PROCEEDINGS OF THE INTERNET ANALYSIS SEMINAR ON HAUSDORFF GEOMETRY AND SINGULAR INTEGRAL OPERATORS

Contents

Overview of the Workshop

This workshop was part of the Internet Analysis Seminar that is the education component of the National Science Foundation – DMS # 0955432 held by Brett D. Wick. The Internet Analysis Seminar consists of three phases that run over the course of a standard academic year. Each year, a topic in complex analysis, function theory, harmonic analysis, or operator theory is chosen and an internet seminar will be developed with corresponding lectures. The course will introduce advanced graduate students and post-doctoral researchers to various topics in those areas and, in particular, their interaction.

This was a workshop that focused on the connections between Hausdorff geometry and singular integral operators. How one can discern order in a seemingly very disordered set? If the set in question has some self-similarity then a dynamical systems approach can be of use. But suppose there is no a priori structure. We consider one such situations when the full a priori knowledge about the set is the following: 1) its Hausdorff dimension is given, and we know that the Hausdorff measure in this dimension is (positive) and finite, 2) the set is a singularity set of a non-constant Lipschitz function satisfying some (fractional) Laplace equation. Or, instead of 2) one can say that singular integrals from a small collection (e.g., Riesz transforms) are bounded in L^2 with respect to Hausdorff measure. Or, instead of 2) one can say that a certain Calderón-Zygmund capacity of the set is positive. Then what geometry, if any, is imposed on the set by these conditions? It turns out (or conjectured) that automatically we can "connect" points from a non-trivial part of the set by a smooth manifold. In other words, the points of such a set should "feel" each others presence in a very quantitative and geometric way. This multi-dimensional analytic traveling salesman problem is the subject of the lectures. This is because several (but not all) such problems were recently solved, and they turned out to be entangling PDE, Harmonic Analysis and Geometric Measure Theory into one knot.

First, there is a particular (but very interesting and important) family of problems on the plane. These are problems posed by Painlevé, Denjoy, Ahlfors, Vitushkin, and were solved in the last 12 years by the efforts of a large group of mathematicians. Sets of finite Hausdorff measure $\mathcal{H}^1$ and positive analytic capacity on the plane must contain a subset of positive $\mathcal{H}^1$-measure of a rectifiable curve. This "analysis-to-geometry" statement was known as "Denjoy's problem" and was solved almost simultaneously, and by different methods by David-Mattila-Léger and Nazarov-Treil-Volberg.

The higher dimensional analogue of this question is a very interesting area of current research, and is the following: Is it true that the sets of finite Hausdorff measure $\mathcal{H}^m$, $1 \leq m \leq d$, m an integer, and positive $\gamma(m, d)$-capacity must contain a non-trivial m-rectifiable subset? This is known as the David-Semmes problem and is completely analogous to Denjoy's problem in dimension greater than 2. Unfortunately, in higher dimensions the main geometric tool, called Menger's curvature, is "cruelly missing". The topic of the Internet Analysis Seminar this year will focus on the machinery necessary to understand the David-Semmes problem and the recent work of Nazarov, Tolsa and Volberg in the co-dimension one case. The lectures will touch upon the themes connecting analysis (singular integral operators, operator capacity) with geometry (geometric measure theory).

The participants that presented, presented one of the following papers:

[1] Vladimir Eiderman, Fedor Nazarov, and Alexander Volberg, *The s-Riesz transform of an s-dimensional measure in $\mathbb{R}^2$ is unbounded for* $1 < s < 2$, J. Anal. Math. **122** (2014), 1–23. ↑

[2] Vladimir Eiderman, Alexander Reznikov, and Alexander Volberg, *Almost-additivity of analytic capacity and Cauchy independent measures*, available at http://arxiv.org/abs/1401.0407. ↑

[3] Benjamin Jaye and Fedor Nazarov, *Reflectionless measures and the Mattila-Melnikov-Verdera uniform rectifiability theorem*, available at http://arxiv.org/abs/1307.1156. ↑

[4] ———, *Three revolutions in the kernel are worse than one*, available at http://arxiv.org/abs/1307.3678. ↑

[5] Peter W. Jones, *Rectifiable sets and the traveling salesman problem*, Invent. Math. **102** (1990), no. 1, 1–15. ↑

[6] J. C. Léger, *Menger curvature and rectifiability*, Ann. of Math. (2) **149** (1999), no. 3, 831–869. ↑

[7] Fedor Nazarov, Xavier Tolas, and Alexander Volberg, *On the uniform rectifiability of AD regular measures with bounded Riesz transform operator: the case of Codimension 1*, available at http://arxiv.org/abs/1212.5229. ↑

[8] ———, *The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions*, available at http://arxiv.org/abs/1212.5431. ↑

[9] Kate Okikiolu, *Characterization of subsets of rectifiable curves in* $\mathbf{R}^n$, J. London Math. Soc. (2) **46** (1992), no. 2, 336–348. ↑

[10] Xavier Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), no. 1, 105–149. ↑

They were then responsible to prepare two one-hour lectures based on the paper and an extended abstract based on the paper. This proceedings is the collection of the extended abstract prepared by each participant.

The following people participated in the workshop:

Philip Benge	Georgia Institute of Technology
Tyler Bongers	Michigan State University
Lucas Chaffee	University of Kansas
Amalia Culiuc	Brown University
Ishwari Kunwar	Georgia Institute of Technology
Michael Lacey	Georgia Institute of Technology
Pan Ma	Vanderbilt University
Mishko Mitkovski	Clemson University
Robert Rahm	Georgia Institute of Technology
Guillermo Rey	Michigan State University
Rishika Rupam	Texas A&M University
Fangye Shi	University of Indiana
Prabath Silva	University of Indiana
Raghavendra Venkatraman	Indiana University
Brett D. Wick	Georgia Institute of Technology

RECTIFIABLE SETS AND THE TRAVELING SALESMAN PROBLEM

PETER JONES

presented by Tyler Bongers

Abstract. In this paper, a characterization of subsets of rectifiable sets in $\mathbb{R}^2$ is given. A measurement of the local deviation of a set from its best local linear approximation will be defined, which will then be studied at all scales and in all locations. When this measurement is finite, a rectifiable curve containing the set is explicitly constructed. On the other hand, it is also shown that all rectifiable curves satisfy this same geometric condition; the proof of this fact is carried out in the special case of Lipschitz curves, and complex analysis techniques are used to reduce the general case to the particular.

1. Introduction and Main Results Presented

The classic Traveling Salesman Problem asks to find the shortest possible route visiting each of (finitely many) cities and returning home; this leads to the problem of finding the shortest connected set containing a given finite set of points. In this paper, we will study a characterization of subsets of the plane for which there does exist a connected superset of finite length - that is, characterize subsets of rectifiable curves.

Let us begin by giving a scale invariant measure of the deviation of a set K from linearity at a given location. We call a square Q dyadic if it is of the form $[j \cdot 2^{-n}, (j+1) \cdot 2^{-n}] \times [k \cdot 2^{-n}, (k+1) \cdot 2^{-n}]$ for integers j, k, n; its sidelength is denoted $l(Q) = 2^{-n}$. For $\lambda > 0$, we say that λQ is the square with the same center as Q but sidelength $\lambda l(Q)$. We then define

$$\beta_K(Q) = \frac{\omega(Q)}{l(Q)}$$

where ω is the width of the smallest infinite strip (line, in the degenerate case) containing $K \cap 3Q$. Alternatively, we could define

$$\beta_K(Q) = 2 \inf_L \sup_{z \in 3Q \cap K} \frac{d(z, L)}{l(Q)}$$

where the infimum is taken over all lines L. We then use this to define

$$\beta^2(K) = \sum_Q \beta_K^2(Q) l(Q) = \sum_Q \beta_K(Q) \omega(Q)$$

where the sum is taken over all dyadic squares in the plane.

Thus it is seen that $\beta_K(Q)$ measures the deviation of the set from the best approximating line. We have that $0 \leq \beta_K(Q) \leq 3$ for all K, Q; the closer $\beta_K(Q)$ is to zero, the less deviation from linearity K has on the scale and location of Q. Then $\beta^2(K)$ measures non-linear behaviour at all scales and at all locations in the plane. This allows us to state one of the main results of this paper.

Theorem 1. *If $\Gamma \subset \mathbb{C}$ is connected, then*

$$\beta^2(\Gamma) \lesssim l(\Gamma)$$

The proof of this will proceed in several parts, beginning with a geometric study of the boundaries of Lipschitz domains (which are of independent importance). Let us consider a curve Γ, parameterized by $r(\theta)e^{i\theta}$, where

$$\frac{1}{C} \leq r(\theta) \leq 1$$

and r is Lipschitz. Morally, such almost-smooth-enough curves ought to behave linearly at least on small scales and at most locations; thus it is reasonable to expect that $\beta^2(K)$ is finite here. This is made precise by studying the distance between the curve Γ and an approximating polygon with 2^n sides; after rotating the dyadic grid and using the triangle inequality appropriately, this distance can be directly compared to the beta numbers.
In order to use the result on Lipschitz curves, we will use a general geometric theorem of independent interest:

Theorem 2. *If Ω is a simply connected domain and $l(\partial\Omega) < \infty$, there is a rectifiable curve Γ such that*

$$\Omega \setminus \Gamma = \bigcup_j \Omega_j$$

where each Ω_j is a C_0 Lipschitz domain, and

$$\sum_j l(\partial\Omega_j) \leq C_0 l(\partial\Omega)$$

This is proven using complex analysis techniques. We begin by conformally mapping Ω to the unit disk by a Riemann map F, and decomposing the disk into (a polar version of) dyadic squares. The assumption that Ω has finite boundary length implies that F lies in the Hardy space H^1, making a number of complex analytic tools available. Letting φ be the logarithm of F', an elementary argument with Green's theorem shows that

$$\int\int_{\mathbb{D}} |F'(z)||\varphi'(z)|^2 \log\frac{1}{|z|} dA \lesssim \int_{\partial\mathbb{D}} |F'(e^{it})|dt = l(\partial\Omega)$$

We then perform a stopping time argument, based on the behaviour of φ, tiling the disk with regions derived from dyadic squares; the images of these regions under the conformal map will (almost) be the disjoint C_0 Lipschitz domains. The regions so selected will also be 6-chord arc domains; that is, given any two points x, y in the boundary of such a region, there is a subarc of the boundary of length at most $6|x-y|$ containing x and y; this is almost preserved by the conformal map. Letting γ_Q denote the boundary of one of these regions, we estimate

$$\int_{\gamma_Q} |F'(z)|ds(z)$$

in order to estimate $\sum_j l(\partial\Omega_j)$. This is done in several ways, according to the exact behaviour of the region. Slightly smaller regions are then chosen in a manner which does not increase the boundary length too much, which are then mapped to actual C_0 Lipschitz domains; the Koebe distortion theorem or a Cantor-like construction is used here.

Now given a rectifiable curve Γ, we attach a large circle S and a line segment L meeting these two; applying the theorem to each of the bounded components of $\mathbb{C} \setminus (\Gamma \cup S \cup L)$, we can prove

Lemma 3. *If Γ is connected, there is a connected $\tilde{\Gamma}$ containing Γ with $l(\tilde{\Gamma}) \lesssim l(\Gamma)$, every bounded component of $\mathbb{C} \setminus \tilde{\Gamma}$ is a C_0 Lipschitz domain, and the unbounded component is the complement of a disk. Moreover, given $x, y \in \tilde{\Gamma}$, there is a subarc γ containing x, y with $l(\gamma) \lesssim |x - y|$.*

Denote the components as Ω_j with boundaries Γ_j, and set $d_j = \operatorname{diam}(\Gamma_j)$. To apply the lemma, we will use the fact that different parts of the curve will act almost independently on certain scales: in particular, let us consider two sets

$$\mathcal{F}(Q) = \{\Gamma_j : \Gamma_j \cap 4Q \neq \emptyset, d_j \geq l(Q)\}$$
$$G(Q) = \{\Gamma_j : \Gamma_j \cap 5Q \neq \emptyset, d_j < l(Q)\}$$

That is, $\mathcal{F}(Q)$ and $G(Q)$ count the pieces of Γ which are close to Q, classified according to the size of the region bounded by Γ. Morally, since the elements of $\mathcal{F}(Q)$ correspond to features of Γ which are large relative to the square Q, these pieces will act independently when computing the beta numbers (after all, large scale features must be somewhat separated due to their size). This is not quite true, and must be corrected to account for behaviour of the curve on scales smaller than Q, leading to

Lemma 4. *If Q^* is the dyadic parent of Q,*

$$\beta_\Gamma^2(Q) \lesssim \sum_{\mathcal{F}(Q)} \beta_{\Gamma_j}^2(Q^*) + \sum_{G(Q)} \frac{\operatorname{Area}(\Omega_k)}{l(Q)^2}$$

Notice how this contrasts with the fact that β is (in general) far from sublinear: It is not true that $\beta_{A\cup B}^2(Q) \lesssim \beta_A^2(Q) + \beta_B^2(Q)$. However, by considering the curve on the slightly larger scale of Q^* and adding a correction due to the small scale behaviour, we are left with a viable estimate.
By noticing the (almost) scale-invariance on both sides of the inequality, we may assume that Q has sidelength 1, in which case the estimate is reduced to

$$\omega_\Gamma(Q)^2 \lesssim \sum_{\mathcal{F}(Q)} \omega_{\Gamma_j}(Q^*)^2 + \sum_{G(Q)} \operatorname{Area}(\Omega_k)$$

The lemma is proved by studying a few cases, based on the size of $\mathcal{F}(Q)$:

- If $\mathcal{F}(Q)$ is empty, then exploit the fact that the region bounded by Γ is a disk - whose area is directly comparable to the square of its diameter.
- If $\mathcal{F}(Q)$ is large, containing at least 3 curves, then there must be some $\Gamma_j \in \mathcal{F}(Q)$ with $\beta_{\Gamma_j}^2(Q^*) > \epsilon$ (in which case, this case is finished). If not, then each $\Gamma_j \in \mathcal{F}(Q)$ is almost linear on the scale of Q^*, in which case two curves Γ_j must collide or cross - but the regions Ω_j are disjoint.
- If $\mathcal{F}(Q)$ contains only one or two curves, then the argument is more subtle, and requires studying $G(Q)$ as well.

From here, the proof of the main theorem is almost finished: We have

$$\begin{aligned}\sum_{Q}\sum_{\mathcal{F}(Q)} \beta^2_{\Gamma_j}(Q^*)l(Q) &\leq \sum_{j}\sum_{Q} \beta^2_{\Gamma_j}(Q^*)l(Q)\\ &\sim \sum_{j}\sum_{Q} \beta^2_{\Gamma_j}(Q^*)l(Q^*)\\ &\lesssim \sum_{j} l(\Gamma_j)\\ &\sim l(\Gamma)\end{aligned}$$

For the second term, we must estimate

$$\sum_{Q}\sum_{G(Q)} \frac{\text{Area}(\Omega_k)}{l(Q)}$$

The key point here is that, fixing a curve Γ_j, there are only a few squares Q at each level for which $\Gamma_j \in G(Q)$ - in particular, there are only C such squares, for a large constant C. Thus, changing the order of summation and summing over levels instead of squares,

$$\begin{aligned}\sum_{k} \text{Area}(\Omega_k) \sum_{\substack{Q\\ \Omega_k \in G(Q)}} l(Q)^{-1} &\lesssim \sum_{k} \text{Area}(\Omega_k) \sum_{\substack{n\\ 2^{-n} > d_k}} 2^n\\ &\sim \sum_{k} \frac{\text{Area}(\Omega_k)}{d_k}\\ &\sim \sum_{k} d_k \sim \sum_{k} l(\Gamma_k) \lesssim l(\Gamma)\end{aligned}$$

where we have used the fact that $\text{Area}(\Omega_k) \sim d_k^2$ and $l(\Gamma_k) \sim d_k$ for Lipschitz domains. This finishes the proof of the theorem.

