Since pointwise $s_L(x) \leq S_L(x)$, then the main result follows from lemmas 1, 2 and 4.

References

[1] Auscher, P., Russ, E. *Hardy spaces and divergence operators on strongly Lipschitz domain of* $\mathbb{R}^n$. J. Funct. Anal. **201**, 148184 (2003)

[2] Chang, D.C. *The dual of Hardy spaces on a bounded domain in* $\mathbb{R}^n$. Forum. Math. **6**, 6581 (1994).

[3] Dziubański, J. *Atomic representation of* H^p *spaces associated with some Schrödinger operators*. Indiana Univ. Math. J. **47**, 75-98 (1998).

[4] Dziubański, J., Garrigós, G., Martínez,T., Torrea, J.L., Zienkiewicz, J. *BMO spaces related to Schrödinger operator with potential satisfying reverse Hölder inequality*. Math. Z. **249**, 329356 (2005).

[5] Dziubański, J., Zienkiewicz, J. *Hardy spaces associated with some Schrödinger operators*. Studia. Math. **126**, 149160 (1997)

[6] Huang, J.Z. *Hardy spaces associated to the Schrödinger operator on strongly Lipschitz domains of* $\mathbb{R}^d$. Math. Z. **266** , no. 1, 141168 (2010).

[7] Huang, J.Z., Liu, H.P. *Littlewood-Paley functions associated with the Schrödinger operators*. Mathematische Annalen (submitted).

[8] Huang, J.Z., Liu, H.P. *Area integral associated with the Schrödinger operators*. Indiana Univ. Math. J. (submitted).

[9] Li, H. *Estimations* L^p *des opérateurs de Schrödinger sur les groupes nilpotents*. J. Funct. Anal. **161**, 152218 (1999).

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THE RIESZ TRANSFORM FOR $1 \leq p \leq 2$

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Presented by Shahaf Nitzan

Abstract. The authors prove that the Riesz transform is L^p bounded, $1 < p \leq 2$, when considered over complete non–compact Riemannian manifolds which satisfy some rather weak requirements. Further, they show that the result does not hold for $p > 2$.

The assumptions made on the Riemannian manifolds, namely the doubling property of the Riemannian measure and an upper estimate over the diagonal of the heat kernel, include as particular cases several different settings which were considered in previous studies. The technique used to obtain the main result combines the application of weighted estimates of the heat kernel with a Calderón-Zygmund decomposition argument. In particular, this approach allows the authors to remove any requirements on the derivatives of the heat kernel.

In addition, the authors prove a version of their main result with a local doubling condition replacing the global one.

1. Introduction and main results presented

Let M be a complete Riemannian manifold. In this section we first define the Laplace-Beltrami operator and the Riesz transform over M, and then formulate and discuss the main results of the paper.

1.1. **Preliminaries.** Here we recall how the classical notions of gradient, divergence, the Laplace operator, and the Riesz transform are extended to the setting of a Riemannian manifold.
Gradient. Let $\phi \in C^\infty(M)$. For $x \in M$ the differential $d\phi_x$ defines a linear functional on the tangent space T_xM. It follows that there exists an element of T_xM, denoted $\nabla\phi(x)$ and called the gradient of ϕ at x, such that,

$$\langle \nabla\phi(x), V_x\rangle_{T_xM} = d\phi_x(V_x) \qquad \forall V_x \in T_xM.$$

The gradient of ϕ is the vector field on M obtained by this definition, it can be shown that it is a C^∞ vector field and that basic properties of the classical gradient in an euclidian space hold also in this setting.
Divergence. (For simplicity, this definition is given over an oriented manifold, though a similar definition can be given also for non–oriented manifolds). Denote by w the volume form on M and by n its dimension. Given a C^∞ vector field Φ on M let $\imath_\Phi w$ be the $n-1$ form defined by

$$\imath_\Phi w(V_1, ..., V_{n-1}) = w(\Phi, V_1, ..., V_{n-1}),$$

for vector fields $V_1, ..., V_{n-1}$ over M. It follows that $d(\imath_\Phi w)$ is an n form and therefore that there exists a function $\mathrm{div}\Phi \in C^\infty(M)$, called the divergence of Φ, which satisfies,

$$d(\imath_\Phi) = \mathrm{div}\Phi \cdot w.$$

It can be shown that basic properties of the classical divergence in an euclidian space hold also in this setting.
The Laplace-Beltrami operator. Similar to the definition of the Laplace operator over an euclidian space, the Laplace-Beltrami operator over a Riemannian manifold is defined as the divergence of the gradient. We denote

$$\triangle\, \phi = -(\mathrm{div} \circ \nabla)\phi$$

for a function $\phi \in C^\infty(M)$, (with the minus sine chosen to make the operator positive). It can be shown that basic properties of the classical Laplace operator in an euclidian space hold also in this setting. In particular, integration by parts gives

$$\langle \triangle\, f, g\rangle_{L^2(M)} = \langle \nabla f, \nabla g\rangle_{L^2(M)} \qquad \forall f, g \in C_0^\infty(M). \tag{1}$$

It now follows that since M is complete $\triangle$ can be extended to a non-negative self adjoint operator on $L^2(M)$ and spectral theory can be applied, [9]. It can be proved that the *heat semigroup* $e^{-t\triangle}$ is a family of integral operators,

$$e^{-t\triangle} f(x) = \int_M p_t(x, y) f(y) d\mu(y),$$

where $d\mu$ is the Riemannian measure [9]. The kernel $p_t(x, y)$ associated with this family is called the *heat kernel*, it can also be defined using the terminology of solutions to the heat equation

$$\frac{\partial u}{\partial t} = -\triangle\, u.$$

The Riesz transform. One can easily check, applying the Fourier transform for example, that the Riesz transform in $\mathbb{R}^n$ can be formally identified with the operator $\nabla\, \triangle^{-\frac{1}{2}}$. This motivates the similar definition of the Riesz transform over Riemannian manifolds. We denote by

$$T := \nabla\, \triangle^{-\frac{1}{2}}$$

the Riesz transform over M.

1.2. **Results presented in the paper.** A fundamental property of the Riesz transform is that it is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. It is natural to ask whether this result can be extended to a reasonable class of non–compact complete Riemannian manifolds. This question was raised in [9] and several partial results have been obtained since. In this paper the authors consider the class of complete Riemannian manifolds which satisfy the following two conditions.
By $V(x, r)$ we denotes the measure of the geodesic ball $B(x, r)$ with center at x and radius r.
Condition A. The manifold M satisfies the doubling property, i.e. there exits a constant $C > 0$ such that,

$$V(x, 2r) \le CV(x, r) \qquad \forall x \in M, r > 0.$$

Condition B. The heat kernel satisfies the following on diagonal estimate,

$$p_t(x, x) \le \frac{C}{V(x, \sqrt{t})} \qquad \forall x \in M, t > 0,$$

where C is a positive constant.

To formulate the main result of the paper we will need the following notations. For $f \in C_0^\infty(M)$ we denote by $\|f\|_p$ the L^p norm of f with respect to $d\mu$, and $\|f\|_{1,\infty} := \sup_{\lambda>0} \lambda\mu(\{x : |f(x)| > \lambda\})$.