The other main result of the paper is

Theorem 5. *If $\beta^2(K) < \infty$, there is a connected set Γ with $K \subset \Gamma$, such that*

$$l(\Gamma) \leq (1+\delta)\,\text{diam}(K) + C(\delta)\beta^2(K)$$

where C is used to indicate a universal constant dependent on δ. By adjusting the factor δ (and increasing it sufficiently), it is possible to make Γ have some useful properties, such as uniform local connectedness: that is, if $z_1, z_2 \in \Gamma$, there is a connected subset γ containing z_i with $l(\gamma) \lesssim |z_1 - z_2|$.

The proof of this is carried out inductively. We begin with a sequence of sets $\mathcal{L}_n$ which are uniformly distributed throughout K on scale 2^{-n}. If Γ_{n-1} has already been constructed, containing many line segments, we efficiently replace line segments making up Γ_{n-1} to connect the points in $\mathcal{L}_n \setminus \mathcal{L}_{n-1}$.

We now study $\beta^2_K(Q)$ for a dyadic square on a scale slightly larger than 2^{-n} containing a point in $\mathcal{L}_n$; based on this study, we replace certain line segments (adding a length which can be bounded in terms of $\beta^2_K(Q)l(Q)$) and add new line segments, preserving certain segments I_Q contained in Γ_{n-1}. Letting D_n denote the set of all such squares, the length added in constructing Γ_n will then be bounded by

$$C \sum_{Q \in D_n} \beta_K^2(Q) l(Q) + \frac{1}{2} \sum_{Q \in D_n} l(I_Q)$$

Summing and telescoping, this will lead to

$$l(\Gamma_N) \leq 2l(\Gamma_0) + C \sum_{Q} \beta_K^2(Q) l(Q)$$

and taking limits will finish the proof.
Combining the two main results, we are left with a complete answer to the question: A set K is contained in a rectifiable curve in the plane if and only if $\beta^2(K)$ is finite, and $\beta^2(K)$ is comparable to the length of the shortest such curve.

References

[1] Jones, P.W. Rectifiable sets and the Traveling Salesman Problem. *Invent. Math.*, **102**, 1-15 (1990)
[2] Jones, P.W. The Traveling Salesman Problem and Harmonic Analysis. *Publicacions Matematiques*, **35**, 259-267 (1991)

Tyler Bongers, Michigan State University, East Lansing MI 48823
E-mail address: `bongerst@msu.edu`

ON THE UNIFORM RECTIFIABILITY OF AD REGULAR MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: THE CASE OF CODIMENSION 1

FEDOR NAZAROV, XAVIER TOLSA, AND ALEXANDER VOLBERG

presented by Lucas Chaffee

Abstract. It is shown that if μ is a d-dimensional Alfhors-David regular measure in $\mathbb{R}^{d+1}$, and the Riesz transform is bounded in $L^2(\mu)$, then the non-BAUP David-Semmes cells form a Carleson family. This result, along with previous results of David and Semmes, yields as an immediate corollary that μ is uniformly rectifiable.

1. Introduction and Main Results Presented

In short, this paper examines an interesting piece of the much more general issue of how the geometry of the support of a d-dimensional measure, μ, in $\mathbb{R}^n$ relates to the boundedness of certain singular integral operators in $L^2(\mu)$. It has been known for some time that for a d-dimensional Alfhors-David regular measure in $\mathbb{R}^n$, uniform rectifiability is sufficient to obtain $L^2(\mu)$ boundedness of many d-dimensional Calderón Zygmund operators, however the necessity of this is much more difficult to examine, particularly in higher dimensions where certain curvature methods are unavailable. This paper sets forth, in a manner that's as self contained as possible, to examine the necessity of uniform rectifiabilty for a d-dimensional AD regular measure in $\mathbb{R}^{d+1}$ for the boundedness of the d-dimensional Riesz transform,

$$f \mapsto K * (f\mu), \text{ with } K(x) = \frac{x}{|x|^{d+1}}.$$

The main result of this paper is the following theorem,

Theorem 1. *Let μ be an AD regular measure of dimension d in $\mathbb{R}^{d+1}$. If the associated d-dimesional Riesz transform is bounded in $L^2(\mu)$, then the non-BAUP cells in the David-Semmes lattice associated with μ form a Carleson family*

The BAUP condition on a set is defined in by David and Semmes in [5], the term being short for what they call the bilateral approximation of unions by d-planes condition. For the purpose of this paper it is more useful to simply define the non-BAUP cells.

Definition 2. Let $\delta > 0$. We say that a cell $P \in \mathcal{D}$ is δ-non-BAUP if there exists a point $x \in P \cap \text{supp } \mu$ such that for every hyperplane L passing through x, there exists a point $y \in B(x, \ell(P)) \cap L$ for which $B(y, \delta\ell(P)) \cap \text{supp } \mu = \emptyset$.

It was shown in [5] that this result is sufficient to show that the measure is, in fact, uniformly rectifiable, and the authors make a special point of noting that their work in a sense deals only with the 'analytic' passing of the operator's boundedness to the measure's rectifiability, but that all credit for the 'geometric' aspect firmly belong to David and Semmes.
As mentioned above, this paper is as self contained as reasonably possible, and as such, the paper begins a long path beginning with building up preliminaries, and then smoothly

transitioning into the proof itself. In the next section we examine this path. For shorter definitions and lemmata I will transcribe their statements, but for the longer ones I will merely describe the ideas behind them.

2. Path to the Proof

After a brief section of notation, the authors begin by familiarizing the reader with AD regular measures and the Riesz transforms associated with them. Since the singularity in the kernel of the Riesz transform makes it difficult to impossible to immediately approach for an arbitrary Borel measure, the paper defines the regularized kernels,

$$K_\delta(x) = \frac{x}{\max\{\delta, |x|\}^{d+1}},$$

with the corresponding operator being $R_\delta \nu = K_\delta * \nu$. The paper then goes on to define the following useful properties a measure can have.

Definition 3. A positive Borel measure μ in $\mathbb{R}^{d+1}$ is called C-nice if $\mu(B(x,r)) \leq Cr^d$ for every $x \in \mathbb{R}^{d+1},\ r > 0$. It is called C-good if it is C-nice and $\|R_{\mu,\delta}\|_{L^2(\mu)\to L^2\mu)} \leq C$ for every $\delta > 0$.

With these concepts of nice and good measures, one can indeed discuss and develop the Riesz transform. That being said, the ultimate result requires the use of not only the upper bound of nice measures, but a lower bound as well, and they provide the following definition.

Definition 4. Let U be an open subset of $\mathbb{R}^{d+1}$. a nice measure μ is called Alfhors-David regular (or just AD regular) in U with lower regularity constant $c > 0$ if for every $x \in$supp $\mu \cap U$ and every $r > 0$ such that $B(x,r) \subset U$, we have $\mu(B(x,r)) \geq cr^d$.

With these notions in hand, the paper then proceeds to develop the Riesz transform for a sufficiently smooth measure on a fixed affine hyperplane L in $\mathbb{R}^{d+1}$ in the following sense: let m_L be the d-dimensional Lebesgue measure on L in $\mathbb{R}^{d+1}$, and let $\nu = fm_L$, where f is a C^2 compactly supported density with respect to m_L. They project the Riesz transform to the hyperplane, H, parallel to L, passing through the origin. They then show that with this smooth measure, $R^H\nu$ exists as a limit of the regularized operators R^H_δ, that it is Lipschitz in $\mathbb{R}^{d+1}$ and harmonic outside of the support of ν, and obtain some bounds for the L^∞ and Lipschitz norms. Seeing that a smooth measure produces a smooth measure, they tackle a partial converse, something they call a toy flattening lemma, and see that if $R^H(fm_L)$ is smooth in a ball on L, then f must be slightly less smooth on a smaller ball. While the authors consider this lemma to be rather elementary, through the use of some weak limiting techniques they later on are able to obtain a full flattening lemma which can be used on measures not supported just on a hyperplane. In the meantime, however, they move on to carefully developing the Riesz transform for arbitrary good measures, as well as showing that if a sequence of good measures, μ_k, converges weakly to some other measure, μ, then μ is also good, and for f and g Lipschitz, with f scalar and g vector valued, we have $\int\langle R_{\mu_k} f, g\rangle d\mu_k \to \int\langle R_\mu f, g\rangle d\mu$. Noting how this paper began by working on measures supported on a hyperplane, it should come as no surprise that a measure being 'flat' in some sense might be beneficial, and indeed, in the next section the authors define the following concepts of flatness.

Definition 5. We say that a measure μ is geometrically (H, A, α)-flat at the point z on the scale ℓ if every point of supp $\mu \cap B(z, A\ell)$ lies within distance $\alpha\ell$ from the affine hyperplane

L containing z and parallel to H, and every point of $L \cap B(z, A\ell)$ lies within distance $\alpha\ell$ from supp μ.
We say that a measure μ is (H, A, α)-flat at the point z on the scale ℓ if it is geometrically (H, A, α)-flat at the point z on the scale ℓ and, in addition, for every Lipschitz function f supported in $B(z, A\ell)$ such that $\|f\|_{Lip} \leq \ell^{-1}$ and $\int f dm_L = 0$, we have

$$\left| \int f d\mu \right| \leq \alpha \ell^d.$$

This notion of flatness seems to suggest that for suitably behaved functions, one might be able to replace a flat measure with a multiple of m_L and suffer only minor error, and indeed, the paper goes on to make this error explicit in two lemmata for the integrals of Lipschitz functions against particular types of nice flat measures. For the next piece of the puzzle, we need to fix parameters $0 < r < R$ and a continuous function $\psi : [0, \infty) \to [0, 1]$ such that $\psi(x) = 1$ for $x \leq 1$ and $\psi(x) = 0$ for $x \geq 2$, and define $psi_{z,\delta,\Delta}(x) = \psi\left(\frac{|x-z|}{R}\right) - \psi\left(\frac{|x-z|}{r}\right)$. With these in mind, the paper then shows that for $A, \alpha, \beta, \tilde{c}, \tilde{C} > 0$ fixed, there exists some $\rho = \rho(A, \alpha, \beta, \tilde{c}, \tilde{C}, d)$ such that if μ is $\tilde{C}$-good and AD regular on a ball $B(x, R)$ for some point $x \in$supp μ with lower regularity $\tilde{c}$, and

$$|[R(\psi_{z,\delta R,\Delta R}\mu)(z)]| \leq \beta$$

for all $\rho < \delta < \Delta < \frac{1}{2}$ and all $z \in B(x, (1-2\Delta))$, then for some $\ell > \rho R$, $z \in B(x, R-(A+\alpha)\ell)$, and hyperplane H, we have that μ is geometrically (H, A, α)-flat at the point z on the scale ℓ. With this tool in hand, the authors are ready to prove the vital Flattening Lemma (whose lengthly statement I will omit in the interest of brevity) which allows them to move from a lack of oscillation on $R^H\mu$ at some fixed point in the support of μ on scales comparable to ℓ, to flatness of μ at z on the scale ℓ. Recall that we are proving a theorem which, in turn, gives us uniform rectifiability, and one of the first things one needs to show uniform rectifiability is that the support is, essentially, geometrically flat, and so by proving this property, even though they don't prove the rest of the fact that their measure is uniformly rectifiable directly, this flattening lemma will allow them to take advantage of the much stronger analytic condition of flatness.

With these measure related preliminaries taken care of, the authors now fix an AD regular d-dimensional measure in $\mathbb{R}^{d+1}$ for the remainder of the paper and develop the David-Semmes lattice for it, and then define a what it means for a family of sets to be Carleson, as well as state a useful lemma about what it means if a family is not Carleson. They also denote by z_Q the 'center' of the cell Q with respect to the support of the measure μ, and they say that a cell is (H, A, α)-flat if μ is (H, A, α)-flat at z_Q at scale $\ell(Q)$. They then move on to define what it means for a family of functions to be a Riesz system and show how they can be used to show that a family of cells are Carleson. Recalling how often the measure theoretic preliminaries dealt with Lipschitz functions we begin to see some more strands of logic tie together as they show that the Lipschitz wavelet system associated to a cube Q, denoted $\Psi_Q(A)$, which is simply the family of all Lipschitz functions with mean zero supported on $B(z_Q, A\ell(Q))$ and with Lipschitz norm less than $C\ell(Q)^{-\frac{d}{2}-1}$, forms a Riesz system.

It is here that the meat of the proof can now begin, as the authors show that there is an integer N, a finite set of linear hyperplanes, and a Carleson family depending on parameters A and α, such that for every cell P in the lattice which is not in the Carleson family, there exists a cell Q at most N levels below it and a hyperplane in the aforementioned set such that Q is (H, A, α)-flat (using the flattening lemma). With this result in hand the original goal of showing that all non-BAUP cells is a Carleson family may be tweaked in the sense that it is now sufficient to show that we can find parameters $A, \alpha > 0$ such that for a fixed hyperplane, H, the family $\mathcal{F} = \mathcal{F}(A, \alpha, H, N)$ of all non-BAUP cells containing an (H, A, α)-flat cell at most N layers below it, is Carleson. With this in mind, we state the following important lemma.

Lemma 6. *If $\mathcal{F}$ is not Carleson, then for every positive integer K and every $\eta > 0$, there exists a cell $P \in \mathcal{F}$ and $K+1$ alternating pairs of finite layers $\mathfrak{B}_k,\ \mathfrak{Q}_k \subset \mathcal{D}$ $(k = 0, ..., K)$ such that*

- $\mathfrak{B}_0 = \{P\}$
- $\mathfrak{B}_k \subset \mathcal{F}_P = \{Q \in \mathcal{D} : Q \subset P\}$ *for all* $k = 0, ..., K$.
- *All layers $\mathfrak{Q}_k$ consist of (H, A, α)-flat cells only.*
- *Each individual layer consists of pairwise disjoint cells.*
- *If $Q \in \mathfrak{Q}_k$ then there exists some $P' \in \mathfrak{B}_k$ such that $Q \subset P'$ $(k = 0, ..., K)$.*
- *If $P' \in \mathfrak{B}_{k+1}$ then there exists some $Q \in \mathfrak{D}_k$ such that $P' \subset Q$ $(k = 0, ..., K-1)$.*
- $\sum_{Q \in \mathfrak{Q}_k} \mu(Q) \geq (1-\eta)\mu(P)$

The proof of this lemma in no way uses the actual definition of what it means to be non-BAUP, however it is this lemma that will allow us to finally prove our theorem. The remainder of the paper assumes that $\mathcal{F}$ is non-Carleson and non-BAUP and closely examines these layers to eventually arrive at a contradiction much further down the road. They first examine the flat layers, $\mathfrak{Q}_k$, and apply the what was developed in the beginning for R^H to obtain an almost orthogonality result for the Riesz transforms of the different layers which I will state after the following setup. Let $\phi \in C^\infty$ such that supp $\phi \subset B(0,1)$ and $\int \phi dm = 1$. For a cube Q, define $Q_\epsilon = \{x \in Q : \operatorname{dist}(x, \mathbb{R}^{d+1} \setminus Q) \geq \epsilon \ell(Q)\}$, and

$$\phi_Q = \chi_{Q_{2\epsilon}} * \frac{1}{(\epsilon \ell(Q))^d} \phi\left(\frac{\cdot}{\epsilon \ell(Q)}\right).$$

Define $\nu_Q = a_Q \phi_q m_{L(Q)}$ with a_Q such that $\nu_Q(\mathbb{R}^{d+1}) = \int \phi_Q d\mu$. Define

$$G_k = \sum_{Q \in \mathfrak{Q}_k} \phi_Q R^h[\phi_Q \mu - \nu_Q] \text{ for } k = 0, ..., K,$$

and $F_k = G_k - G_{k+1}$. Then we have that if $\epsilon < \frac{1}{48},\ A > 5$ and $\alpha < \epsilon^8$, then

$$|\langle F_k, G_k \rangle_\mu| \leq \sigma(\epsilon, \alpha)\mu(P), \quad \text{where } \lim_{\epsilon \to 0}\left(\lim_{\alpha \to 0} \sigma(\epsilon, \alpha)\right) = 0$$

recalling that α is one of the flatness parameters, and by previous work, we can take it to be as small as we want. This means, essentially, that there is no real lower bound to this inner product, and this lack of lower bound is where we will obtain our contradiction. Note the following,

$$\|G_0\|^2_{L^2(\mu)} = \left\|\sum_{k=0}^{K} F_k\right\|^2_{L^2(\mu)} = \sum \|F_k\|^2_{L^2(\mu)} + 2\sum_{k=0}^{K-1} \langle F_k, G_{k+1} \rangle_\mu.$$

In the proving of the almost orthogonality result, the authors show that $\|G_0\|^2_{L^2(\mu)} \leq C\mu(P)$, and so if we can find a uniform lower bound for $\|F_k\|^2$ independent of K, ϵ, and α, and take K sufficiently large we will see a clear lower bound for $|\langle F_k, G_{k+1}\rangle_\mu|$ arise, which, by taking ϵ and then α sufficiently small we can arrive at a contradiction. The remainder of the paper is devoted to showing that such a uniform lower bound exists. It does this by first making the reduction to densely packed cells. A cell, $Q \in \mathfrak{Q}_k$, is densely packed if

$$\sum_{Q' \in \mathfrak{Q}_{k+1},\ Q' \subset Q} \mu(Q') \geq (1-\epsilon)\mu(Q).$$

Dealing with these cells is a little easier, and it is sufficient to obtain a lower bound for an F^Q defined similarly to F_k, but in terms of Q and its children. With this cell and by using (finally) the non-BAUP layer $\mathfrak{B}_{k+1}$, they are able to construct a vector field and a slightly modified measure (similar in spirit to how we saw that good flat measures could be replaced with a multiple of m_L if done carefully) which finally allows them to obtain their uniform lower bound, and hence, a contradiction.

References

[1] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61, 1990, pp. 601-628

[2] G. David, Unrectifiable 1-sets have vanishing analytic capacity, Revista Mat. Iberoamericana, 14(2), 1998, pp. 369-479

[3] G. David Uniformly rectifiable sets, notes of a Summer school on Harmonic Analysis at Park City, to appear in IAS/Park City Mathematics series AMS, also available from http://www.math.u-psud.fr/ gdavid

[4] G. David, Morceaux de graphes lipschitziens et int egrales singulières sur une surface, Rev. Mat. Iberoamericana, Volume 4, 1, 1988, pp. 73-114

[5] G. David, S. Semmes, Analysis of and on uniformly rectifiable sets, Mathematical Surveys and Monographs, Volume 38, 1993, AMS, Providence, RI

[6] V. Eiderman, F. Nazarov, A. Volberg, The s-Riesz transform of an s-dimensional measure in $\mathbb{R}^2$ is unbounded for $1 < s < 2$, available from http://arxiv.org/abs/1109.2260

[7] B. Jaye, F. Nazarov, A. Volberg, The fractional Riesz transform and an exponential potential, available from http://arxiv.org/abs/1204.2135

[8] S. Hofmann, J. M. Martell, S. Mayboroda, Uniform rectifiability and harmonic measure III: Riesz transform bounds imply uniform rectifiability of boundaries of 1-sided NTA domains, http://arxiv.org/abs/1207.1527

[9] P. Mattila, M. Melnikov, J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. (2) 144, 1996, pp. 127-136

[10] J. Mateu, X. Tolsa, Riesz transforms and harmonic Lip_1 capacity in Cantor sets, Proc. London math. Soc., 89(3), 2004, pp. 676-696

[11] P. Mattila and J. Verdera, Convergence of singular integrals with general measures, J. European Math. Soc. 11, 2009, pp. 257-271

[12] P. Mattila and D. Priess, Rectifiable measures in $\mathbb{R}^n$ and existence of principle values for singular integrals, J. London Math. Soc. (2) 52, 1995 no. 3, pp. 482-496

[13] D. Priess, Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities, Ann. of Math. (2) 125, 1987, pp. 537-643

[14] X. Tolsa, Uniform rectifiability, Calderón-Zygmund operators with odd kernels, and quasiorthogonality, Proc. London Math. Soc. 98(2), 2009, pp. 393-426

[15] X. Tolsa, Principal values for Riesz transform and rectifiability, J. Func. Anal. vol. 254(7), 2008, pp. 1811-1863

405 Snow Hall, 1460 Jayhawk Blvd, Lawrence, Kansas 66045-7594

E-mail address: `chaffel@ku.edu`

MENGER CURVATURE AND RECTIFIABILITY

J.C.LÉGER

presented by Amalia Culiuc

ABSTRACT. A well-known conjecture by Vitushkin states that a 1-set in $\mathbb{C}$ is removable from the bounded analytic functions if and only if it is purely unrectifiable. Using the Menger curvature and a construction similar to Jones' Traveling Salesman Theorem, G. David proved this conjecture in 1998. The final argument of his proof is, however, only presented in detail in an unpublished manuscript. In the present paper, Léger provides an alternative construction that proves David's final argument and can extend naturally to higher dimensions.

1. Introduction and Main Results Presented

We begin by introducting the relevant terminology. Let x, y, z be three points in $\mathbb{R}^n$. The Menger curvature of the triple (x, y, z), denoted by $c(x, y, z)$, is the inverse of the radius of the circumcircle of the triangle (x, y, z) if such a triangle exists and 0 otherwise. Given this definition, if E is a Borel set in $\mathbb{R}^n$, its total Menger curvature is the nonnegative number $c(E)$ satisfying

$$c^2(E) = \int\int\int_{E^3} c^2(x, y, z) d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z),$$

where $\mathcal{H}^1$ represents the 1-dimensional Hausdorff measure on $\mathbb{R}^n$.
A Borel set $E \subset \mathbb{R}^n$ is said to be rectifiable if there exists a countable collection of Lipschitz functions $\gamma_i : \mathbb{R} \to \mathbb{R}^n$ such that

$$\mathcal{H}^1 \left(E \setminus \cup \gamma_i(\mathbb{R})\right) = 0,$$

and it is said to be purely unrectifiable if for any Lipschitz function $\gamma : \mathbb{R} \to \mathbb{R}^n$,

$$\mathcal{H}^1 \left(E \cap \gamma(\mathbb{R})\right) = 0$$

By these definitions, any 1-set can be partitioned into a rectifiable subset and a purely unrectifiable subset.
The main result of this paper relates the concepts of Menger curvature and rectifiability as follows:

Theorem 1. *If $E \subset \mathbb{R}^n$ is a Borel set of finite, nonzero 1-dimensional Hausdorff measure and $c^2(E) < \infty$, then E is rectifiable.*

The interest in rectifiability comes from its relation to the problem of removable sets from the bounded analytic functions. Recall that a compact set $E \subset \mathbb{C}$ is said to be removable from the bounded analytic functions if the only bounded analytic functions on $\mathbb{C} - E$ are the constants. In the 1880s, Painlevé considered the question of finding necessary and sufficient conditions for a set in the complex plane to be removable. He was able to prove that a sufficient condition was that the set E have 1-dimensional Hausdorff measure 0. In the 1960s, Vitushkin [Vi67] conjectured that a compact 1-set E is removable from the bounded

analytic functions if and only if it is purely unrectifiable. This conjecture was proved in 1996 by Mattila et.al.[MMV96], but the proof required the additional assuption of Ahlfors regularity. In 1998, G.David [Da98] removed this assumption using the theorem above.
In what follows, we present a brief summary of the arguments used in the proof of Theorem 1. We conclude this section by stating Léger's higher dimensional version of this theorem.

Theorem 2. *Let $E \subset \mathbb{R}^n$ be a Borel set, d a positive integer, and*

$$c^{d+1}(E) = \int_{x\in E}\int_{y_)\in E} \dots \int_{y_d\in E} \left(\frac{d(x, < y_0, \dots y_d >)}{d(x, y_0)\dots d(x, y_d)}\right)^{d+1} d\mathcal{H}^d(y_0)d\mathcal{H}^d(y_d)\dots d\mathcal{H}^d(x),$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^n$ and $d(x, < y_0, \dots y_d >)$ represents the distance between x and the plane going through the points $y_0, \dots y_d$. If $\mathcal{H}^d(E) < \infty$ and $c^{d+1}(E) < \infty$, then E is contained in a countable collection of images of Lipschitz function up to a set of $\mathcal{H}^d$ measure 0.

2. Proof of the main result

The proof of Theorem 1 relies on two propositions. Proposition 1 states that under the assumptions of Theorem 1, there exists a subset F of E with some desirable geometric properties. Proposition 2 shows that for a set with these given properties, if $c^2(F)$ is small, a very large portion of F is included in the graph of a Lipschitz function.

Proposition 1. *If $E \subset \mathbb{R}^n$ is a 1-set and $c^2(E) < \infty$, then for all $\eta > 0$, there exists a subset F of E such that:*

1. *F is compact*
2. *$c^2(F) \leq \eta \cdot diam F$*
3. *$\mathcal{H}^1(F) > \frac{diam_F}{40}$*
4. *For all $x \in F$ and all $t > 0$, $\mathcal{H}^1(F \cap B(x, t)) \leq 3t$, where $B(x, t)$ represents the ball of center x and radius t.*

The proof of this statement is based on the fact that

$$\frac{1}{2} \leq \limsup_{t\to 0} \frac{E \cap B(x, t)}{2t} \leq 1$$

and requires a construction relying on Vitali's covering theorem.
Before stating Proposition 2, we must define the total Menger curvature of a Borel measure μ on $\mathbb{R}^n$. We say that the nonnegative number $c(\mu)$ is the total Menger curvature of μ if

$$c^2(\mu) = \int\int\int c^2(x, y, z)d\mu(x)d\mu(y)d\mu(z).$$

Given this definition, we state the following:

Proposition 2. *If $C_0 \geq 10$, there exists a number $\eta > 0$ such that if μ is a compactly supported Borel measure on $\mathbb{R}^n$ and the following hold:*

1. *$\mu(B(0, 2)) \geq 1$ and $\mu(\mathbb{R}^n \backslash B(0, 2)) = 0$,*
2. *For any ball B, $\mu(B) \leq C_0 \cdot diam B$,*
3. *$c^2(\mu) \leq \eta$,*

then there exists a Lipschitz graph Γ *such that*

$$\mu(\Gamma) \geq \frac{99}{100}\mu(\mathbb{R}^n)$$

Assume for now that Proposition 2. We can use it together with Proposition 1 to prove Theorem 1.

Proof. Let E be a Borel set that satisfies the assumptions of Proposition 1 and let E_{irr} be the unrectifiable part of E. Suppose $\mathcal{H}^1(E_{irr}) > 0$ (so E is not rectifiable). Then E_{irr} satisfies the hypoteses of Proposition 1 and we can apply it to find $F \subset E_{irr}$. It is not difficult to check that the measure $40 \times \mathcal{H}^1$ restricted to a rescaled copy of F satisfies the conditions of Proposition 2. Applying Proposition 2, we find that there exists a Lipschitz graph Γ such that $40 \times \mathcal{H}^1(\Gamma) > 0$. Thus $\mathcal{H}^1(E_{irr} \cap \Gamma) = \mathcal{H}^1(\Gamma) > 0$. But E_{irr} was assumed to be unrectifiable, so we have reached a contradiction. Therefore, Theorem 1 is proved. $\square$

We return now to the proof of Proposition 2. In what follows, μ will always be a measure satisfying the assumptions of Proposition 2 and F will denote its support. Our goal is to find a Lipschitz function $A : \mathbb{R} \to \mathbb{R}^{n-1}$ whose graph is the desired Γ. Before doing so, we define some functions that describe the geometry of F.
To measure the degeneracy of μ, for a ball B with center $x \in \mathbb{R}^n$ and radius t, let the density of B be,

$$\delta(B) = \delta(x,t) = \frac{\mu(B(x,t))}{t}$$

and set

$$\tilde{\delta}(B) = \tilde{\delta}(x,t) = \sup_{y \in B(x,k_0 t)} \delta(y,t).$$

Note that for any ball B, $\delta(B) < 2C_0$.
If $k > 0$ is fixed, for any $x \in \mathbb{R}^n$, $t > 0$, and any line D in $\mathbb{R}^n$, we also define the P. Jones β functions:

$$\beta_1^D(x,t) = \frac{1}{t}\int_{B(x,kt)} \frac{d(y,D)}{t} d\mu(y)$$

$$\beta_2^D(x,t) = \left(\frac{1}{t}\int_{B(x,kt)} \left(\frac{d(y,D)}{t}\right)^2 d\mu(y)\right)^{\frac{1}{2}}$$

$$\beta_1(x,t) = \inf_D \beta_1^D(x,t)$$

$$\beta_2(x,t) = \inf_D \beta_2^D(x,t)$$

The functions β_1^D and β_2^D provide a measurement of the average distance between F and the line D inside the ball $B(x,kt)$. Since this interpretation is not valid if $\delta(x,y)$ is very low, we also introduce a uniform lower bound $\delta > 0$, the density threshold, and investigate the behavior only in balls with $\delta(B) > \delta$.
The construction of the Lipschitz graph is performed using a stopping time argument. Notice, however, that the familiar setting of "dyadic cubes" cannot be employed here, since those may not exist in the support of μ. This issue is resolved by considering overlapping balls and applying the Besicovitch covering lemma. Fix δ, $k > 10$, $k_0 > 10$, a β_1 threshold $\varepsilon > 0$, a small angle $\alpha > 0$ and a number $\eta > 0$, which will be chosen appropriately after the

construction. Also fix a point $x_0 \in F$ and a line D_0 such that $\beta_1^{D_0}(x_0, 1) \leq \varepsilon$. This will be the domain of the function A. Now define

$$S_{total} = \left\{ (x,t) \in F \times (0,5), \begin{cases} \delta(x,t) \geq \frac{1}{2}\delta \\ \beta_1(x,t) < 2\varepsilon \\ \exists D_{x,t} \text{ s.th } \beta_1^{D_{x,t}}(x,t) \leq 2\varepsilon \text{ and angle}(D_{x,t}, D_0) \leq \alpha \end{cases} \right\}$$

We make the observation that S_{total} is not a coherent region, that is, if a ball B is in S_{total}, there is no gurantee that any of its dilations will be in S_{total}. To resolve this issue, define, for any $x \in F$,

$$h(x) = \sup \left\{ t > 0, \exists y \in F, \exists \tau, \frac{t}{3} \geq \tau \geq \frac{t}{4}, x \in B\left(y, \frac{\tau}{3}\right), \text{ and } (y,\tau) \notin S_{total} \right\}$$

and set

$$S = \{(x,t) \in S_{total}, t \geq h(x)\}.$$

It can be checked that the set S has the coherence property. We now divide F into four pieces. We aim to show that one piece is "good" for our purposes, while the other three, which are "bad", carry only a small portion of the measure μ. Consider the following partition:

$$Z = \{x \in F, h(x) = 0\}$$

$$F_1 = \left\{ x \in F \backslash Z, \begin{cases} \exists y \in F, \exists \tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right], x \in B\left(y, \frac{\tau}{2}\right) \\ \text{and} \\ \delta(y,t) \leq \delta \end{cases} \right\},$$

$$F_2 = \left\{ x \in F \backslash (Z \cup F_1), \begin{cases} \exists y \in F, \exists \tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right], x \in B\left(y, \frac{\tau}{2}\right) \\ \text{and} \\ \beta_1(y,t) \geq \varepsilon \end{cases} \right\},$$

$$F_3 = \left\{ x \in F \backslash (Z \cup F_1 \cup F_2), \begin{cases} \exists y \in F, \exists \tau \in \left[\frac{h(x)}{5}, \frac{h(x)}{2}\right], x \in B\left(y, \frac{\tau}{2}\right) \\ \text{and} \\ \text{angle}(D_{y,\tau}, D_0) \geq \frac{3}{4}\alpha \end{cases} \right\}.$$

One can show that the set F is the disjoint union of the four subsets above.
We will define a Lipschitz function $A : D_0 \to D_0^\perp$ such that Z is in the graph of A and show that $\mu(Z) \geq \frac{99}{100}\mu(F)$. For the latter it suffices to prove that $\mu(F_i) \leq 10^{-6}$ for all F_i.
The bound on $\mu(F_2)$ is the easiest to handle due to the control on β_1 implied by the proposition below:

Proposition 3. *There exists a constant C depending on δ, C_0, k, and k_0 such that*

$$\int\int_0^\infty \beta_1(x,t)^2 \mathbb{1}_{(\tilde{\delta}(x,t) \geq \delta)} \frac{d\mu(x,t)dt}{t} \leq Cc^2(\mu)$$

The bounds on the other two sets are less straightforward and they require some control on the size of sets where $h > 0$. Their proof are the consequence of a series of geometric arguments and technical lemmas. Before considering any of those arguments, we begin the construction of the function A. Consider the following functions:

$$d(x) = \inf_{(X,t)\in S}(d(X,x)+t), \text{ for all } x \in \mathbb{R}^n$$
$$D(p) = \inf_{x\in\pi^{-1}(p)} d(x) = \inf_{(X,t)\in S}(d(\pi(X),p)+t), \text{ for } p \in D_0,$$

where π is the orthogonal projection map onto D_0. The function D above associates to each point $p \in D_0$ a "good" point in F. Notice that d and D are Lipschitz, $h(x) \geq d(x)$, and $Z = \{x \in F, d(x) = 0\}$, since F is closed. To construct A, we attempt to invert the projection $\pi : F \to D_0$. This is the outcome of the following lemma:

Lemma 3. *There exists a constant C_2 such that if $x, y \in F$ and $t \geq 0$ satisfy dist$(\pi(x), \pi(y)) \leq t$, $d(x) \leq t$, and $d(y) \leq t$, then $d(x,y) \leq C_2 t$.*

A consequence of this result obtained by setting $t = 0$ is that the restriction of π to Z is injective. Therefore, we can define A on $\pi(Z)$ by $A(\pi(z)) = \pi^{\perp}(x), \forall x \in Z$, and it is possible to prove that this restriction is Lipschitz. What remains then is to extend A to the entire domain D_0.