Theorem 1. *Let M be a complete Riemannian manifold which satisfies conditions A and B. Then the Riesz transform T is weak $(1,1)$ and bounded on L^p, $1 < p \leq 2$. Explicitly, there exists C_p, $1 \leq p \leq 2$, such that*

$$\||\nabla f|\|_p \leq C_p \| \Delta^{\frac{1}{2}} f\|_p \quad \forall f \in C_0^\infty(M),$$

and

$$\||\nabla f|\|_{1,\infty} \leq C_1 \| \Delta^{\frac{1}{2}} f\|_1 \quad \forall f \in C_0^\infty(M).$$

The conditions in Theorem 1 cover the cases previously considered in [1], [2], [6], [8], [4]. It should be mentioned though, that in some of these more specific settings the Riesz transform was shown to be bounded for $1 < p < \infty$.

In [4] it is shown that, for $1 \leq p < 2$, it is enough to get a pointwise estimate on the first derivative of the heat kernel. This covers several settings but there are many situations in which this condition does not hold. In [5] it is shown that a weighted estimate on the first derivative can be obtained from condition B above, with no additional assumptions. The main observation of the authors is that estimates of the type obtained in [5] are enough to apply the approach from [4].

The authors give several examples of complete Riemannian manifolds which satisfy conditions A and B, for some of them the boundedness of the Riesz transform was not previously known.

A natural question is whether Theorem 1 holds also for $p > 2$. For this, the authors have a counter example.

Next, the authors obtain a local version of Theorem 1 replacing conditions A and B on M with the following.

Condition A'. The manifold M satisfies the local doubling property, i.e. for every $r_0 > 0$ there exits a constant $C(r_0) > 0$ such that,

$$V(x, 2r) \leq C(r_0)V(x, r) \qquad \forall x \in M, 0 < r < r_0.$$

and its volume growth at infinity is at most exponential in the sense that

$$V(x, \theta r) \leq Ce^{c\theta}V(x, r) \qquad \forall x \in M, 0 < r \leq 1, \theta > 1.$$

Condition B'. The heat kernel satisfies the following on diagonal estimate,

$$p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})} \qquad \forall x \in M, 0 < t \leq 1,$$

where C is a positive constant.

Theorem 2. *Let M be a complete manifold which satisfies conditions A' and B'. Then there exists C_p, $1 \leq p \leq 2$, such that*

$$\||\nabla f|\|_p \leq C_p(\| \Delta^{\frac{1}{2}} f\|_p + \|f\|_p) \quad \forall f \in C_0^\infty(M),$$

and

$$\||\nabla f|\|_{1,\infty} \leq C_1(\| \Delta^{\frac{1}{2}} f\|_1 + \|f\|_1) \quad \forall f \in C_0^\infty(M).$$

We note that if in addition one has $\Delta^{-\frac{1}{2}}$ is bounded on L^p then we obtain the boundedness of the Riesz transform. Here too Theorem 2 extends several previous results [1],[7].

2. Main result 1

In this section we sketch the main ideas in the proof of Theorem 1.

2.1. **The tools.**

2.1.1. *Calderón-Zygmund decomposition.* A proof of the following Calderón-Zygmund decomposition can be found, for example, in [3].

Let (M, d, μ) be a metric measured space and suppose that M satisfies the doubling condition (Condition A for a metric measured space). Then there exists $c = c(M)$ such that, given $f \in L^1(M) \cap L^2(M)$ and $\lambda > 0$, one can decompose f as $f = g + b = g + \sum b_i$, so that
(a) $|g(x)| \leq c\lambda$ for almost all $x \in M$;
(b) there exists a sequence of balls $B_i = B(x_i, r_i)$ so that the support of each b_i is contained in B_i:

$$\int |b_i(x)| d\mu(x) \leq c\lambda\mu(B_i) \quad \text{and} \quad \int b_i(x) d\mu(x) = 0;$$

(c) We have,

$$\sum \mu(B_i) \leq \frac{c}{\lambda} \int |f(x)| d\mu(x)$$

(d) There exists a positive integer k such that each point of M is contained in at most k balls B_i.

2.1.2. *An estimate on the space derivative of the heat kernel.* Let d be the geodesic distance on M and fix $0 < \alpha < 1/4$. Theorem 1.1 in [5] states that the on-diagonal estimate in Condition B implies that

$$p_t(x, y) \leq \frac{C_\alpha \exp(-\alpha \frac{d^2(x,y)}{t})}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}} \tag{2}$$

for all $x, y \in M$ and $t > 0$.
Combining this estimate with the doubling property (Condition A) the authors obtain the following estimate of the space derivative of the heat kernel.

Lemma 3. *There exists $\beta > 0$ such that,*

$$\int_{d(x,y) \geq \sqrt{t}} |\nabla_x p_s(x, y)|^2 d\mu(x) \leq C e^{-\frac{\beta t}{s}} s^{-\frac{1}{2}} \quad \forall y \in M; \; s, t > 0.$$

2.2. **Sketch of the proof of Theorem 1.** We first note that for $p = 2$ Theorem 1 follows immediately from (1), so our goal is to prove the weak (1,1) estimate, the result for $1 < p < 2$ will then follow from an interpolation argument. We therefore want to prove that for all $f \in L^1(M)$ and $\lambda > 0$ we have,

$$\mu(x : |Tf(x)| > \lambda) \leq C \frac{\|f\|_1}{\lambda}.$$

Fix $f \in L^1(M) \cap L^2(M)$ and $\lambda > 0$ and write the corresponding Calderón-Zygmund decomposition for f and λ, $f = g + b$.

We need to estimate $\mu(x : |Tg(x)| > \lambda/2)$ and $\mu(x : |Tb(x)| > \lambda/2)$ by $C\frac{\|f\|_1}{\lambda}$. The first estimate follows easily from the fact that T is bounded on L^2 and $|g(x)| \leq c\lambda$. To obtain the second estimate we first write,

$$Tb_i = Te^{-t_i\triangle}b_i + T(I - e^{-t_i\triangle})b_i$$

where $t_i^2 = r_i$, is the radius of the ball B_i.
Applying (2), the doubling property (property A), and the properties of the Calderón-Zygmund decomposition, the authors obtain the estimate

$$\|\sum e^{-t_i\triangle}b_i\|_2^2 \leq \|f\|_1.$$

The L^2 boundedness of T implies now the required estimate on $\mu(x : |T\sum e^{-t_i\triangle}b_i| > \lambda/4)$ in much the same way as with the function g above.
To complete the proof it remains to estimate

$$\mu(x : |T\sum(I - e^{-t_i\triangle})b_i| > \lambda/4)$$

$$\leq \sum \mu(2B_i) + \mu(x \in M \setminus \cup 2B_i : |T\sum(I - e^{-t_i\triangle})b_i| > \lambda/4).$$

The left term satisfies the required estimate due to the doubling condition and property (c) of the C-Z decomposition. The estimate for the right term will follow easily if we obtain the following inequality.

$$\int_{M\setminus 2B_i} |T(I - e^{-t_i\triangle})b_i(x)|d\mu(x) \leq \|b_i\|_1.$$

To this end the authors find an explicit form of the kernel $k_t(x, y)$ of the operator $T(I-e^{-t\triangle})$,

$$k_t(x, y) = \int_0^\infty (\frac{1}{\sqrt{s}} - \frac{\mathbb{1}_{s>t}}{\sqrt{s-t}})\nabla_x p_s(x, y)ds$$

and then apply Lemma 3 to make the estimates which provide the required inequality.

References

[1] D. Bakry, *Etude des transformations de Riesz dans les varietes riemanniennes ‘a courbure de Ricci minoree*, Seminaire de Probabilites XXI, Springer L.N. no 1247, 137-172, (1987).

[2] J.-C. Chen, *Heat kernels on positively curved manifolds and applications*, Ph. D. thesis, Hanghzhou University, 1987.

[3] R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homog‘ enes*, Springer L. N. no 242, (1971).

[4] X. T. Duong, A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Revista matematica iberoamericana, 15(2), 233-265, (1999).

[5] A. Grigor’yan, *Gaussian upper bounds for the heat kernel on arbitrary manifolds*, J. Di.. Geom., 45, 33–52, (1997).

[6] J. Li, *Gradient estimate for the heat kernel of a complete Riemannian manifold and its applications*, J. Funct. Anal., 97, 293-310, (1991).

[7] N. Lohoue, *Comparaison des champs de vecteurs et des puissances du laplacien sur une variete riemannienne a courbure non positive*, J. Funct. Anal., 61, 2, 164–201, (1985).

[8] L. Saloff-Coste, *Analyse sur les groupes de Lie a croissance polynomiale* Arkiv f0r Matematik, 28(1-2), 315-331,(1990).

[9] R. S. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold* Journal of functional analysis, 52(1), 48-79, (1983).

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THE SOLUTION OF THE KATO SQUARE ROOT PROBLEM FOR SECOND ORDER ELLIPTIC OPERATORS ON $\mathbb{R}^n$

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presented by Yumeng Ou

Abstract. We prove the Kato conjecture for elliptic operators on $\mathbb{R}^n$. More precisely, we show that the domain of the square root of a uniformly complex elliptic operator $L = -\operatorname{div}(A\nabla)$ with bounded measurable coefficients in $\mathbb{R}^n$ is the Sobolev space $H^1(\mathbb{R}^n)$ in any dimension with the estimate $\|\sqrt{L}f\|_2 \approx \|\nabla f\|_2$.

1. Introduction and Main Results Presented

Let $A = A(x) = (a_{i,j}(x))$ be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, satisfying the ellipticity condition

$$\lambda|\xi|^2 \le \operatorname{Re} A\xi \cdot \xi^* \quad \text{and} \quad |A\xi \cdot \zeta^*| \le \Lambda|\xi||\zeta|, \tag{1}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some $0 < \lambda \le \Lambda < \infty$. Here $u \cdot v^* = u_1\bar{v}_1 + \cdots + u_n\bar{v}_n$ is the usual inner product in $\mathbb{C}^n$. Then a second order divergence form operator

$$Lf := -\operatorname{div}(A\nabla f)$$

can be defined in the usual weak sense. Moreover, the ellipticity (1) enables one to define a square root $\sqrt{L}$, which is the operator satisfies $\sqrt{L}\sqrt{L} = L$ and has the following resolution formula:

$$\sqrt{L}f = a\int_0^\infty (1+t^2L)^{-3}t^3L^2f\,\frac{dt}{t}, \tag{2}$$

for some constant a.
The "Kato square root problem" is: (K) Show that there exists $C = C(n, \lambda, \Lambda)$ such that for any such L, there holds $\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \le C\|\nabla f\|_{L^2(\mathbb{R}^n)}$. Note that since the hypotheses are also satisfied by L^*, an immediate consequence is the following main theorem.

Theorem 1. *For any operator as above the domain of $\sqrt{L}$ coincides with the Sobolev space $H^1(\mathbb{R}^n)$ and $\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}$.*

This provides the complete solution to a well known problem originated in the work of Kato [4], where he had been motived by the applicability of a positive result to the perturbation theory for parabolic and hyperbolic evolution equations. It has been a long-standing open problem and a number of people have contributed to its solution. Among others along the way, some key notes of the methods proposed are (1) reduction to Carleson measure estimate, (2) $T(b)$ theorem for square roots, and (3) stopping-time argument. (See for instance [2].) Combining these ideas, Hofmann, Lacey and McIntosh in [3] prove (K) under a restriction of sufficient pointwise decay of the heat kernel, which is available for all real operators and two dimensional complex operators. In the paper [1] we present here, (K) is finally solved in its full generality. The key new idea that compensates for the missing of the heat kernel bounds is that there is enough decay, in an averaged sense, to carry out the program developed in [3]. More precisely, one can use the resolvent $(1+t^2L)^{-1}$ in place of the heat operator e^{-t^2L} in [3]. The key estimates that are involved is in the following lemma.

Lemma 2. *Let E and F be two closed sets of $\mathbb{R}^n$ and set $d = \mathrm{dist}(E,F)$. Then for f that is supported in E,*

$$\int_F |(1+t^2L)^{-1}f(x)|^2\,dx \le Ce^{-\frac{d}{ct}}\int_E |f(x)|^2\,dx,$$
$$\int_F |t\nabla(1+t^2L)^{-1}f(x)|^2\,dx \le Ce^{-\frac{d}{ct}}\int_E |f(x)|^2\,dx,$$
$$\int_F |(1+t^2L)^{-1}t\,\mathrm{div}\,\vec{f}(x)|^2\,dx \le Ce^{-\frac{d}{ct}}\int_E |\vec{f}(x)|^2\,dx,$$

where $c>0$ depends only on λ and Λ, and C on n, λ and Λ.

We also observe that the operators $(1+t^2L)^{-1}$, $t\nabla(1+t^2L)^{-1}$, $(1+t^2L)^{-1}t\,\mathrm{div}$ and $t^2\nabla(1+t^2L)^{-1}\,\mathrm{div}$ are uniformly bounded on L^2. In the rest of the note, we will sketch the main steps of the proof of (K). We remark that one can assume that the coefficients $a_{i,j}$ are C^∞ as long as it is not used quantitatively in the estimates. Then, one removes this assumption using a slight variant of Chapter 0, Proposition 7 in [2].

2. Reduction to quadratic estimate and Carleson measure estimate

We start with the resolution formula (2), where the integral converges normally in L^2 for $f \in C_0^\infty$. Take $g \in C_0^\infty(\mathbb{R}^n)$ with $\|g\|_{L^2} = 1$. By duality and Cauchy-Schwarz inequality

$$|\langle\sqrt{L}f, g\rangle|^2 \le a^2\int_0^\infty \|(1+t^2L)^{-1}tf\|_2^2\,\frac{dt}{t}\int_0^\infty \|V_tg\|_2^2\,\frac{dt}{t},$$

where $V_t = t^2L^*(1+t^2L^*)^{-2}$ is uniformly (in t) bounded on $L^2(\mathbb{R}^n)$. Using the standard orthogonality arguments of Littlewood-Paley theory, one can prove

$$\int_0^\infty \|V_tg\|_2^2\,\frac{dt}{t} \le C\|g\|_2^2 = C$$

through the following lemma.