Fix a family of dyadic intervals on D_0 and for any $p \in D_0$ that is not a boundary point of some dyadic interval, if $D(p) > 0$, let R_p be the largest dyadic interval such that $p \in R_p$ and $\mathrm{diam} R_p \leq \frac{1}{2} \inf_{u\in R_p} d(u)$. Now consider a relabeling of the collection of intervals R_p, $\{R_i, i \in I\}$. These intervals have disjoint interiors, and we can show that the family $\{2R_i\}_{i\in I}$ covers $D_0 \backslash \pi(Z)$. Set $U_0 = D_0 \cap B(0,10)$ and $I_0 = \{i \in I, R_i \cap U_0 = \emptyset\}$.

The construction of the extension amounts to defining a partition of unity. For each i, define a function $\tilde{\phi}_i \in C^{\infty}(D_0)$ such that $0 < \tilde{\phi}_i < 1$, $\tilde{\phi}_i = 1$ on $2R_i$, $\tilde{\phi}_i = 0$ outside $3R_i$, $|\partial\tilde{\phi}_i| \leq \frac{C}{\mathrm{diam} R_i}$, and $|\partial^2\tilde{\phi}_i| \leq \frac{C}{(\mathrm{diam} R_i)^2}$. We can then define a partition of unity for $V = \bigcup_{i\in I_0} 2R_i$ by

$$\phi_i(p)\frac{\tilde{\phi}_i(p)}{\sum_j \tilde{\phi}_j(p)}.$$

This partition has the property that $|\partial\phi_i| \leq \frac{C}{\mathrm{diam} R_i}$, and $|\partial^2\phi_i| \leq \frac{C}{(\mathrm{diam} R_i)^2}$.

Now for $p \in V$, let

$$A(p) = \sum \phi_i(p)A_i(p),$$

where A_i is the affine function $D_0 \to D_0^{\perp}$ whose graph is $D_i = D_{B_i}$. Note that V is disjoint from $\pi(Z)$ and contains $U_0 \backslash \pi(Z)$, so A is now defined for all of U_0. A few facts remain to be checked. First of all, one must verify that this definition produces, indeed, a Lipschitz extension. It is also necessary to show that most of F lies near the graph of A, which involves bounding the subsets F_1 and F_3 defined previously. Settling these issues completes the proof of Proposition 2, and thus the proof of the main theorem.

References

[Da98] G. David, Unrectifiable sets have vanishing analytic capacity, Revista Mat. Iberoamericana, **14** (1998), 369–479.

[Vi67] A. G. Vitushkin, The analytic capacity of sets in problems of approximation theory, Uspekhi Mat. Nauk **22** (1967), 141–199; English translation in Russian Math. Surveys **22** (1967), 139-200.

[MMV96] M. S. Melnikov, P. Mattila, and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiablity, Ann. of Math. **144** (1996), 127–136.

Brown University Mathematics Department
E-mail address: amalia@math.brown.edu

CHARACTERIZATION OF SUBSETS OF RECTIFIABLE CURVES IN $\mathbb{R}^n$

KATE OKIKIOLU

presented by Ishwari Kunwar

Abstract. This paper characterizes the subsets of rectifiable curves in $\mathbb{R}^n$.

1. Introduction and Main Results Presented

The main result of this paper is the following theorem which was proved for the special case of $n = 2$ in [3] by Peter W. Jones.

Theorem 1. *If Γ is a connected set in $\mathbb{R}^n$, then there exists a constant $C = C(n)$ such that*

$$\sum_{Q\in\mathcal{D}} \frac{r_{3Q}^2}{l_Q} \leq C\, l(\Gamma).$$

Here, $\mathcal{D}$ denotes the set of all dyadic cubes $Q = \prod_{j=1}^{n}[m_j 2^{-k}, (m_j+1)2^{-k}]$ $(k, m_j \in \mathbb{Z})$, l_Q denotes the sidelength of Q, $r_Q = r_Q(\Gamma)$ is the cylinder radius of Γ in Q and $l(\Gamma)$ is the one-dimensional (outer) Hausdorff measure of Γ. 3Q denotes the cube concentric to Q with sidelength $l_{3Q} = 3l_Q$.

Let Q^0 be a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. By choosing a new origin and coordinate axes in which $Q^0 = [0,1]^n$, we can define the *dyadic decomposition* of Q^0 to be the set of all cubes contained in Q^0 that are dyadic with respect to the new coordinates. Let $\langle Q^0\rangle$ denote this set, and let $\langle Q^0\rangle_k := \langle Q^0\rangle \cap \mathcal{D}_k$, where $\mathcal{D}_k$ is the k-th generation of dyadic cubes in $\mathbb{R}^n$.

The following lemma shows, for given $\lambda > 1$, how to associate to a cube Q^0 a finite number of larger cubes containing Q^0 such that if $Q \in \langle Q^0\rangle_k$, then λQ is contained in some cube $Q^* = Q^*(Q, Q^0, \lambda)$ in the k-th generation of one of these larger cubes. The lemma also shows that the number of cubes Q in $\langle Q^0\rangle$ giving rise to the same Q^* under this association is bounded.

Lemma 2. (*a*) *For given $\lambda > 0$ and $F \subseteq \mathbb{R}^n$,*

$$\#\{Q \in \mathcal{D}_k : F \cap \lambda Q \neq \varnothing\} \leq \left(\frac{diameter(F)}{2^{-k}} + \lambda + 1\right)^n \quad (k = 0, 1, 2, ...)$$

(*b*) *Let $\lambda > 1$ and Q^0 be a cube in $\mathbb{R}^n$. Then for $k = 0, 1, 2, ...$ and each cube $Q \in \langle Q^0\rangle_k$ there exists a cube $Q^* = Q^*(Q, Q^0, \lambda)$ such that*

$$\lambda Q \subseteq Q^* \in \bigcup_{e\in V} \langle Q^0(\lambda, e)\rangle_k\,,$$

where V is the set of 2^n vertices of the cube $[0,1]^n$ and

$$Q^0(\lambda, e) = 4\lambda Q^0 + \frac{4\lambda l_{Q^0}}{3} e.$$

If

$$\hat{Q} \in \bigcup_{e \in V} \left\langle Q^0(\lambda, e) \right\rangle_k,$$

then

$$\#\{Q \in \left\langle Q^0 \right\rangle : Q^* = \hat{Q}\} \leq (4N)^n.$$

This lemma reduces the theorem to proving the following statement:

If Γ is a connected set in $\mathbb{R}^n$ and Q^0 is a cube in $\mathbb{R}^n$, then there exists a constant $C = C(n)$ such that

$$\sum_{Q \in \langle Q^0 \rangle} \frac{r_Q^2}{l_Q} \leq C\, l(\Gamma),$$

.

For closed Γ, there exists an arclength preserving surjective map $\gamma : T \to \Gamma$, from a circle T of length $2l(\Gamma)$ such that γ hits almost every point of Γ twice. For a cube Q in $\mathbb{R}^n$, let T^α be the connected components of $\gamma^{-1}(Q)$, where $\alpha \in \Lambda_Q$, the indexing set, and let $\Gamma^\alpha = \gamma(T^\alpha)$. Let $L^\alpha = \varnothing$ if $T^\alpha = T$ and $L^\alpha = [\gamma(x), \gamma(y)]$ if $T^\alpha \neq T$, where x and y are the endpoints of the arc T^α, and define

$$s_Q = \sup_{\alpha \in \Lambda_Q} s_\alpha,$$

where $s_\alpha = r_Q$ if $T^\alpha = T$, and $s_\alpha = \sup_{z \in \Gamma^\alpha} dist(z, L^\alpha)$ if $T^\alpha \neq T$.
Let $\mathscr{A} = \{Q \in \langle Q^0 \rangle\} : s_{Q^*} < \delta r_Q\}$, and $\mathscr{B} = \{Q \in \langle Q^0 \rangle\} : s_{Q^*} \geq \delta r_Q\}$, where $\delta = \delta(n) > 0$.

Lemma 2 together with the following lemma enables us to bound

$$\sum_{Q \in \mathscr{B}} \frac{r_Q^2}{l_Q}.$$

Lemma 3. *If Γ is a connected set in $\mathbb{R}^n$ with $l(\Gamma) < \infty$ and Q^0 is a cube in $\mathbb{R}^n$, then there exists a constant $C = C(n)$ such that*

$$\sum_{Q \in \langle Q^0 \rangle} \frac{s_Q^2}{l_Q} \leq C\, l(\Gamma \cap Q^0).$$

Lemma 2 together with the following lemma enables us to bound

$$\sum_{Q \in \mathscr{A}} \frac{r_Q^2}{l_Q}.$$

Lemma 4. *If Γ is a connected set in $\mathbb{R}^n$ and Q^0 is a cube in $\mathbb{R}^n$, then*

$$\sum_{Q\in\mathscr{A}} r_Q \leq C\, l(\Gamma \cap 2Q^0),$$

where $\mathscr{A} = \{Q \in \langle Q^0\rangle : s_{Q^*} < \delta r_Q\}$, $Q^* = Q^*(Q, Q^0, \lambda)$, $\lambda = \lambda(n)$, $\delta = \delta(n)$, and $C = C(n)$.

Finally, to prove

$$\sum_{Q\in\langle Q^0\rangle} \frac{r_Q^2}{l_Q} \leq C\, l(\Gamma),$$

we write $\sum_{Q\in\langle Q^0\rangle} \frac{r_Q^2}{l_Q} = \sum_{Q\in\mathscr{A}} \frac{r_Q^2}{l_Q} + \sum_{Q\in\mathscr{B}} \frac{r_Q^2}{l_Q}$.

References

[1] K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbb{R}^n$, J. London Math. Soc., (2)46(1992) pp. 336-348.

[2] Christopher J. Bishop and Peter W. Jones, Harmonic measure and arclength, Ann. of Math. 142 (1990) 511-547.

[3] Peter W. Jones, Rectifiable sets and the traveling salesman problem, Invent. Math. 102(1990)1-15

Ishwari Kunwar, School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia, USA
E-mail address: ikunwar3@math.gatech.edu

THREE REVOLUTIONS IN THE KERNEL ARE WORSE THAN ONE

BENJAMIN JAYE AND FEDOR NAZAROV

presented by Robert Rahm

ABSTRACT. In this paper, the authors provide a kernel, $K(z) = \frac{\overline{z}}{z^2}$ and a measure μ such that the operator given by

$$(T_\mu f)(z) = \int_{\mathbb{C}} f(\xi)K(z-\xi)d\mu(\xi)$$

is bounded and $\operatorname{supp}(\mu)$ is purely unrectifiable. This is in contrast to the situation when $K(z) = \frac{1}{z}$.

1. Introduction and Main Results Presented

For a kernel $K : \mathbb{C} - \{0\} \to \mathbb{C}$ and a finite measure μ, we can define the (singular) integral operator:

$$(T_\mu f)(z) = \int_{\mathbb{C}} K(z-\xi)f(\xi)d\mu(\xi).$$

We want to determine geometric properties of μ (that is, geometric properties of the support of μ) from properties of the operator T_μ. For example, if $K(z) = 1/z$ is the Cauchy kernel, it was proven by Lèger in [1] that if μ is a one dimensional measure, then μ is rectifable if $\|T_\mu(1)\|_{L^\infty-\operatorname{supp}(\mu)} < \infty$ (this is eqivilent to the boundedness of T_μ as an operator on $L^2(\mu)$, see [3].) The main result of the paper shows that not all kernels can be used to give geometric information about finite measures. In particular, for the remainder of the abstract, set:

$$K(z) = \frac{\overline{z}}{z^2}.$$

The main result of the paper is:

Theorem 1. *There exists a 1-dimensional purely unrectifiable probability measure μ with the property that $\|T_\mu(1)\|_{L^\infty-\operatorname{supp}(\mu)} < \infty$.*

They also show that T_μ fails to exists in the sense of p.v. $\mu -$ a.e.

2. Notation and Terminology

We will define some terms that we will use (and some that we have already used). A measure μ is rectifiable if its support is rectifiable and it is unrectifiable if its support is unrectifiable. Let m_1 and m_2 denote 1 and 2 dimensional Lebesgue measure with m_2 normalized so that $m_2(B(0,1)) = 1$. A collection of squares is said to be essentially pairwise disjoint if their interiors are pairwise disjoint. The letters "c" and "C" will denote small (< 1) and large (≥ 1) constants. If B is a ball with radius r, λB is the ball concentric with B with radius λr. Finally, $A(z,r)$ is the annulus $B(z,r) - B(z,r/2)$.

Date: August 3, 2014.