Lemma 3. *Let $U_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $t>0$, be a family of bounded operators with $\|U_t\|_{op} \le 1$. If $\|U_tQ_s\|_{op} \le \left(\inf(\frac{t}{s},\frac{s}{t})\right)^\alpha$, $\alpha>0$, for a family Q_s, $s>0$, defined as the convolution operator with $\frac{1}{s^n}\psi(\frac{x}{s})$ for a real-valued $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\int\psi = 0$, then for some constant C depending only on α,*

$$\int_0^\infty \|U_tg\|_2^2\,\frac{dt}{t} \le C\|g\|_2^2.$$

Indeed, apply this to V_t. Since $V_tQ_s = -(1+t^2L^*)^{-2}t^2\,\mathrm{div}\,A^*\nabla Q_s$, we have

$$\|V_tQ_s\|_{\mathrm{op}} \le \|(1+t^2L^*)^{-2}t^2\,\mathrm{div}\,A^*\|_{\mathrm{op}}\|\nabla Q_s\|_{\mathrm{op}} \le cts^{-1},$$

with c depending only on n,λ and Λ. On the other hand, choose $\psi = \Delta\phi$ for some smooth radial function ϕ such that $\psi = \mathrm{div}\,\vec{h}$. This implies $Q_s = s\,\mathrm{div}\,\mathbf{R_s}$ for some uniformly bounded $\mathbf{R_s}$, which yields

$$\|V_tQ_s\|_{\mathrm{op}} \le \|t^2L^*(1+t^2L^*)^{-2}\,\mathrm{div}\,\|_{\mathrm{op}}\|s\mathbf{R}_s\|_{\mathrm{op}} \le ct^{-1}s,$$

with c depending only on n, λ and Λ.
We are left with proving

$$\int_0^\infty \|(1+t^2L)^{-1}tLf\|_2^2 \frac{dt}{t} \le C \int_{\mathbb{R}^n} |\nabla f|^2,$$

which by defining $\theta_t := -(1+t^2L)^{-1}t \operatorname{div} A$ is equivalent to

$$\int_0^\infty \|\theta_t \nabla f\|_2^2 \frac{dt}{t} \le C \int_{\mathbb{R}^n} |\nabla f|^2. \tag{3}$$

Particularly useful to us is the following $\mathbb{C}^n$-valued function $\gamma_t(x)$ defined as

$$\gamma_t(x) := (\theta_t \mathbf{1}) = ((-(1+t^2L)^{-1}t\partial_j a_{j,k})(x))_{1\le k\le n},$$

where $\mathbf{1}$ is the $n \times n$-identity matrix, the action of θ_t on $\mathbf{1}$ being columnwise.

Lemma 4. *The inequality (3) (and thus (K)) follows from the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dxdt}{t} < \infty, \tag{4}$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

Sketch of the proof. (4) and Carleson's inequality imply

$$\int_{\mathbb{R}^n} \int_0^\infty |\gamma_t(x) \cdot (P_t^2 \nabla f)(x)|^2 \frac{dxdt}{t} \le C \int_{\mathbb{R}^n} |\nabla f|^2,$$

where P_t denotes the operator of convolution with $\frac{1}{t^n} p(\frac{x}{t})$ for some smooth real-valued function p supported in the unit ball of $\mathbb{R}^n$ with $\int p = 1$. We then claim that

$$\int_{\mathbb{R}^n} \int_0^\infty |\gamma_t(x) \cdot (P_t^2 \nabla f)(x) - (\theta_t \nabla f)(x)|^2 \frac{dxdt}{t} \le C \int_{\mathbb{R}^n} |\nabla f|^2,$$

which will imply (3). To obtain this estimate, we begin by writing

$$\gamma_t(x) \cdot (P_t^2 \nabla f)(x) - (\theta_t \nabla f)(x) = (U_t P_t \nabla f) + (\theta_t (P_t^2 - I)\nabla f)(x),$$

where $U_t \vec{f}(x) := \gamma_t(x) \cdot (P_t \vec{f})(x) - (\theta_t P_t \vec{f})(x)$. The first term is taken care of by the Littlewood-Paley theory and standard Fourier analysis techniques. We refer to Lemma 4.3 and 4.4 in [1] for details. The second term, by observing that P_t commutes with partial derivatives and $\|\theta_t \nabla\|_{\mathrm{op}} = \|(1+t^2L)^{-1}tL\|_{\mathrm{op}} \le Ct^{-1}$, satisfies

$$\int_{\mathbb{R}^n} \int_0^\infty |(\theta_t (P_t^2 - I)\nabla f)(x)|^2 \frac{dxdt}{t} \le C^2 \int_{\mathbb{R}^n} \int_0^\infty |((P_t^2 - I)f)(x)|^2 \frac{dt}{t^3} dx \le C^2 c(p) \|\nabla f\|_2^2,$$

by the Plancherel theorem with C depending only on n, λ and Λ. □

3. Prove the Carleson measure estimate (4) via a $T(b)$ argument

In the last part of the note, we sketch the proof of the Carleson measure estimate (4) in Lemma 4, where the integral tool is a variant of the local $T(b)$ theorem for square roots in [2]. We first construct the functions that will act as the "b_Q" in the $T(b)$ theorem. Fix a cube Q, $\epsilon \in (0,1)$, a unit vector ω in $\mathbb{C}^n$ and define a scalar-valued function

$$f^\epsilon_{Q,\omega} = (1 + (\epsilon \ell(Q))^2 L)^{-1} (\Phi_Q \cdot \omega^*),$$

where, denoting by x_Q the center of Q, $\Phi_Q(x) := x - x_Q \in \mathbb{R}^n$. The key estimates are the following, which follows from Lemma 2.

$$\int_{5Q} |f^{\epsilon}_{Q,\omega} - \Phi_Q \cdot \omega^*|^2 \le C_1 \epsilon^2 \ell(Q)^2 |Q|, \tag{5}$$

$$\int_{5Q} |\nabla(f^{\epsilon}_{Q,\omega} - \Phi_Q \cdot \omega^*)|^2 \le C_2 |Q|, \tag{6}$$

where C_1, C_2 are independent of ϵ, Q and ω.

We will also need a dyadic averaging operator in our argument. Let Q be a cube in $\mathbb{R}^n$ and consider a collection of dyadic cubes of $\mathbb{R}^n$ that contains Q. Define

$$S^Q_t \vec{f}(x) = \frac{1}{|Q'|} \int_{Q'} \vec{f}(y)\, dy$$

for x in the dyadic cube Q' and $\frac{1}{2}\ell(Q') < t \le \ell(Q')$. Then, (4) follows immediately from the combination of the next two lemmata.

Lemma 5. *There exists an $\epsilon > 0$ depending on n, λ, Λ, and a finite set W of unit vectors in $\mathbb{C}^n$ whose cardinality depends on ϵ and n, such that*

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dxdt}{t} \le C \sum_{\omega \in W} \sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x) \cdot (S^Q_t \nabla f^{\epsilon}_{Q,\omega})(x)|^2 \frac{dxdt}{t},$$

where C depends only on ϵ, n, λ and Λ.

Lemma 6. *For C depending only on n, λ, Λ and $\epsilon > 0$, we have*

$$\int_Q \int_0^{\ell(Q)} |\gamma_t(x) \cdot (S^Q_t \nabla f^{\epsilon}_{Q,\omega})(x)|^2 \frac{dxdt}{t} \le C|Q|.$$

The proof of Lemma 6 involves estimates (5), (6), and repeated use of Lemma 2. It also utilizes a quadratic estimate of operator S^Q_t of the form

$$\int_Q \int_0^{\ell(Q)} |\gamma_t(x) \cdot ((S^Q_t - P_t^2)\vec{f})(x)|^2 \frac{dxdt}{t} \le C \int_{\mathbb{R}^n} |\vec{f}|^2,$$

which results from a Littlewood-Paley type argument and the fact that S^Q_t is an orthogonal projection. We refer to Lemma 4.7 in [1] for details.

The proof of Lemma 5 relies heavily on a stopping-time argument, which is given as the following lemma.

Lemma 7. *There exists a small $\epsilon > 0$ depending on n, λ and Λ, and $\eta = \eta(\epsilon) > 0$ such that for each unit vector ω in $\mathbb{C}^n$ and cube Q, one can find a collection $\mathcal{S}'_\omega = \{Q'\}$ of nonoverlapping dyadic sub-cubes of Q with the following properties:*

(i) *The union of the cubes in $\mathcal{S}'_\omega$ has measure not exceeding $(1-\eta)|Q|$;*

(ii) *If $Q'' \in \mathcal{S}''_\omega$, the collection of all dyadic sub-cubes of Q not contained in any $Q' \in \mathcal{S}'_\omega$, then*

$$\frac{1}{|Q''|} \int_{Q''} \mathrm{Re}(\nabla f^{\epsilon}_{Q,\omega}(y) \cdot \omega)\, dy \ge \frac{3}{4}$$

and

$$\frac{1}{|Q''|}\int_{Q''} |\nabla f^{\epsilon}_{Q,\omega}(y)|^2\, dy \le (4\epsilon)^{-2}.$$

We continue with the proof of Lemma 5 admitting Lemma 7. Let $\epsilon > 0$ to be chosen later and cover $\mathbb{C}^n$ with a finite number, depending on ϵ and n, of cones $\mathcal{C}_\omega$ associated with unit vectors ω in $\mathbb{C}^n$ and defined by

$$|u - (u\cdot\omega^*)\omega| \le \epsilon |u\cdot\omega^*|.$$

It thus suffices to argue for each ω fixed and to obtain a Carleson measure estimate for $\gamma_{t,\omega}(x) := \chi_{\mathcal{C}_\omega}(\gamma_t(x))\gamma_t(x)$. Fix a cube Q and let $Q'' \in \mathcal{S}''_\omega$ as defined in Lemma 7. Set

$$v = \frac{1}{|Q''|}\int_{Q''} \nabla f^{\epsilon}_{Q,\omega}(y)\, dy \in \mathbb{C}^n.$$

If $x \in Q''$ and $\frac{1}{2}\ell(Q'') < t \le \ell(Q'')$, then $v = (S^Q_t \nabla f^\epsilon_{Q,\omega})(x)$. Hence, according to Lemma 7 and a simple geometric observation (Lemma 8) stated at the end of this section, one has

$$|\gamma_{t,\omega}(x)| \le 4|\gamma_{t,\omega}(x)\cdot (S^Q_t\nabla f^\epsilon_{Q,\omega})(x)| \le 4|\gamma_t(x)\cdot(S^Q_t\nabla f^\epsilon_{Q,\omega})(x)|. \tag{7}$$

One then uses (7) to estimate

$$A_\omega := \sup_Q \frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |\gamma_{t,\omega}(x)|^2\, \frac{dxdt}{t}$$

after partitioning the Carleson box $Q\times(0,\ell(Q)]$ into the Carleson boxes $Q'\times(0,\ell(Q')]$ for Q' in $\mathcal{S}'_\omega$ and the Whitney rectangles $Q''\times(\frac{1}{2}\ell(Q''),\ell(Q'')]$ for Q'' in $\mathcal{S}''_\omega$. Note that (7) will be useful in the estimate over the Whitney rectangles, while the estimate over the part of the Carleson boxes will follow from the definition of A_ω (which can be assumed to be finite), the disjointness of $\{Q'\}$, and property (i) in Lemma 7.

Lemma 8. *Let ω be a unit vector in a Hilbert space H, u, v be vectors in H and $0 < \epsilon \le 1$ be such that*

$$|u - (u\cdot\omega^*)\omega| \le \epsilon|u\cdot\omega^*|, \quad \mathrm{Re}(v\cdot\omega) \ge \frac{3}{4}, \quad |v| \le (4\epsilon)^{-1}.$$

Then, $|u| \le 4|u\cdot v|$.

References

[1] A. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. **156** (2002), 633-654.

[2] P. Auscher and Ph. Tchamitchian, *Square root problem for divergence operators and related topics*, Astérisque **249**, Soc. Math. France, Paris, 1998.

[3] S. Hofmann, M. Lacey and A. McIntosh, *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Ann. of Math. **156** (2002), 622-631.

[4] T. Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), 246-274.

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ATOMIC DECOMPOSITIONS OF H^p SPACES ASSOCIATED WITH SOME SHRÖDINGER OPERATORS

JACEK DZIBUAŃSKI

presented by Robert S. Rahm

1. Overview

Let

$$A = -\Delta + V$$

where is a $V(x) = \sum_{\gamma \leq \alpha} c_\gamma x^\gamma$ positive polynomial on $\mathbb{R}^d$. The purpose of the paper is to use the operator, A to define a certain Hardy space and then to prove a certain atomic decomposition of the Hardy space and then to prove that the atomic–decomposition norm is the same as the maximal function norm (both of these will be defined below.)

In particular, let $\{T_t\}$ be the semi-group associated to $-A$. That is, $T_t = e^{-tA}$. Define the following maximal function based on $\{T_t\}$:

$$\mathcal{M}f(x) = \left| \sup_{0<t} T_t f(x) \right| .$$

Using this maximal function, we define H^p_A in the expected way:

$$\|f\|_{H^p_A} := \|\mathcal{M}f\|_{L^p} .$$

The results of this paper apply to the H^p_A spaces for $0 < p \leq 1$, but the ideas are contained in the case $p = 1$ and so for the purposes of this extended abstract, we will be concerned only with this case.

We will next define an atomic space associated to A, denoted by $H^1_{A,\text{atomic}}$. To do this, we first define an auxillary function:

$$m(x) = \sum_{\beta:\beta\leq\alpha} \left| D^\beta V(x) \right|^{1/|\beta|+2} .$$

Since V is positive, there is a $c > 0$ such that $m(x) \geq c$ for all $x \in \mathbb{R}^d$. Define:

$$\mathcal{B}_0 := \{c \leq m < 1\}$$

$$\mathcal{B}_n := \{2^{(n-1)/2} \leq m < 2^{n/2}\}.$$

Note that $\mathbb{R}^d = \cup \mathcal{B}_n$. We are now able to give the definition of an atom: A function a is an atom for H^1_A associated to the ball $B(x,r)$ if:

$$\text{supp } a \subset B(x,r)$$

$$\|a\|_{L^\infty} \leq (\text{vol} B(x,r))^{-1}$$

$$\text{if } x \in \mathcal{B}_n, \text{ then } r \leq 2^{-n/2}$$

$$\text{if } x \in \mathcal{B}_n \text{ and } r \leq 2^{-1-n/2}, \text{ then } \int a = 0.$$

Now, the norm for $H^1_{A,\text{atomic}}$ has the expected definition:

$$\|f\|_{H^1_{A,\text{atomic}}} := \inf \sum |c_j|\,,$$

where the infimum is taken over all sets of coefficients $\{c_j\}$ such that $f = \sum c_j a_j$, with a_j atoms.