3. Properties of K

First, note that K is the same size as the Cauchy kernel. If we write $z = re^{i\theta}$ then:

$$K(z) = \frac{1}{re^{i\theta}e^{i\theta}e^{i\theta}}$$

whereas for the Cauchy kernel:

$$\frac{1}{z} = \frac{1}{re^{i\theta}}.$$

So, K has three "revolutions" in the kernel, while the Cauchy kernel has only one. This means that K has a certain mean value zero property that the Cauchy kernel does not have. Obviously, this statement needs some clarification, since the Cauchy kernel does have mean value zero. In particular, the following is proven:

Theorem 2. *If* $|\omega| < 1$, *then:*

$$\int_{B(0,1)} K(\omega - \xi) dm_2(\xi) = 0.$$

Clearly, once this is shown, a similar result holds for $z \in \mathbb{C}$, $r > 0$ and $\omega \in B(z, r)$. There won't be too many proofs included in this extended abstract, but we include this proof to show that three rotations are needed. That is, Theorem 2 isn't true if the kernel has only one or two rotations in the kernel.

proof of Theorem 2. We want to show the following:

$$\int_0^1 \int_0^{2\pi} K(\omega - re^{i\theta}) r d\theta dr = \int_0^{|\omega|} \int_0^{2\pi} K(\omega - re^{i\theta}) r d\theta dr + \int_{|\omega|}^1 \int_0^{2\pi} K(\omega - re^{i\theta}) r d\theta dr = 0.$$

For the second integral, $|\omega| < |\xi|$ and so there holds:

$$K(\omega - re^{i\theta}) = \frac{\overline{\omega - re^{i\theta}}}{r^2 e^{2i\theta}} \sum_{l=0}^{\infty} (l+1) \left(\frac{\omega}{\xi}\right)^l. \tag{1}$$

For fixed, r, the integral of this in the variable θ is zero since the integral of $\overline{e^{ki\theta}}$ is equal to 0 for all $k > 0$. For the first integral, $|\xi| < |\omega|$ and there holds:

$$K(\omega - re^{i\theta}) = \frac{\overline{\omega - re^{i\theta}}}{\omega^2} \sum_{l=0}^{\infty} (l+1) \left(\frac{re^{i\theta}}{\omega}\right)^l. \tag{2}$$

Therefore, there holds:

$$\begin{aligned}\int_0^{|\omega|} \int_0^{2\pi} K(\omega - re^{i\theta}) r d\theta dr &= \int_0^{|\omega|} 2\pi r \left(r\frac{\overline{\omega}}{\omega^2} - 2\frac{r^2}{\omega^3} \right) dr \\ &= \int_0^{|\omega|} \frac{2\pi}{\omega^3} \left(r |\omega|^2 - 2r^3 \right) dr = 0.\end{aligned}$$

Now, if we consider the Cauchy kernel, then the expression analogous to (1) is:

$$\frac{1}{re^{i\theta}} \sum_{l=0}^{\infty} \left(\frac{\omega}{re^{i\theta}}\right)^l,$$

and integrating this in the varaible θ gives 0. However, the expression that is analogous to (2) is

$$\frac{1}{\omega}\sum_{l=0}^{\infty}\left(\frac{re^{i\theta}}{\omega}\right),$$

and integrating this over θ is equal to $\frac{1}{\omega^2}$ and integrating over $\frac{r}{\omega^2}$, in the variable r from 0 to $|\omega|$ is not zero. Perhaps we might try working with $\frac{\overline{\omega-\xi}}{\omega-\xi}$. The integral of this is not 0, but even if it was, it wouldn't be desirable to use this kernel since it does not have the same size as the Cauchy kernel. □

4. The Measure

The measure μ is obtained as a weak limit of measures that are supported on increasingly sparse sets. (This will be made more precise.) In order to construct these sets, the authors begin with a square packing lemma:

Lemma 3. *Fix $r, R \in (0,\infty)$ with $r < \frac{R}{16}$ and $\frac{R}{r} \in \mathbb{N}$. Then one can pack $\frac{R}{r}$ pairwise essentially disjoint squares of side length $\sqrt{\pi r R}$ into a disc of radius $R(1+4\sqrt{\frac{r}{R}})$.*

The proof is superimpose the square lattice with mesh size $\sqrt{\pi r R}$ over the plane. There are M squares that intersect $B(0,R)$ and $M > R/r$. Now just throw out $M - R/r$ squares.
The next step is to construct a sparse cantor set using the square packing lemma, Lemma 3. Pack 100 squares into the ball $B(0,(1.4))$ to get a collection $Q_1^1, \cdots Q_{100}^1$ of 100 essentially pairwise disjoint squares, each with center $z_1^1, \cdots, z_{100}^1$. Now, consider each $B(z_j^1, 1/100)$ and $B(z_j^1, 1.4/100)$. For each $B(z_j^1, 1/100)$, repeat the above construction, packing 100 squares into it. So, on the n^{th} step, we get 100^n new balls centered at points $z_1^n, \cdots, z_{100^n}^n$. Again, consider the balls $B(z_j^n, 1/100^n)$ and the balls $B(z_j^n, 1.4/100^n)$. Let E_n be the union of the balls $B(z_j^n, 1.4/100^n)$.
So, the idea is that at each step, we take a large collection of balls, and within each ball, we put 100 cubes. Deep within each cube, we put a ball with radius $\frac{1}{100}^{th}$ the radius of its parent ball. Doing this indefinitely, we obtain a collection of cubes, call them $Q_j^{(n)}$, and a collection of balls that are "rapidly nesting" call them $\widetilde{B}_j^{(n)}$ where $n \in \mathbb{N}$ and $j = 1, \cdots, 100^n$. Also, consider the collection $B_j^{(n)}$ where $B_j^{(n)} = 1.4\widetilde{B}_j^{(n)}$. Let $E^{(n)} = \cup_j B_j^{(n)}$. The following properties hold:

(a) $\cup_l Q_l^{(n+1)} \subset E^{(n)}$;

(b) $B_j^{(n)} \subset Q_j^{(n)}$; Moreover, $\mathrm{dist}(B_j^{(n)}, \partial Q_j^{(n)}) \geq \frac{1}{2}\sqrt{\frac{1}{100}^{n(n-1)}}$;

(c) $\mathrm{dist}(B_j^{(n)}, B_k^{(n)}) \geq \frac{1}{2}\sqrt{\frac{1}{100}^{n(n-1)}}$ whenever $j \neq k$, $n \geq 0$.

Properties (a) and (b) tell us that $E^{(n+1)} \subset E^{(n)}$. Let $E = \cap_n E^{(n)}$. As usual, we can easily obtain an upper bound for $\mathcal{H}^1(E)$. Indeed, for $m \geq n \geq 0$, $E \cap B_j^{(n)}$ is covered by the 100^{m-n} disks, $B_k^{(m)}$ that are contained in $B_j^{(n)}$. Each of these disks has radius $1.4(100)^{-m} \leq 2(100)^m$. Thus, there holds:

$$\mathcal{H}^1(E \cap B_j^{(n)}) \leq 2100^{m-n}(100)^m = \frac{2}{100^n}.$$

Taking $n = 0$, there holds:

$$\mathcal{H}^1(E) = \mathcal{H}^1(E \cap 1.4B(0,1)) \leq 2.$$

This also implies that E has Hausdorff dimension at most 1.
We are now ready to define a sequence of measures, μ_n, and μ will be the weak limit of these measures. First, let $\mu_j^{(n)} = 100^n \chi_{B_j^{(n)}} m_2$ and set $\mu^{(n)} = \sum_{j=1}^{100^n} \mu_j^{(n)}$. Note that $\text{supp}(\mu^{(n)}) \subset E^{(n)}$ and $\mu^{(n)}(\mathbb{C}) = \sum_{j=1}^{100^n} \frac{1}{100^n} m_2(B_j^{(n)}) = \sum_{j=1}^{100^n} 100^n 1.4 \left(\frac{1}{100^n}\right)^2 = \sum_{j=1}^{100^n} 1.4 \frac{1}{100^n} = 1.4$. This is true for every n, so there is a measure μ that is the weak limit of a subsequence of the measures μ_n. Note that $\mu(\mathbb{C}) = 1.4$ and $\text{supp}(\mu) \subset E$. The following three properties hold:

(i) $\text{supp}(\mu^{(m)}) \subset \cup_j B_j^{(n)}$ whenever $m \geq n$;
(ii) $\mu^{(m)} = r_n$ for $m \geq n$; and
(iii) there exists C_0 such that $\mu^{(n)}(B(z,r)) \leq C_0 r$ for any $z \in \mathbb{C}$, $r > 0$ and $n \geq 0$.

Property (iii) and the fact that μ is a weak limit of a subsequence of the measure $\mu^{(n)}$ implies that $\mu(B(z,r)) \leq C_0 r = C_0 \mathcal{H}^1(B(z,r))$. In particular, there holds:

$$\mathcal{H}^1(E) \geq \frac{1}{c_0} \mu(E) > 0.$$

This implies that E has Hausdorff dimension at least 1. Thus, E must have Hausdorff dimension 1.

5. E is purely unrectifiable

The point of the paper is to provide a measure μ such that T_μ is bounded and $\text{supp}(\mu)$ is purely unrectifiable. We have constructed the measure, we now show that $\text{supp}(\mu)$ is purely unrectifiable. Since $\text{supp}(\mu) \subset E$, it suffices to show that E is purely unrectifiable. Recall that the lower $\mathcal{H}^d$ density of a A at a point a set is given by:

$$\Theta_*^d(A,a) = \liminf_{r \searrow 0} (2r)^{-d} \mathcal{H}^d(A \cup B(a,r)).$$

In [2], it is shown that a set E is purely $\mathcal{H}^d$–unrectifiable if and only if $\Theta_*^d(E,a) < 1$ for $\mathcal{H}^d$–a.e. $a \in E$. Since $\text{dist}(B_j^{(n)}, B_k^{(n)}) \geq \frac{1}{2}\sqrt{100^{-n(n-1)}}$, then $B(z, \frac{1}{4}\sqrt{100^{-n(n-1)}})$ can only intersect one of the $B_j^{(n)}$. But this implies that:

$$\mathcal{H}^1(E \cap B(z, \frac{1}{4}\sqrt{100^{-n(n-1)}})) \leq 1.4(100)^{-n} < 2(100)^{-n}.$$

This implies that:

$$\Theta_*^1(E,z) \lesssim \frac{\sqrt{100^n 100^{n-1}}}{100^n} = \sqrt{\frac{100^{n-1}}{100^n}} = \sqrt{.01} = .1.$$

6. $T_\mu(1)$ is bounded off the support of μ

For every $n \in \mathbb{N}$, each $z \in E^{(n)}$ is contained in a unique $B_j^{(n)}$. Call this $B^{(n)}(z)$. We want to show: $\text{dist}(z, \text{supp}(\mu)) = \epsilon$, and if m is chosen so that $100^{-m} < \frac{\epsilon}{4}$ then the following is true:

$$\left| \int_{\mathbb{C}} K(z-\xi) d\mu^{(m)}(\xi) \right| \leq C, \tag{3}$$

where the C does not depend on ϵ or z. First, pick a $w \in \operatorname{supp}(\mu)$ with $\operatorname{dist}(z, w) = \epsilon$ and let q be the least integer with $100^{-q} < \epsilon$. This means that $q \leq m$. The idea is to write (3) as:

$$\int_{\mathbb{C}} K(z-\xi)d\mu^{(m)}(\xi) = \int_{B^q(w)} K(z-\xi)d\mu^{(m)}(\xi) + \sum_{n=1}^{q} \int_{B^{n-1}(w)-B^n(w)} K(z-\xi)d\mu^{(m)}(\xi).$$

The first term is controlled using size properties of K and the measure of $B^q(w)$. The second term is controlled uniformly over q. The second term is controlled by the "mean value zero"–type property possessed by $K(z-\xi)$ over balls that contain z. The property can't be applied directly because z is not in all of the balls $B^{n-1}(w)$, so a slightly more delicate argument is needed.

References

[1] Léger, J. C., *Menger curvature and rectifiability*, Ann. of Math. (2), **149**, (1999), no. 3, 831–869

[2] Mattila, Pertti *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995. xii+343

[3] Nazarov, F., Treil, S., Volberg, A., *The Tb-theorem on non-homogeneous spaces*, Acta Math., **190**, (2003), no. 2, 151–239

School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332-0160 USA

E-mail address: rrahm3@math.gatech.edu

THE s-RIESZ TRANSFORM OF AN s-DIMENSIONAL MEASURE IN $\mathbb{R}^2$ IS UNBOUNDED FOR $1 < s < 2$

VLADIMIR EIDERMAN, FEDOR NAZAROV, AND ALEXANDER VOLBERG

presented by Guillermo Rey

Abstract. The authors show that no totally lower irregular finite positive Borel measure μ in $\mathbb{R}^2$ with $\mathcal{H}^s(\operatorname{supp}\mu) < \infty$ can have bounded Riesz transform. This, combined with previous results of Prat and Vihtilä, shows that for any $s \in (0,1) \cup (1,2)$ and any finite positive Borel measure μ in $\mathbb{R}^2$ with $\mathcal{H}^s(\operatorname{supp}\mu) < \infty$, we have $\|R\mu\|_{L^\infty(m_2)} = \infty$.

1. Introduction and Main Results Presented

To introduce the main results, we will need some definitions first. Let E be a subset of $\mathbb{R}^d$ and let $s > 0$, define

$$\mathcal{H}^s_\epsilon(E) := \inf\Big\{\sum_{i=1}^\infty (\operatorname{diam} E_i)^s : \, E \subseteq \bigcup_{i=1}^\infty E_i, \, \operatorname{diam} E_i < \epsilon\Big\}.$$

With this, we can define the s-dimensional Hausdorff measure of E as

$$\mathcal{H}^s(E) = \sup_{\epsilon > 0} \mathcal{H}^s_\epsilon(E).$$

So, if a set E has s-dimensional Hausdorff measure $H < \infty$, it means that for all $\epsilon > 0$ we can find a countable sequence of balls $B_i = B(c_i, r_i)$ such that $r_i \le \epsilon$,

$$\sum_{i=1}^\infty r_i^s \le H$$

and

$$\bigcup_{i=1}^\infty B(c_i, r_i) \supseteq E.$$

We will say that a positive measure has dimension s if the support of μ has finite s-dimensional Hausdorff measure, that is: for every $\epsilon > 0$ there exists a countable collection of balls $B_i = B(c_i, r_i)$ with $r_i \le \epsilon$,

$$\sum_{i=1}^\infty r_i^s \le H$$

and

$$\mu\Big(\mathbb{R}^d \setminus \bigcup_{i=1}^\infty B_i\Big) = 0.$$

If μ is a finite (signed) measure on $\mathbb{R}^d$, its s-dimensional (vector) Riesz transform $R\mu$ is defined by

$$R\mu(x) = \int_{\mathbb{R}^d} \frac{x - y}{|x - y|^{s+1}} \, d\mu(y).$$

If $0 < s < d$, it is easy to see that the integral converges almost everywhere with respect to the d-dimensional Lebesgue measure m_d on $\mathbb{R}^d$. We say that R is bounded in $L^2(\mu)$ if

$$\|R(f\mu)\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}$$

Observe that when $s \in (0, d] \cap \mathbb{N}$ and $\mu = \mathcal{H}^s|_V$ is the s-dimensional Hausdorff measure restricted to an s-dimensional linear subspace V, then

$$f \mapsto R(f\mu)$$

can, essentially, be seen as a standard Calderón-Zygmund operator on $\mathbb{R}^s$ and hence bounded in $L^2(\mu)$.
Things can get much more complicated than this and one in general cannot hope to have boundedness for "all" s-dimensional measures. In fact the characterization of those measures for which the Riesz transforms (of the appropriate order) are bounded in L^2 is the so-called "David-Semmes problem", which asks to relate the $L^2(\mu)$ boundedness of certain singular integral operators with the geometry of the support of μ. In the case of integer s the conjecture is that if the support of μ is Alfohrs-David regular and R is bounded, then μ is uniformly rectifiable (see [1] for some recent results and definitions in this direction).
As we just saw, the s-dimensional Riesz transform can be bounded in $L^2(\mu)$ for *some* s-dimensional measures, but the situation for non-integer s changes dramatically. The example given above certainly cannot be translated to non-integer s and in fact, another of the conjectures of David and Semmes states that

> Let μ be an s-dimensional finite measure. If the s-dimensional Riesz transform is bounded in $L^2(\mu)$, then s is an integer.

Laura Prat in [4] attacked this conjecture in the plane for $s \in (0, 1)$ using Menger's curvature techniques, however for $s > 1$ one cannot use Menger's curvature due to non-positivity issues. In higher dimensions the conjecture was proved by Merja Vihtilä in [5] under condition that

$$\mu(\{x : \liminf_{r\to 0^+} r^{-s}\mu(B(x, r)) > 0\}) > 0.$$

The present work gives the last partial result needed to settle the conjecture in the case of $d - 1 < s < d$ by studying the case where the measure μ is *totally lower irregular*:

$$\mu(\{x : \liminf_{r\to 0^+} r^{-s}\mu(B(x, r)) > 0\}) = 0.$$

Precisely, the statement of the main result is the following:

Theorem 1 (Main Theorem)**.** *Let $s \in (d-1, d)$, and let μ be a strictly positive finite totally irregular Borel measure in $\mathbb{R}^d$ such that $\mathcal{H}^s(\operatorname{supp}\mu) < \infty$, then $\|R\mu\|_{L^\infty(m_d)} = \infty$.*

Actually, one can see in the article that the authors also prove that, under the same conditions, R cannot be bounded in $L^2(\mu)$. Hence, this articles settles the David-Semmes conjecture alluded to earlier (the one treating the case of non-integer s) in dimension 2.
This article introduced one of the main ideas used in [1] and [2] and is very flexible. One of the techniques used is a kind of maximum principle which is used in an essential way in the proof. This is precisely what stops the argument from working in higher codimensions and it would be very interesting to prove this result (as well as those in [1] and [2], of course) without using the recourse to the maximum principle, or else by showing that the corresponding maximum principle works for all $0 < s \leq d$.