The main theorem is the following:

Theorem 1.1. *Let $f \in H^1_A$, then f admits an atomic decomposition and every function, f that has $\|f\|_{\mathfrak{a}} < \infty$ is in H^1_A. Futhermore, $\|f\|_{H^1_A} \simeq \|f\|_{H^1_{A,atomic}}$.*

Note several differences between atoms defined here and atoms defined on the classical space H^1, which is $H^1_{-\Delta}$. For example, only those atoms supported on small enough balls need to satisfy the mean value zero condition.

The "structure" of the proof is familiar: there is a partition of unity that is related to the atoms and the sets $\mathcal{B}_n$, and f times the functions in the partition of unity will be the "candidate atoms". In the present case, the atoms will be some sort of average of f on a given scale.

Showing that the "candidate atoms" are indeed atoms requires the use of two ideas.

First is the notion of "local Hardy spaces" of Goldberg, [5]. In essence, these are Hardy spaces "at a fixed scale" and the point is that all of the candidate atoms at a certain scale form a "local Hardy space" atomic decomposition of f at the given scale. We then sum over all scales.

To complete the estimates, we use the fact that the kernels of the operators T_t look like the usual local Gaussian kernels associated with $-\Delta$ locally, and for large values of t, they are small. These estimates are deduced by relating the operator A to a certain operator on a certain nilpotent Lie group, the details of which are carried out in [1].

In Section 2, we will briefly describe the local Hardy spaces and the ideas surrounding the Lie group business. In Section 3, we give some lemmas and estimates. Finally, in Section 4, we discuss how the author pulls everything together to prove the main theorem.

Remark 1.2. We would like to mention that in [2], the theorem actually contains another maximal function characterisation and, as mentioned before, the characterisation holds for p in the range $0 < p \leq 1$. (The atoms have a slightly different definition in the $p \neq 1$ case. In particular, there are more moment conditions.) Due to space and time constraints, we will concentrate on the theorem mentioned above.

2. Some Background

2.1. **Local Hardy Spaces.** We first give a brief definiton of the local Hardy spaces. First, let $\{\widetilde{T}_t\}$ be the semigroup of operators generated by Δ. Define the following "local maximal function":

$$\widetilde{\mathcal{M}}_n f(x) := \sup_{0<t\leq 2^{-n}} \left| \widetilde{T}_t f(x) \right|.$$

The local Hardy space h_n^1 is defined in the obious way. We say that a is an atom for h_n^1 if:

$$\begin{aligned} \text{supp } a \subset B(x,r) \quad r \leq 2^{-n/2}, \\ \|a\|_{L^\infty} \leq (\text{vol} B(x,r))^{-1}, \\ \text{if } r \leq 2^{-1-n/2} \text{ then } \int a = 0. \end{aligned}$$

Note the similarity between h_n^1 atoms and H_A^1 atoms. Indeed, every h_n^1 atom is a H_A^1 atom and every H_A^1 atom is an h_n^1 atom for some n. The space $h^1_{n,atomic}$ is given the obvious definition. The theorem of Goldberg [5] that we will use is essentially that every function f in h_n^1 has an atomic decomposition and that:

$$\|f\|_{h^1_{n,\text{atomic}}} \simeq \|f\|_{H^1_{A,\text{atomic}}}$$

2.2. **Lie Algebra and Kernel Estimates.** Some background information is contained in [4,6,7]. The part about the Lie algebra and Lie group is contained in [1], but we will at least attempt explain some of the ideas involved. Create a Lie algebra, $\mathfrak{g}$ with the following basis: $\{X_1 \ldots, X_d, Y_\beta : 0 \leq \beta \leq \alpha\}$ and associated Lie group, G. The idea is that the vectors X_i correspond to the differential operators D_i and the vectors Y_β correspond to multiplication by V and derivatives of V.

In particular, $\mathfrak{g}$ is represented as operators acting on $C^\infty(G)$ through via the map Π (defined in the paper, but not defined here.) Let $W = -X_1^2 - \cdots - X_n^2 - iY_\beta$. This is in the dual of $L^2(G)$. In the representation, this is essentially sent to A. Thus, it follows (though it is not trivial) that $A = \Pi_W$, where Π_W is defined by the equations:

$$\langle \Pi_W f, g \rangle = \langle W, \varphi_{f,g} \rangle ,$$

where $\varphi_{f,g}(x) = \langle \Pi(x) f, g \rangle$. (Note that we are using $\langle \cdot, \cdot \rangle$ to mean different things and it isn't always an inner product.)

The point is that by studying the operator W, we can deduce:

$$0 \leq T_t(x,y) \leq C_a t^{-d/2} (1 + t^{-1/2} |x-y|)^{-a} (1 + t^{1/2} m(x))^{-a}, \tag{2.1}$$

where the above holds for every $a > 0$.

3. Some Lemmas

Since we are only dealing with the case $p = 1$, we will quote lemmas from an earlier paper by Dzibuański and Zienkiewicz [3].

3.1. **A Covering Lemma and a Partition of Unity.** In this subsection, we give a covering of $\mathbb{R}^d$ and a partition of unity to go along with this covering. The covering is based on the level sets, $\mathcal{B}_n$. The idea is to cover each $\mathcal{B}_n$ with balls at scale 2^{-n}. This allows the authors to exploit the local Hardy spaces of Goldberg.

Lemma 3.1 (Covering Lemma)**.** *There is a constant C and a collection of balls $B_{(n.k)} = B(x_{(n,k)}, 2^{-n/2})$ such that $x_{(n,k)} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \cup_k B_{(n,k)}$ and $\#\{(n',k') : B(x_{(n,k)}, R2^{-n/2}) \cap B(x_{(n',k')}, R^{-n'/2} \neq \{\}) \leq R^C$.*

Using this covering, we have the partition of unity:

Lemma 3.2 (Partition of Unity)**.** *There are non–negative functions $\psi_{(n,k)}$ such that:*

$$\psi_{(n,k)} \in C_c^\infty(B(x_{(n,k)}, 2^{1-n/2})),$$

$$\sum_{(n,k)} \psi_{(n,k)}(x) = 1,$$

$$\left\| \nabla \psi_{(n,k)} \right\|_{L^\infty} \leq C 2^{n/2}.$$

3.2. **Some Estimates.** Recall that $\{\widetilde{T}_t\}$ is the semigroup generated by Δ. The following lemma gives a way to compare $\mathcal{M}_n$ with $\widetilde{\mathcal{M}}_n$.