2. Ideas of the proof of the Main Result

The proof proceeds by contradiction: one assumes that $\|R\mu\|_{L^\infty} \leq 1$ from which it follows that R should be bounded in $L^2(\mu)$, then it is shown that R is unbounded in $L^2(\mu)$, arriving at a contradiction.

The first step in the proof is to notice that the $L^\infty(m_d)$ bound implies that R is of polynomial growth. Indeed, it is shown that

$$\mu(B(c,r)) \leq C\|R\mu\|_{L^\infty(m_d)} r^s.$$

With this bound on μ one can apply the general machinery of Calderón-Zygmund operators on non-homogeneous spaces from [3] to deduce that the maximal singular integral operator

$$R^\sharp(f\mu)(x) := \sup_{B \ni x}\Big|\int_{\mathbb{R}^d \setminus 2B} \frac{x-y}{|x-y|^{s+1}} f(y)\, d\mu(y)\Big|$$

is bounded in $L^2(\mu)$.

Next one constructs a Cantor-type structure on the support of μ. This is the most difficult part of the paper, and where more time is spent. We refer the reader to the original article for all the details, but let us sketch here the argument in the case where $\mu = \mathcal{H}^s|_K$, where K is a sparse s-dimensional Cantor square.

For $x \in K$ let $K^{(n)}(x)$ be the square of the n-th generation containing x. Define

$$R^{(n)}\mu(x) = \int_{K^{(n)}(x)\setminus K^{(n+1)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y).$$

Then

$$\sum_{n=0}^{N-1} R^{(n)}\mu(x) \lesssim R^\sharp\mu(x) + 1.$$

Indeed, by telescoping the series, we obtain:

$$\sum_{n=0}^{N-1} R^{(n)}\mu(x) = \int_{\mathbb{R}^d\setminus K^{(N)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y)$$

since μ is supported on $K^{(0)}$

If one takes B to be the ball circumscribing $K^{(N)}(x)$ then one can proceed by:

$$\int_{\mathbb{R}^d\setminus K^{(N)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y) = \int_{\mathbb{R}^d\setminus 2B} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y) + \\ \int_{\mathbb{R}^d\setminus K^{(N)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y) - \int_{\mathbb{R}^d\setminus 2B} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y)$$

which we can rewrite as

$$\int_{\mathbb{R}^d\setminus 2B} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y) + \int_{2B\setminus K^{(N)}(x)} \frac{x-y}{|x-y|}\, d\mu(y).$$

The first term is, by definition, bounded by $R^\sharp$. For the second term observe that we have

$$|x-y|^{-s} \lesssim l(K^{(N)}(x))^s$$

while $\mu(K^{(N)}) \lesssim l(K^{(N)}(x))^s$. As long as we construct the Cantor set in a way that the squares of each generation are separated by significantly more than their diameters then $2D$ only contains $K^{(N)}(x)$ and we are set.

We continue the argument by noting that for each n we have

$$\|R^{(n)}\mu\|_{L^2(\mu)} \gtrsim 1.$$

This is a bit involved but relatively straightforward in the particular case of a Cantor square. The main idea here is that if $x \in K^{(n+1)}(x)$ and $y \in K^{(n)}(x) \setminus K^{(n+1)}(x)$, then the differences $x - y$ have mostly constant sign, so there is very little cancellation.
Finally, we use the fact that the functions $R^{(n)}\mu$ are "almost-orthogonal" so we end-up being able to show that

$$\int_{\mathbb{R}^2} \Big| \sum_{n=0}^{N-1} R^{(n)}\mu \Big|^2 \, d\mu \gtrsim N,$$

but this contradicts the fact that $R^\sharp$ is bounded since μ is finite.
Let us mention a slimmer of the ideas behind the use of the "maximum principle". The kind of result that is used is the following:

Lemma 2. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and $d-1 < s < d$, then*

$$\max_{x \in \mathbb{R}^d} R_j f(x) = \max_{x \in \operatorname{supp} f} R_j f(x)$$

as long as the left hand side exists and is positive. Here R_j denotes the jth component of the s-dimensional Riesz transform.

We will give a sketch of the proof here, but ignore all non-zero constants for briefness.
Observe that

$$R_j f = f * \frac{x_j}{|x|^{s+1}} = f * \Big[\frac{\partial}{\partial x_j} \Big(\frac{-1}{s-1} \frac{1}{|x|^{s-1}} \Big) \Big].$$

Using the fact that $f \in C_c^\infty(\mathbb{R}^d)$, we can put the derivatives on f to obtain (up to a non-zero constant):

$$R_j f(x) = \Big(\partial_j f * \frac{1}{|x|^{s-1}} \Big)(x).$$

Under the Fourier transform, this expression looks like (again, up to constants)

$$\widehat{R_j f}(\xi) = \widehat{\partial_j f}(\xi) |\xi|^{-d-1+s},$$

so

$$|\xi|^{d+1-s} \widehat{R_j f}(\xi) = \widehat{\partial_j f}(\xi).$$

Taking the inverse Fourier transform we arrive at

$$\partial_j f(x) = \mathcal{F}^* \Big(\xi \mapsto |\xi|^{d+1-s} \widehat{u}(\xi) \Big),$$

where $u = R_j f$ and $\mathcal{F}^*$ denotes the inverse Fourier transform.
The multiplier $|\xi|^{d+1-s}$ corresponds to "taking $(d+1-s)$ derivatives", so

$$\partial_j f(x) = \big(\sqrt{-\Delta}\big)^{d+1-s} u(x).$$

We can use the representation formula for the square root of the Laplacian which can be written as:

$$\big(\sqrt{-\Delta}\big)^{\alpha} f(x) = p.v. \int_{\mathbb{R}^d} \frac{f(x) - f(x-y)}{|y|^{d+\alpha}} \, dy$$

for $0 < \alpha < 2$, where $p.v.$ denotes the principal value. In our setting, this becomes:

$$\partial_j f(x) = p.v. \int_{\mathbb{R}^d} \frac{u(x) - u(x-y)}{|y|^{2d+1-s}} \, dy,$$

which is where we need the condition that $s > d - 1$.
Let x_0 be a point where the maximum of $R_j f$ is attained, then

$$\partial_j f(x_0) = p.v. \int_{\mathbb{R}^d} \frac{u(x_0) - u(x_0 - y)}{|y|^{2d+1-s}} \, dy.$$

The right hand side is strictly negative, hence we must have $x_0 \in \operatorname{supp} \partial_j f \subseteq \operatorname{supp} f$.
It is unclear how one could avoid the use of the representation formula above (and hence the requirement of $s > d - 1$, but a possible conjecture of Alexander Volberg and Vladimir Eiderman [6] states

> Let μ be a finite (signed) measure with compact support in $\mathbb{R}^d$ which has a C^∞ density with respect to the Lebesgue measure m_d. Then
>
> $$|R\mu(x)| < C \max_{y \in \operatorname{supp} \mu} |R\mu(y)| \quad \forall x \in \mathbb{R}^d, \; 0 < s \leq d,$$
>
> where $C = C(s, d)$ depends only on s and d, and R denotes the (vector) s-dimensional Riesz transform.

References

[1] F. Nazarov, X. Tolsa, and A. Volberg. On the uniform rectifiability of AD regular measures with bounded Riesz transform operator: the case of codimension 1. *arXiv:1212.5229*, 2012.

[2] F. Nazarov, X. Tolsa, and A. Volberg. The riesz transform, rectifiability, and removability for lipschitz harmonic functions. *Publ. Mat.*, to appear, 2012.

[3] Fedor Nazarov, Sergei Treil, and Alexander Volberg. Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces. *International Mathematics Research Notices*, 1998(9):463–487, 1998.

[4] Laura Prat. Potential theory of signed Riesz kernels: capacity and Hausdorff measure. *Int. Math. Res. Not.*, (19):937–981, 2004.

[5] Merja Vihtilä. The boundedness of Riesz s-transforms of measures in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 124(12):3797–3804, 1996.

[6] A. L. Volberg and V. Ya. Eiderman. Nonhomogeneous harmonic analysis: 16 years of development. *Uspekhi Mat. Nauk*, 68(6(414)):3–58, 2013. Translation in Russian Math. Surveys 68 (2013), no. 6, 9731026.

Michigan State University
E-mail address: reyguill@math.msu.edu

REFLECTIONLESS MEASURES AND THE MATTILA-MELNIKOV-VERDERA UNIIFORM RECTIFIABILITY THEOREM

BENJAMIN JAYE AND FEDOR NAZAROV

presented by Rishika Rupam

Abstract. The aim of this paper is to provide a new proof of a theorem by Mattila-Melnikov-Verdera, on the uniform rectifiability of an Ahlfors-David regular measure μ whose Cauchy transform operator is bounded in $L^2(\mu)$.

1. Introduction and Main Results Presented

The motive and content of this paper is to prove the following

Theorem 1. *An Ahlfors-David regular measure μ whose associated Cauchy transform operator is bounded in $L^2(\mu)$ is uniformly rectifiable.*

This theorem was first stated and proved by Mattila, Melnikov and Verdera in [2]. The proof in this paper is a departure from the proof given there. In this extended abstract, we sketch the outline of the proof. We start with definitions.

Definition 1. *A measure μ is a c_0-* ***nice*** *if $\mu(B(z,r)) \leq C_0 r$ for any disc $B(z,r) \subset \mathbb{C}$.*

Definition 2. *A nice measure μ is called* ***AD-regular****, with regularity constant $c_0 > 0$, if*

$$\mu(B(z,r)) \geq c_0 r,$$

for any disc $B(z,r) \subset \mathbb{C}$ with $z \in supp(\mu)$.

Definition 3. *Let $K(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$. For a measure ν, the Cauchy transform of μ is formally defined by*

$$C(\nu)(z) = \int_{\mathbb{C}} K(z-\xi) d\nu(\xi),$$

for $z \in \mathbb{C}$. For $\delta > 0$, define

$$K_\delta(z) = \frac{\bar{z}}{\max(\delta, |z|)^2}.$$

The δ-regularized Cauchy transform of ν is defined by

$$C_\delta(\nu)(z) = \int_{\mathbb{C}} K_\delta(z-\xi) d\nu(\xi),$$

for $z \in \mathbb{C}$.
We say that μ is a C_0 ***good measure*** *if it is C_0 nice and $\sup_{\delta>0} \|C_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)} \leq C_0$.*

Definition 4. ***A*** *set $E \subset \mathbb{C}$ is called* ***uniformly rectifiable*** *if there exists $M > 0$ such that for any dyadic square Q, there exists a Lipschitz mapping $F : [0,1] \to \mathbb{C}$ with $\|F\|_{Lip} \leq Ml(Q)$ and $E \cap Q \subset F[0,1]$.*
A measure μ is called uniformly rectifiable if the set $E =$supp(μ) is uniformly rectifiable.

We can now restate theorem 1 as follows,

Theorem 2. *A good AD-regular measure μ is uniformly rectifiable.*

The paper is divided into several sections, some providing small lemmas to supply the final proof and some providing evolved versions of the above theorem. In the rest of the abstract, we follow the main results in these sections. In section 3, the authors construct a Carleson family for the following purpose.

The construction of a Lipschitz mapping: The authors identify a local property in order to prove uniform rectifiability. Let's choose a dyadic box $\mathcal{P}$. The authors approximate the support of $\mu = E$ by a sequence of sets of points. At each step in the sequence, they construct a graph connecting the points at that stage. The idea is to connect these points in an economical fashion, such that at each step, the length of the graph is bounded by $Cl(\mathcal{P})$. The conclusion of this procedure is that it is enough to show that

Proposition 1. *Suppose that μ is a C_0-good measure with AD regularity constant c_0. There is a constant $C > 0$ such that for each $\mathcal{P}$,*

$$\sum_{Q \in B^\mu, Q \subset \mathcal{P}} l(Q) \leq Cl(\mathcal{P}),$$

where B^μ is the set of 'bad boxes', which we will elucidate in the next section.
In the following section, we take a short detour to introduce Riesz families

Definition. *A system of functions ψ_Q (one ψ for each dyadic square Q) is called a Riesz system if: 1.$\psi_Q \in L^2(Q)$*
2.$\| \sum_{Q \in D} a_Q \psi_Q \|^2_{L^2(\mu)} \leq C \sum_{Q \in D} |a_Q|^2$, for every sequence a_Q.

For each square Q, let Ψ_Q be a set of functions in $L^2(\mu)$. Then, Ψ_Q is a Riesz family if for any choice of functions $\psi_Q \in \Psi_Q$, the system ψ_Q forms a C-Riesz system.
For example, if we fix $A > 0$, and define

$$\Psi^\mu_{Q,A} = \left\{ \psi_Q \in L^2(\mu) : supp(\mu_Q) \subset B(z_Q, Al_Q), \|\psi_Q\|_{Lip} \leq l(Q)^{-3/2}, \int_{\mathbb{C}} \psi_Q = 0 \right\},$$

then this forms a Riesz family. Choose $A' > 1$, $A' \leq A$. Consider the Riesz family $\Psi^\mu_{Q,A}$ introduced above. For each $Q \in D$, we define

$$\Theta_{A,A'}(Q) = \Theta^\mu_{A,A'}(Q) = \inf_{F \supset B(z_Q, A'l(Q))} \sup_{\psi \in \Psi^\mu_{Q,A}} l(Q)^{-1/2} | < C_\mu(\chi_F), \psi >_\mu |.$$

For some choice of functions ψ_Q, we'll have

$$\sum_{Q \subset \mathcal{P}} \Theta_{A,A'}(Q)^2 l(Q) \leq 2 \sum_{Q \subset \mathcal{P}} | < C_\mu(\chi_{B(z_P, 2A'l(P))}), \psi_Q >_\mu |^2.$$

Since the ψ_Q form a Riesz family, we'll have that

$$\sum_{Q \subset \mathcal{P}} \Theta_{A,A'}(Q)^2 l(Q) \leq C(C_0, A, A')l(\mathcal{P}).$$

In order to prove proposition 1 above, it is sufficient to prove

Proposition 2. *Suppose μ is a $C_0 -$ good measure with AD regularity constant $c_0 > 0$. There exist constants $A, A' > 1$, and $\gamma > 0$, such that for any square $Q \in B^\mu$,*

$$\Theta^\mu_{A,A'}(Q) \geq \gamma.$$

If the above proposition fails, then we'll have the following consequences:

Lemma 3. *Suppose that the proposition 2 fails. Then, there exists a $C_0 -$ good measure μ with AD-regularity constant c_0, such that*

$$| < \tilde{C}_\mu(1), \psi >_\mu | = 0, \tag{1}$$

for all $\psi \in \Phi^\mu$, and there exist $\xi, \zeta \in B(\bar{0}, 20) \cap supp(\mu)$, with $|\xi - \zeta| \geq \frac{1}{2}$, such that $0 \in [\zeta, \xi]$ and $B(0, \tau) \cap supp(\mu) = \emptyset$.

Here $\tilde{C}_\mu(1)$ is defined as the following operator

$$\tilde{C}_\mu(1)(z) = \int_{\mathbb{C}} \left[\frac{1}{z - \xi} - \frac{1}{\xi}\right] d\mu(\xi).$$

Measures that satisfy 1 are called *reflectionless.* It turns out there are few relectionless measures that are also good and AD-regular, thus proving proposition 2.

Proposition 3. *Suppose that μ is a non-trivial reflectionless good AD- regular measure. Then $\mu = c\mathcal{H}^1_L$ for a line L, and a positive constant $c > 0$.*

In order to prove this, we have the following

Lemma 4. *For every $z \notin supp(\mu)$,*

$$[\tilde{\mathcal{C}}(1)(z)]^2 = 2\varkappa.\tilde{\mathcal{C}}_\mu(1)(z).$$

Lemma 5. *Suppose that $z \notin supp(\mu)$. Let $\tilde{z}$ be a closest point in $supp(\mu)$ to z, and set $e = \frac{\tilde{z}-z}{|\tilde{z}-z|}$. For each $\alpha \in (0, 1)$, there is a radius $r_\alpha > 0$ such that $B(\tilde{z}, r_\alpha) \cap C_{\tilde{z},e}(\alpha)$ is disjoint from $supp(\mu)$.*

Lemma 6. *Suppose that $z \notin supp(\mu)$, and $\tilde{z}$ is a closest point on the support of μ to z. Let $e = \frac{\tilde{z}-z}{|\tilde{z}-z|}$. Then $supp(\mu) \subset H_{\tilde{z},e}$.*

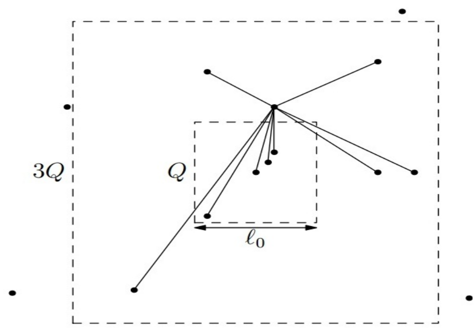
2. Main Result 1

Proof (of proposition 1) In order to prove this proposition, we carry out the following steps.