Lemma 3.3. *There is a constant C such that for every non–negative integer, n, there holds:*

$$\left\| \sup_{0<t\leq 2^{-n}} \left| \widetilde{T}_t(\psi_{(n,k)} f)(x) - T_t(\psi_{(n,k)} f)(x) \right| \right\|_{L^1(dx)} \leq C \left\| \psi_{(n,k)} f \right\|_{L^1}.$$

Let $w_N(x) = (1+|x|)^N$. The author shows that for N big enough:

$$\sum_{(n,k)} w_N(2^{n/2}(x - x_{(n,k)})) \in L^\infty(dx).$$

Define the following maximal function:

$$\mathcal{M}_{(n,k)} f(x) := \sup_{0<t\leq 2^{-n}} \left| [T_t, \psi_{(n,k)}] f(x) \right|.$$

Lemma 3.4. *For every $N > 0$, there is a constant C such that*

$$\left\| \mathcal{M}_{(n,k)} f \right\| \leq C \left\| f(x) w_N(2^{n/2}(x - x_{(n,k)})) \right\|_{L^1(dx)}.$$

4. Bringing Everything Together

In this section, we explain the proof of Theorem 1.1.

First, fix an index pair, (n,k). Note that if $f \in H_A^1$, then the function $\psi_{(n,k)} f$ is in the local Hardy space h_n^1. Therefore, we have that:

$$\psi_{(n,k)} f = \sum_j c_j^{(n,k)} a_j^{(n,k)},$$

where the $a_j^{(n,k)}$ are h_n^1 atoms and therefore H_A^1 atoms. Also by the theory surrounding local Hardy spaces, it follows that:

$$\sum_j |c_j| \lesssim \left\| \widetilde{\mathcal{M}}_n(\psi_{(n,k)} f) \right\|_{L^1}.$$

Using the fact that the $\psi_{(n,k)}$ form a partition of unity, we can compute:

$$f = \sum_{(n,k)} \psi_{(n,k)} f = \sum_{(n,k)} \sum_j c_j^{(n,k)} a_j^{(n,k)}$$

and

$$\sum_{(n,k)} \sum_j |c_j| \lesssim \sum_{(n,k)} \left\| \widetilde{\mathcal{M}}_n(\psi_{(n,k)} f) \right\|_{L^1}.$$

Using the estimates from the previous section, the authors deduce that:

$$\sum_{(n,k)} \left\| \widetilde{\mathcal{M}}_n(\psi_{(n,k)} f) \right\|_{L^1} \lesssim (\|\mathcal{M} f\|_{L^1} + \|f\|_{L^1}),$$

which implies that:

$$\sum_{(n,k)} \sum_j |c_j| \lesssim \|\mathcal{M} f\|_{L^1} = \|f\|_{H^1_A},$$

as desired.

To prove the other inequality, we need to show that:

$$\|\mathcal{M} a\|_{L^1} \lesssim 1$$

for all H^1_A atoms, with constant independent of a. Let a be an atom at scale $2^{-n/2}$. That is, a is associated to the ball $B(x_0, r)$ where $x \in \mathcal{B}_n$ and $r \leq 2^{-n/2}$. Locally, we can use Lemma 3.3 and the local Hardy space theory to deduce that:

$$\|\mathcal{M}_n a\|_{L^1} \lesssim \left\| \widetilde{\mathcal{M}}_n a \right\|_{L^1} \lesssim 1.$$

And so now we need to show:

$$\left\| \sup_{t > 2^{-n}} |T_t a(x)| \right\|_{L^1(dx)}.$$

Here we will use the kernel estimates. In particular, by (2.1), for every $a > 0$, we have:

$$\begin{aligned} T_t(x,y) &\leq C_a t^{-d/2} (1 + t^{-1/2} |x-y|)^{-a} (1 + t^{1/2} m(x))^{-a} \\ &\leq \sum_{m \geq 2} C_a (1+m)^{-a} t^{-d/2} \chi_{[0,m]}(t^{-1/2} |y-x|) \chi_{[0,m]}(t^{1/2} m(x)) \\ &=: \sum_{m \geq 2} b_m P_m^t(x,y). \end{aligned}$$

(The best way to deduce the penultimate estimate above is to consider separetley the cases in which $\chi_{[0,m]}(t^{-1/2})$ "turns on" first and $\chi_{[0,m]}(t^{1/2} m(x))$ "turns on" first.) Note that $b_m \leq c_a (1+m)^{-a}$ for all $a > 0$. We therefore deduce that:

$$|T_t a(x)| \leq \sum_{m \geq 2} b_m P_m^t a(x),$$

where P_m^t is the operator with kernel $P_m^t(x,y)$. To complete the proof, the authors first prove that $P_m^t a = 0$ for $t > m^{C_1} 2^{-n}$, where $C_1 \geq 1$ is some constant independent of n. Using this fact, we complete the proof:

$$\left\| \sup_{t>2^{-n}} |T_t a(x)| \right\|_{L^1(dx)} \leq \sum_{m\geq 2} b_m \left\| \sup_{2^{-n}<\leq 2^{-n}m^{C_1}} \left| P_m^t a(x) \right| \right\|_{L^1(dx)}$$
$$\lesssim \sum_{m\geq 2} b_m m^{dC_1} \|a\|_{L^1(dx)} \lesssim 1.$$

References

[1] Jacek Dziubański, *A note on Schrödinger operators with polynomial potentials*, Colloq. Math. **78** (1998), no. 1, 149–161. MR1658167 (2000c:35029) ↑2, 3

[2] ———, *Atomic decomposition of H^p spaces associated with some Schrödinger operators*, Indiana Univ. Math. J. **47** (1998), no. 1, 75–98. ↑2

[3] Jacek Dziubański and Jacek Zienkiewicz, *Hardy spaces associated with some Schrödinger operators*, Studia Math. **126** (1997), no. 2, 149–160. ↑3

[4] G. B. Folland and Elias M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982. ↑3

[5] David Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. MR523600 (80h:46052) ↑2, 3

[6] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), no. 4 (106), 57–110 (Russian). ↑3

[7] Alexander Kirillov Jr., *An introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics, vol. 113, Cambridge University Press, Cambridge, 2008. ↑3

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L^p BOUNDS FOR RIESZ TRANSFORMS AND SQUARE ROOTS ASSOCIATED TO SECOND ORDER ELLIPTIC OPERATORS

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presented by Guillermo Rey

ABSTRACT. The authors prove a weak-type estimate for the Riesz transforms associated to certain second order elliptic operators.

1. Introduction and Main Results Presented

Let Δ be the Laplacian on $\mathbb{R}^d$:

$$\Delta u := \sum_{j=1}^{d} \partial_i^2 u.$$

If we let $Lu := -\Delta u$ then L is a positive operator and hence has a "square root" which we will denote by $L^{1/2} = \sqrt{-\Delta}$. There are several ways to define what this means and, perhaps, the appropriate for this paper is through the use of Ψ *functional calculus*.
However, for the purposes of exposition, let us instead use the Fourier transform, since for this operator it is simpler:

$$\mathcal{F}\Big[\sqrt{-\Delta} f\Big](\xi) := 2\pi|\xi|\widehat{f}(\xi),$$

where $\mathcal{F}$ is the Fourier transform:

$$\widehat{f}(\xi) := \mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx.$$

Recall that

$$\mathcal{F}\big[\partial_i f\big](\xi) = 2\pi i\xi\mathcal{F}f(\xi),$$

this makes the definition of $\sqrt{-\Delta}$ natural since

$$\mathcal{F}\Big[-\Delta f\Big](\xi) = 4\pi^2|\xi|^2\mathcal{F}f(\xi).$$

And indeed one has $\sqrt{-\Delta}^2 = -\Delta$.