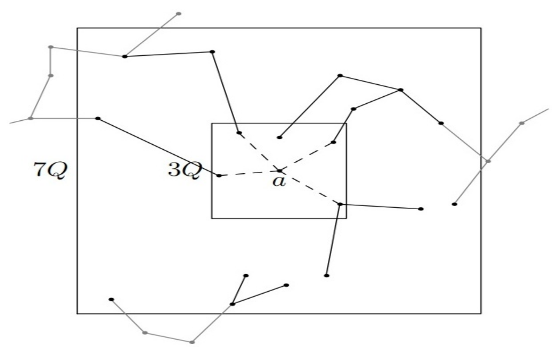
1. Choose and fix a dyadic box. Call it $\mathcal{P}$ - our viewing window.
2. We carry out the following 2 way induction process. For each $n \in \mathbb{N}$, we'll choose a sequence of points $\{x_{n_k}\}$ that form a 2^{-n} net of $E \cap \mathcal{P}$. At each such stage, we have a function $F : [0, 1] \to \mathbb{C}$, with $\|F\|_{Lip} \leq 2L$ and such that $F[0, 1] \supset \{x_{n_k}\} \cap 3\mathcal{P}$. We use the Arzela Ascoli theorem to prove the following

   Lemma 7. *Suppose that there exist $M > 0$ such that $L(l_0) \leq Ml(P)$ for every $l_0 > 0$. Then there exists $F_n : [0, 1] \to \mathbb{C}$ such that $\|F_n\|_{Lip} \leq M.l(\mathcal{P})$ and $F_n[0, 1] \supset \{x_{n_k}\} \cap \mathcal{P}$.*

   The other induction process is as follows: Pick and fix an n as above. Then, in order to create our graph (or web), we'll start with the base step of boxes of side length 2^{-n} and continue to $2^{-n+1}, ..., l(\mathcal{P})/2$.



(3) This is the base step for the induction process. Fix a point in $3Q$. Then join together every other point in $3Q$.
(4) This is the subsequent step in the induction process. Let Q be a box of lenth twice the length as in the previous induction step. If the graph from the previous step has as least two components inside $3Q$, then for each such component, choose a vertex that lies in $3Q$, fix a point inside $3Q$ and join each of the points to this chosen point.



(5) The authors use an economical construction that ensure that the length of the graph in the 2 way induction process remains bounded by $M.l(P)$, where M is a constant that depends on the goodness and regularity constants. They use the rarity of 'bad squares' (squares that are actually used in the induction process) to prove this.

3. Main Result 2

Proof (of lemma 3) If proposition 2 is false, then we have that for each k,

$$|<\tilde{\mathcal{C}}_{\mu_k}(1), \psi>_{\mu_k}| \leq \frac{1}{k} + \frac{CA^3}{k},$$

for all $\psi \in \Phi^{\mu_k}_{A/2}$. Here $\Phi^{\mu_k}_{A/2} = \{\psi : \|\psi\|_{Lip} \leq 1, \int_{\mathbb{C}} \psi d\nu = 0, supp(\psi) \subset B(0, A)\}$ We use the following lemma to conclude the proof.

Lemma 8. *Let ν_k be a sequence of C_0-good measures, with $0 \notin supp(\nu_k)$. Suppose that ν_k converge weakly to ν with $0 \notin supp(\nu)$. Fix non-negative sequences $\tilde{\gamma}_k$ and $\tilde{A}_k$, satisfying*

$\tilde{\gamma}_k \to 0$, *and* $\tilde{A}_k \to \tilde{A} \in (0,\infty]$. *If* $| < \tilde{\mathcal{C}}_{\nu_k}(1), \psi >_{\nu_k} | \leq \tilde{\gamma}_k$ *for all* $\psi \in \Phi^{\nu_k}_{\tilde{A}_k}$, *then* $| < \tilde{\mathcal{C}}_{\nu}(1), \psi >_{\nu} | = 0$, *for all* $\psi \in \Phi^{\nu}_{\tilde{A}}$.

4. Main Result 3

Proof (of lemma 4) The proof relies on the following identity.

$$\left[\frac{1}{z-\xi}+\frac{1}{\xi}\right]\cdot\left[\frac{1}{\xi-\omega}+\frac{1}{\omega}\right]+\left[\frac{1}{z-\omega}+\frac{1}{\omega}\right]\cdot\left[\frac{1}{\omega-\xi}+\frac{1}{\xi}\right]=\left[\frac{1}{z-\xi}+\frac{1}{\xi}\right]\cdot\left[\frac{1}{z-\omega}+\frac{1}{\omega}\right].$$

Integrating both sides of this equality with respect to $d\mu(\xi)d\mu(\omega)$, we get $2\tilde{\mathcal{C}}_\mu(\tilde{\mathcal{C}}_\mu(1))(z) = [\tilde{\mathcal{C}}_\mu(z)]^2$. The rest of the proof is a justification of these formal operations.

Proof (of lemma 5) We divide the plane $\mathbb{C}$ into three different parts, as shown in the

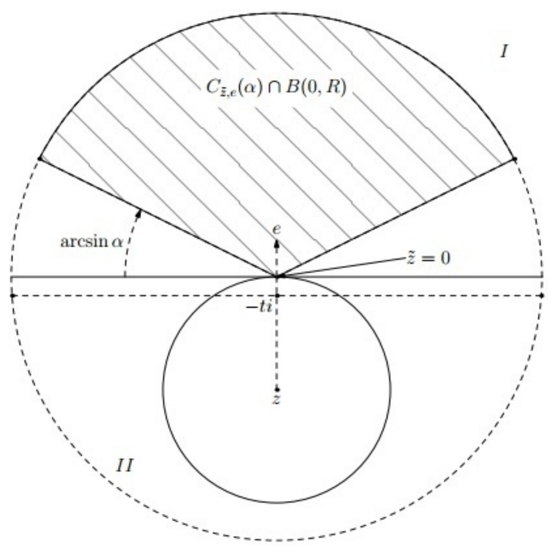


figure. Notice that

$$\Im[\tilde{\mathbb{C}}_\mu(-ti)] = \int_{\mathbb{C}}\left[\frac{\Im\xi+t}{|\xi+it|^2}-\frac{\Im(\xi-z_0)}{|\xi-z_0|^2}d\mu(\xi)\right].$$

Lemma 4 guarantees that $\Im[\tilde{\mathcal{C}}_\mu(1)(-it)] = \Im[\tilde{\mathcal{C}}_\mu(1)(z)]$ for any $t>0$. Let $\xi \in II\cap$ supp(μ). Then an elementary geomtric argument shows that $|\xi - it|^2 \geq -(\Im(\xi)+t)r$, for $|\Im(\xi)| < \frac{r}{2}$ and $t < \frac{r}{2}$. Thus,

$$\int_{II}\frac{\Im\xi+t}{|\xi+it|^2}d\mu(\xi) \geq -\int_{II\cap B(0,r/2)}\frac{1}{r}d\mu(\xi) - \left|\int_{II\setminus B(0,r/2)}\frac{\Im\xi+t}{|\xi+it|^2}d\mu(\xi)\right|.$$

Both terms on the right hand side are bounded in absolute value by $C\frac{\mu(B(0,R))}{r} \leq \frac{CR}{r}$. Similar calculations hold for I. Thus, we can conclude that there is a constant Δ, independent of t such that $\int_{III}\frac{\Im\xi+t}{|\xi+it|^2}d\mu(\xi) \leq \Delta$. Suppose the statement in the lemma is false. Then, there exists $\alpha>0$, along with a sequence $z_j \in C_{0,e}(\alpha)\cap$ supp(μ) with $z_j \to 0$ as $j \to \infty$. For each ball $B_j \in III$, provided $t \leq \frac{\alpha}{2}|z_j|$, we have $\frac{\Im\xi+t}{|\xi+it|^2} \geq \frac{\alpha|z_j|}{8|z_j|^2} = \frac{\alpha}{8|z_j|}$, for $\xi \in B_j$. As a result, we see that

$$\int_{III}\frac{\Im\xi+t}{|\xi+it|^2}d\mu(\xi) \geq \sum_{j:t\leq|z_j|/2}\int_{B_j}\frac{\Im\xi+t}{|\xi+it|^2}d\mu(\xi) \geq \sum_{j:t\leq|z_j|/2}\mu(B_j)\frac{\alpha}{8|z_j|}.$$

But $\mu(B_j) \geq \frac{c_0\alpha|z_j|}{2}$, and so the previous integral over III has size at least $\frac{c_0\alpha^2}{16}$.card$\{j : t \leq |z_j|/2\}$. However, if t is sufficiently small, then this quantity may be made larger than Δ. Since this is not possible, the lemma must be true.

Proof (of lemma 6) We use tangent measures (see [3]) and the previous lemma to show that the measure cannot have any support in a direction that makes an obtuse angle with e.

Proof (of proposition 3) Using lemma 6, we have that for each $z \notin \text{supp}(\mu)$, there is a half space with z on its boundary which does not intersect supp(μ). Now suppose there are three points $z, \xi, \zeta \in \text{supp}(\mu)$, which are not collinear. Then they form a triangle. Since μ is nice, there is a point ω in the interior of this triangle outside the support of μ. But, again by lemma 6, there is a half space, with ω on its boundary, which is disjoint from supp(μ). This half space must contain at least one of the points z, ξ or ζ. But this is not possible. Thus, the proposition is proved.

References

[1] B. Jaye, F. Nazarov *Reflectionless measures and the Mattila-Melnikov-Verdera uniform rectifiability theorem.* ArXiv:1307.1156.

[2] P. Mattila, M. Melnikov, J. Verdera *The Cauchy integral, analytic capacity, and uniform rectifiability.* Ann. Math. 144 (1996), pp 127-136.

[3] D. Preiss *Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities.* Ann. of Math. (2) 125 (1987), no. 3, 537-643.

Department of Mathematics, Texas A&M University, College Station, Texas 77843

E-mail address: rishika@math.tamu.edu

ALMOST-ADDITIVITY OF ANALYTIC CAPACITY AND CAUCHY INDEPENDENT MEASURES

VLADIMIR EIDERMAN, ALEXANDER REZNIKOV, AND ALEXANDER VOLBERG

presented by Fangye Shi

Abstract. In this paper, it is shown that given a family of separated discs centered at a chord-arc curve, the analytic capacity of a union of arbitrary subsets of these discs is comparable with the sum of their analytic capacities. As an application, a necessary and sufficient condition is given for a certain family of Cauchy operator measures to be Cauchy independent.

1. Introduction and Main Results Presented

Given a compact set $F \subseteq \mathbb{C}$,the analytic capacity is defined by

$$\gamma(F) := \sup |f'(\infty)|$$

where the supremum is taken over all analytic functions $f : \mathbb{C}\backslash F \to \mathbb{C}$ with $|f| \leq 1$ and $f'(\infty) := \lim_{z\to\infty} z(f(z) - f(\infty))$. For arbitrary set F,

$$\gamma(F) := sup\{\gamma(K) : K compact, K \subseteq F\}$$

In the celebrated paper [1] Tolsa established the countable semiadditivity of the analytic capacity,

$$\gamma(\cup F_i) \leq C \sum \gamma(F_i)$$

where C is an absolute constant. However, the reverse inequality does not hold in general (even if the sets involved are pairwise disjoint). To see this, consider the n-th generation $E_n = \cup_{k=1}^{4^n} E_{n,k}$ of the corner 1/4-Cantor set in the plane. We know that $\gamma(E_n) \asymp 1/\sqrt{n}$ [2] while $\sum_{k=1}^{4^n} \gamma(E_{n,k}) \asymp 4^n \cdot 4^{-n} = 1$.
Thus a natural question is raised: for what family of sets,the almost-additivity of analytic capacity holds?
We call Γ a chord-arc curve if

$$|t - s| \leq A|z(t) - z(s)|$$

where $t \mapsto z(t)$ is an arc-length parametrization of Γ.
The main result of this paper is the following theorem:

Theorem 1 (Almost Additivity of Analytic Capacity)**.** *Let D_j be discs with centers on a chord-arc curve Γ such that λD_j are pairwise disjoint for some $\lambda > 1$. Let $E_j \subseteq D_j$ be arbitrary compact sets. Then there exists a constant $c = c(\lambda, A)$, such that*

$$\gamma(\cup E_j) \geq c \sum \gamma(E_j).$$

The strict separation condition $\lambda > 1$ is needed in the proof for some technical reasons. It is not known whether the theorem is true or not when $\lambda = 1$.
Let μ be a finite Borel measure with compact support in the complex plane. We say μ is a Cauchy operator measure if the Cauchy operator C_μ is bounded on $L^2(\mu)$ with norm at most 1.Here we define $L^2 - L^2$ norm of C_μ to be $||C_\mu||_\mu := \sup_\epsilon ||C^\epsilon_\mu||_\mu$ where C^ϵ_μ is ϵ-truncations of the Cauchy operator defined by:

$$C^\epsilon_\mu f(z) := \int_{\epsilon<|\xi-z|<\epsilon^{-1}} \frac{f(\xi)}{\xi - z} d\mu(\xi)$$

Let Σ be the class of nonnegative Borel measures of linear growth (i.e. nonnegative Borel measures μ satisfies $\mu(D(x,r)) \leq r$ for every disc $D(x,r)$). In connection to the analytic capacity,we have the following important fact [3]:

$$\gamma(F) \asymp \sup\{||\mu|| : supp(\mu) \subseteq F, \mu \in \Sigma, ||C_\mu||_\mu \leq 1\}$$

We call a colletion $\{\mu_j\}$ of positive measures Cauchy independent measures if μ_j is a Cauchy operator measure for each j and $||C_\mu||_\mu < \infty$ for $\mu := \sum \mu_j$.
It is known that a finite family of Cauchy operator measures is always Cauchy independent, see [4]. Thus, the main interest is in the situations when infinite families are Cauchy independent.As an application of the main result above, the following theorem is proved:

Theorem 2. *Suppose $\lambda > 1$ and measures μ_j are supported on compact sets E_j lying in discs D_j such that λD_j are pairwise disjoint. Assume $||C_{\mu_j}||_{\mu_j} \leq 1$ and $||\mu_j|| \asymp \gamma(E_j)$ with absolute constants. Let $\mu := \sum \mu_j$ and $E := \cup E_j$. Then this family is Cauchy independent if and only if for any disc B,*

$$\mu(B) \leq C\gamma(B \cap E).$$

And a direct corollary of Theorem 2 is the following:

Theorem 3. *Suppose $\lambda > 1$ and measures μ_j are supported on compact sets E_j lying in discs D_j such that λD_j are pairwise disjoint. Assume $||C_{\mu_j}||_{\mu_j} \leq 1$ and $c_1||\mu_j|| \leq \gamma(E_j) \leq c_2||\mu_j||$ Let $\mu := \sum \mu_j$ and $E := \cup E_j$. Suppose for any disc B,*

$$\sum \gamma(B \cap E_j) \leq C_1\gamma(B \cap E),$$

then $||C_\mu||_\mu \leq C < \infty$, here $C = C(c_1, c_2, C_1)$

Note that Theorem 2, Theorem 3 do not have any assumptions on the location of discs D_j. However, under the assumption of Theorem 3, the discs involved must have a very special geometric structure.Recall that a curve Γ in the plane is called Ahlfors regular if for any disc B, we have

$$H^1(\Gamma \cap B) \leq C diam(B)$$

for some absolute constant C. Then we have the following:

Theorem 4. *Under the assumption of Theorem 3, there exists an Ahlfors regular curve Γ such that all discs intersect Γ.Moreover, the Ahlfors constant of Γ depends only on λ and C_1 as in Theorem 3.*

2. Main Tool: The Melnikov-Menger Curvature

A useful tool for proving Theorem 1 is the Melnikov-Menger curvature of a positive Borel measure μ in $\mathbb{C}$ defined as

$$c^2(\mu) := \int\int\int \frac{1}{R^2(x,y,z)} d\mu(x)d\mu(y)d\mu(z),$$

where $R(x,y,z)$ is the radius of the circle passing through x, y, z, with $R(x,y,z) = \infty$ if x, y, z lie on a straight line.