If $-\Delta$ acts like a 2^{nd} order derivative, then $\sqrt{-\Delta}$ should act like a first order derivative. In fact from Plancherel's theorem we have

$$\begin{aligned}\big\|\sqrt{-\Delta}f\big\|_{L^2(\mathbb{R}^d)} &= \big\|2\pi|\xi|\widehat{f}(\xi)\big\|_{L^2_\xi(\mathbb{R}^d)}\\ &= \Big(\sum_{j=1}^{d}\big\|2\pi|\xi_i|\widehat{f}(\xi)\big\|^2_{L^2_\xi(\mathbb{R}^d)}\Big)^{1/2}\\ &= \Big(\sum_{j=1}^{d}\big\|\partial_i f\big\|^2_{L^2(\mathbb{R}^d)}\Big)^{1/2}\\ &= \|\nabla f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

A very natural question is whether this holds in L^p for $p \neq 2$, this is indeed the case (for $p > 1$) but the proof is more involved than the L^2 case.
By duality, in order to show that $\|\sqrt{-\Delta}f\|_{L^p} \sim \|\nabla f\|_{L^p}$ it suffices to prove

$$\|\nabla f\|_{L^p(\mathbb{R}^d)} \lesssim \|\sqrt{-\Delta}f\|_{L^p(\mathbb{R}^d)}.$$

Taking inverses, this would follow from

$$\|\nabla(-\Delta)^{-1/2}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Now observe that the Fourier multiplier associated to $\nabla(-\Delta)^{-1/2}$ is

$$\Big(i\frac{\xi_1}{|\xi|}, \ldots, i\frac{\xi_d}{|\xi|}\Big) = i\frac{\xi}{|\xi|},$$

which is exactly the vector-Riesz transform. Once we are here we can use the following classical theorem (see [2]):

Theorem 1. *Let R be the vector Riesz transform, i.e.:*

$$\mathcal{F}[Rf](\xi) = i\frac{\xi}{|\xi|}\widehat{f}(\xi),$$

then R is of weak-type $(1,1)$ and strong-type (p,p) for all $1 < p < \infty$.

As a consequence we have

Theorem 2. *For $1 < p < \infty$ we have*

$$\|\nabla(-\Delta)^{-1/2}f\|_{L^p(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}.$$

One can obtain similar theorems for variations of the Laplacian; for example one can use Littlewood-Paley theory to prove that for

$$L = -\operatorname{div}\big(A\nabla f\big)$$

one also has

$$\|\nabla L^{-1/2}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for $1 < p < \infty$, as long as A is a $d \times d$ matrix with constant coefficients and

$$\lambda|\xi|^2 \leq \mathrm{Re}\langle A\xi, \xi\rangle \quad \text{and} \quad |\langle A\xi, \zeta\rangle| \leq \Lambda|\xi||\zeta|.$$

However, one runs into serious difficulties when A is allowed to be a function of x.

The *Kato square root problem* is a generalization of this question to operators which act like $-\Delta$. More precisely: let $A : \mathbb{R}^d \to \mathbb{C}^{d\times d}$ be a $d \times d$ matrix L^∞ function, for $f \in H^1$ we can define

$$Lf := -\operatorname{div}\big(A\nabla f\big)$$

in the weak sense.
Suppose A is *accretive*, that is: there exist constants $0 < \lambda \le \Lambda < \infty$ such that

$$\lambda|\xi|^2 \le \operatorname{Re}\langle A\xi, \xi\rangle \quad \text{and} \quad |\langle A\xi, \zeta\rangle| \le \Lambda|\xi||\zeta|,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^d$.
When A is accretive then one can define $\sqrt{L}$ and we can pose a similar question as in Theorem 2:

$$\text{Does} \quad \|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \quad \text{hold for some } p?$$

The *Kato conjecture* was that this was indeed true for $p = 2$. This conjecture was solved in [1]:

Theorem 3 ([1]). *If L is defined as above and A is accretive, then the domain of $\sqrt{L}$ coincides with H^1 and*

$$\|\sqrt{L} f\|_{L^2(\mathbb{R}^d)} \sim \|\nabla f\|_{L^2(\mathbb{R}^d)}.$$

The main theorem of the this work is the following:

Theorem 4 ([3]). *Let $p_d = \frac{2d}{d+2}$. Then the Riesz transform $\nabla L^{-1/2}$ is of weak-type (p_d, p_d) and thus bounded on L^p for $p_d < p \le 2$.*

Compare Theorem 4 with Theorem 2; the result presented in this paper is a generalization of both Theorem 2 and Theorem 3.

2. Main Result

The proof uses Theorem 3 as a black-box in order to apply a Calderón-Zygmund-type decomposition. We refer the reader to the original article [3] for all the details, but here is a schematic description of the proof:
Let $T = \nabla L^{-1/2}$, we want to prove

$$|\{x : |Tf(x)| > \alpha\}| \lesssim \alpha^{-p} \int_{\mathbb{R}^d} |f(x)|^p \, dx.$$

Assume without loss of generality that $f \ge 0$ and $\|f\|_{L^p} = 1$, where

$$p = p_d = \frac{2d}{d+2}.$$

We start by decomposing

$$f = g + b = g + \sum_j b_j,$$

where g is the "good" part and b is the "bad" part. The function g is good in the sense that

$$0 \le g \lesssim \alpha.$$

The function b does not satisfy this pointwise bound, but it instead is a sum of functions b_j which are supported on certain cubes Q_j and for which

$$\int_{Q_j} b_j = 0 \quad \text{and} \quad \frac{1}{|Q_j|}\int_{Q_j} |b_j|^p \lesssim \alpha^p.$$

So far, this is essentially the classical Calderón-Zygmund decomposition.
Now we decompose

$$\begin{aligned}|\{|Tf(x)| > 3\alpha\}| &\leq |\{Tg(x) > \alpha\}| + |\{|Tb(x)| > 2\alpha\}| \\ &= I + II.\end{aligned}$$

As usual, the bound for the first term follows from the L^2 boundedness of T (Theorem 3). Most of the difficulty is in estimating the second term.
The proof for the second term is a little involved but the basic idea is as follows: one is able to reduce to studying certain heat-extensions of the functions b_j. Using the cancellation properties of b_j we can "subtract" their average and then use the Poincaré-Sobolev inequality. Once we have the L^2 norm of the gradient of this heat-extension we use the fact that we are integrating over a dyadic shell which is quantifiably far from the support of b_j, together with a useful off-diagonal estimate, to reduce everything to the weak-type $(1, 1)$ of the Hardy-Littlewood maximal function.

References

[1] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, *The solution of the Kato Square Root Problem for Second Order Elliptic operators on* $\mathbb{R}^n$. Annals of Math. 156 (2002), 633–654.
[2] J. Duoandikoetxea, *Fourier Analysis*, Grad. Stud. Mth. 29, American Math. Soc., Providence 2000.
[3] S. Hofmann and J.M. Martell, L^p *bounds for Riesz transforms and square roots associated to second order elliptic operators.* Publ. Mat. 47 (2003), no. 2, 497–515

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