Lemma 5 (Main Lemma)**.** *Let $D_j := D(x_j, r_j)$ be discs with centers on a chord-arc curve Γ, with λD_j pairwise disjoint for some $\lambda > 1$. Let μ_j be positive measures supported in D_j such that $||\mu_j|| \leq r_j$. Then for $\mu := \sum \mu_j$, we have $c^2(\mu) \leq \sum c^2(\mu_j) + C||\mu||$, $C = C(\lambda, A)$ where A is the constant of Γ.*

Theorem 1 is a direct consequence of Lemma 5 and the fact [3] that for any compact set F,

$$\gamma(F) \asymp \sup\{\mu(F) : supp(\mu) \subseteq F, \mu \in \Sigma, c^2(\mu) \leq \mu(F)\}$$

In proving the main lemma, $c^2(\mu)$ is compared to $c^2(\sigma)$ where σ is (part of) the arc-length measure supported on Γ. The following facts are used along the way: (a) The boundedness of the Cauchy operator on chord-arc curves, see [5]. (b) The connection between the curvature of a 'good' measure and the norm of the Cauchy operator associated to that measure:

$$||C^{\epsilon}_{\mu}1||^2_{L^2(\mu)} = \frac{1}{6}c^2_{\epsilon}(\mu) + O(||\mu||)$$

uniformly in ϵ for any measure $\mu \in \Sigma$, see [6] . Here we take the truncated version of the curvature $c^2_{\epsilon}(\mu)$, defined in the same way as $c^2(\mu)$, but the integral is taken over the set $\{(x,y,z) \in \mathbb{C}^3 : |x-y|, |y-z|, |x-z| > \epsilon\}$.

3. The Necessary and Sufficient Condition in Theorem 2

The necessity is an easy consequence of the fact that boundedness of C_μ implies that $\alpha\mu \in \Sigma$ for some α depending only on $||C_\mu||_\mu$. Note that the extra structure of the support of the measure is not needed.
To prove the sufficiency,the following theorem from [4] will be used several times to compare different measures:

Theorem 6. *Suppose that $\{D_j\}$ are discs on the plane and the dilated discs λD_j are pairwise disjoint for some $\lambda > 1$. Let ν, σ be two positive measures supported in $\cup D_j$ such that $\sigma(D_j) \asymp \nu(D_j)$. Suppose $C_{\nu|D_j}$ are uniformly bounded. Suppose C_σ is bounded on $L^2(\sigma)$, then C_ν is bounded on $L^2(\nu)$.*

The idea is to compare μ with a new measure σ such that C_σ is known to be bounded.Below is the construction of the measure σ.
By a cross, we mean two perpendicular line segments of equal length intersecting in their centers, one of them being horizontal. Note that it is very easy to compare 'length' and analytic capacity of the cross. Indeed, we have

$$\gamma(cross \cap B) \asymp H^1(cross \cap B)$$

for any disc B with absolute constants of comparison.
Let $\lambda > 1$. Let $\lambda' := \frac{1+\lambda}{2}$. For a disc D with radius r, we place a cross of length less than $r/1000$ in the center of D and N disjoint copies of crosses that touch $\partial(\lambda' D)$ on the inside and on equal distance from each other. We choose N to be the minimal integer such that if a disc B intersects D and $\mathbb{C}\backslash(\lambda D)$, then at least one cross lies inside B. Note that $N \asymp \frac{1+\lambda}{\lambda-1}\pi$ depends only on λ. Let L_j denote the union of crosses constructed as above that are associated to D_j with size $H^1(L_j) := \frac{N+1}{1000}\gamma(E_j)$. Let L be the union of L_j. Let $\sigma := H^1|L$.

Lemma 7. *For any disc B,*

$$\gamma(B \cap L_j) \asymp \sigma(B \cap L_j)$$

And if we further assume

$$\mu(B) \leq C\gamma(B \cap E).$$

for every disc B as in Theorem 2, then we have almost additivity

$$\gamma(B \cap L) \geq C \sum \gamma(B \cap L_j),$$

for some absolute constant C.

The proof of first part of Lemma 7 is straight forward. The second part requires an application of Theorem 1 and Theorem 6.
The following theorem from [7] and Lemma 7 thus implies that C_σ is bounded:

Theorem 8. *Let $L \subseteq \mathbb{C}$ be a compact set of positive and finite length. Let $\sigma := H^1|L$. Then C_σ is bounded if and only if there exists a finite constant C such that $\sigma(B \cap L) \leq C\gamma(B \cap L)$ for any disc B.*

Finally, the boundedness of C_σ and Theorem 6 conclude the proof of Theorem 2.

4. 'Sharpness' of Theorem 2

It is pointed out that the condition

$$\mu(B) \leq C\gamma(B \cap E) \tag{1}$$

for every disc B alone is not enough to guarantee the boundedness of C_μ. Indeed, it is proved that:

Theorem 9. *There exists a family of measures $\{\mu_j\}_{j=0}^{\infty}$ with the following properties: (a) $||C_{\mu_j}||_{\mu_j} \leq 1$; (b) $||\mu_j|| \asymp \gamma(E_j)$, where $E_j = supp(\mu_j)$; (c) $\{2E_j\}_{j\geq 1}$ are pairwise disjoint; (d) Let $\mu := \sum_{j=0}^{\infty} \mu_j$, then $\mu(B) \leq C\gamma(B \cap E)$ for each disc B ; (e) $||C_\mu||_\mu = \infty$.*

The idea is to use again a variant of the corner 1/4-Cantor set. The family $\{\mu_j\}_{j=1}^{\infty}$ constructed above satisfies all the assumptions of Theorem 2 except (1). Adding μ_0 changes the structure of μ and fails the separation assumption of theorem 2.

References

[1] X.Tolsa, Painlevé's problem and the semiadditivity of analytic capacity, Acta Math. 190:1 (2003), 105-149.

[2] J.Mateu, X.Tolsa,J.Verdera, The planar Cantor sets of zero analytic capacity and the local T(b)-theorem, J.Amer. Math. Soc. 16(2003), no.1, 19-28.

[3] A.Volberg, Calderón-Zygmund capacities and operators on nonhomogeneous spaces. CBMS Reginal Conf.Ser. in Math. 100, Amer. Math. Soc., Providence, 2003.

[4] F.Nazarov, X.Tolsa and A.Volberg, The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions, Preprint, 2012, arXiv:1212.5431, 15p. To appear in Publ. Mat.

[5] M.Melnikov, J.Verdera, A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs, Internat. Math. Res. Notices 1995, no. 7,325-331

[6] X. Tolsa, Painlevé's problem and analytic capacity, Proceedings of the 7th International Conference on Harmonic Analysis and Partial Differential Equations El Escorial, Madrid (Spain), June 21-25,2004, Collect. Math. (2006), 89-125.

[7] F.Nazarov, A. Volberg, Analytic capacity of the portion of continuum and a question of T. Murai, Proc. of Symposia in Pure Math, v. 79 (2008), pp. 279-292. The volume dedicated to Vladimir Maz'ya's 70th anniversary.

INDIANA UNIVERSITY, BLOOMINGTON, INDIANA, USA
E-mail address: shif@indiana.edu

THE RIESZ TRANSFORM, RECTIFIABILITY, AND REMOVABILITY FOR LIPSCHITZ HARMONIC FUNCTIONS

FEDOR NAZAROV, XAVIER TOLSA, AND ALEXANDER VOLBERG

presented by Prabath Silva

ABSTRACT. In this note we will discuss the results from [4]: given a set $E \subset \mathbb{R}^{n+1}$ with finite Hausdorff measure $\mathcal{H}^n$, if the n-dimentional Riesz transform

$$R_{\mathcal{H}^n\lfloor E} f(x) = \int_E \frac{x-y}{|x-y|^{n+1}} f(y) d\mathcal{H}^n(y),$$

is bounded on $L^2(\mathcal{H}\lfloor E)$, then E is n-rectifiable. As a corollary of this result we get that a compact set $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E)$ is removable for Lipschitz harmonic functions if and only if it is purely n-unrectifiable.

1. Introduction and Main Results Presented

A set $E \subset \mathbb{R}^{n+1}$ is called a n-rectifiable if it is contained in a countable union of C^1 manifolds up to a set of zero $\mathcal{H}^n$ measure. If E does not have any n-rectifiable subsets with positive n-Hausdorff measure then E is called purely unrectifiable.
Next we consider singular integral operators related to measure. Given a Borel measure ν in $\mathbb{R}^{n+1}$ such that

$$\int_{\mathbb{R}^{n+1}} \frac{d|\nu|(x)}{(1+|x|)^n} < \infty,$$

the n-dimensional Riesz transform is defined by

$$R\nu(x) = \int_{\mathbb{R}^{n+1}} \frac{x-y}{|x-y|^{n+1}} d\nu(y)$$

when the integral makes sense.
The main theorem of this work is the following, which relates the geometric notion of rectifiability with the boundedness for the Riesz transform. The proof of this theorem uses deep results from [1], [4] and [2].

Theorem 1. *Let E be a set such that $\mathcal{H}^n(E) < \infty$. If $R_{\mathcal{H}^n\lfloor E}$ is bounded on $L^2(\mathcal{H}^n\lfloor E)$, then E is n-rectifiable.*

A subset $E \subset R^{n+1}$ is removable for Lipschitz functions if, for every open set $\Omega \subset \mathbb{R}^{n+1}$, every Lipschitz function $f : \Omega \to \mathbb{R}$ that is harmonic in $\Omega \setminus E$ is harmonic in Ω.
It is known from [3] that if a compact set $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$ is removable for Lipschitz harmonic functions, then it must be purely n-unrectifiable. Also, from Theorem 2.2 [7], if E is not removable for Lipschitz harmonic functions, then there exists some measure μ supported on E such that R_μ is bounded in $L^2(\mu)$. Then one can use Theorem 1 to show that E is not purely n-unrectifiable. So we have the following corollary to Theorem 1.

Theorem 2. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set such that $\mathcal{H}^n(E) < \infty$. Then E is removable for Lipschitz harmonic functions in $\mathbb{R}^{n+1}$ if and only if E is purely n-unrectifiable.*

2. Proof of Theorem 1

Definition 3. *A d-dimensional set E in $\mathbb{R}^n$ is called d-AD regular if with some $0 < c < C < \infty$,*

$$cr^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq Cr^d, \forall x \in E, \forall r \in (0, \operatorname{diam} E).$$

A Borel measure μ in $\mathbb{R}^d$ has growth of degree n if there exists $c > 0$ such that

$$\mu(B(x,r)) \leq cr^n$$

for all $x \in \mathbb{R}^d, r > 0$. The upper and lower n-dimensional densities of a measure μ are defined by

$$\theta^{n,*}(x,\mu) = \limsup_{r\to 0} r^{-n}\mu(B(x,r)) \text{ and } \theta^n_*(x,\mu) = \liminf_{r\to 0} r^{-n}\mu(B(x,r)),$$

respectively.
By using the main theorem from [2] we can assume that $\theta^n_*(x,\mu) > 0$ for μ-a.e. $x \in \mathbb{R}^{n+1}$.

Lemma 4 (Main Lemma.)**.** *Let μ be a compactly supported finite Borel measure in $\mathbb{R}^d$ with growth degree n such that $\theta^n_*(x,\mu) > 0$ for μ-a.e $x \in \mathbb{R}^d$. Suppose that R_μ is bounded in $L^2(\mu)$. Then there are finite Borel measures $\mu_k, k \geq 1$, such that*

1. *$\mu \leq \sum_{k\geq 1} \mu_k$,*
2. *μ_k is AD-regular for each $k \geq 1$, with the AD regularity constant depend on k,*
3. *for each $k \geq 1$, R_{μ_k} is bounded on $L^2(\mu_k)$.*

Using this lemma with $\mathcal{H}^n\lfloor E$ together with the results from [1] and [4] we get that for each k, $\operatorname{supp}\mu_k$ is n-rectifiable.

3. Proof idea of the main lemma

This is the main technical part of the paper. A sequence of measures satisfying conditions (1) and (2) were already obtained in [6], but in this lemma we have to make sure that the Riesz transform is bounded on $L^2(\mu_k)$.
Let $F \subset \operatorname{supp}\mu$ such that $\theta^n_*(x,\mu) > 0$ for all $x \in F$ and $\mu(\mathbb{R}^d \setminus F) = 0$. Next consider the sets

$$F_p = \{x \in F : \text{ for } 0 < r \leq D, \mu(B(x,r)) \geq \frac{1}{p} r^n\},$$

$$F_{p,s} = \{x \in F_p : \text{ for } 0 < r \leq D, \mu(F_p \cap B(x,r)) \geq \frac{1}{ps} r^n\},$$

where $D = \operatorname{diam}(\operatorname{supp}\mu)$. Note that we have $F = \bigcup_{p\geq 1} F_p$ and $\mu(F_p \setminus \bigcup_{s\geq 1} F_{p,s}) = 0$.
The measures $\mu_{p,s}$ are obtained by adding carefully choosen measures $\sigma_{p,s}$ to $\mu\lfloor F_{p,s}$. The measures $\sigma_{p,s}$ are obtained by first obtaining an at most countable covering of the set $F_p \setminus F_{p,s}$ from collection of balls $\{B(x,d(x)) : x \in F_p \setminus F_{p,s}, d(x) = \operatorname{diet}(x, F_{p,s})\}$, using Besicovitch's covering theorem, and then putting n-dimensional Hausdorff measures on n-dimensional hyperplanes in each ball from the covering. This construction gives us AD regularity of $\mu_{p,s}$. Next we show that the Riesz transform is bounded on $L^2(\mu_{p,s})$. First, using a result from [5], we note that it is enough to show that Riesz transform is bounded on $L^2(\sigma_{p,s})$. This is obtained by approximating the Riesz transform on $L^2(\sigma_{p,s})$ with the Riesz transform on $L^2(\nu)$, where ν is an appropriately normalized version of measure μ defined on the balls.

References

[1] G. David and S. Semmes., *Analysis of and on uniformly rectifiable sets,* Mathematical Surveys and Monographs, 38, American Mathematical Society, Providence, RI, 1993.

[2] V. Eiderman, F. Nazarov and A. Volberg., *The s-Riesz transform of an s-dimensional measure in R2 is unbounded for* $1 < s < 2$*,* to appear in J. Anal. Math.

[3] P. Mattila and P.V. Paramonov., *On geometric properties of harmonic Lip1-capacity* Pacific J. Math. 171:2 (1995), 469490.

[4] F. Nazarov, X. Tolsa and A. Volberg., *The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions,* arXiv:1212.5431, to appear in Publ. Mat.

[5] F. Nazarov, S. Treil and A. Volberg., *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators in nonhomogeneous spaces,* Int. Math. Res. Notices 9 (1998), 463487.

[6] H. Pajot., *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral,* Lecture Notes in Math. 1799, Springer, 2002.

[7] A. Volberg., *Calderón-Zygmund capacities and operators on nonhomogeneous spaces.* CBMS Regional Conf. Ser. in Math. 100, Amer. Math. Soc., Providence, 2003.

Department of Mathematics, Indiana University Bloomington
E-mail address: `pssilva@indina.edu`

PAINLEVÉ'S PROBLEM AND THE SEMIADDITIVITY OF ANALYTIC CAPACITY

XAVIER TOLSA

presented by Raghavendra Venkatraman

1. Introduction and Main Results Presented

Let $\Omega \subset \mathbb{C}$ be an open set, and let $H^\infty(\Omega)$ denote the algebra of bounded analytic functions in Ω. Set $E = \mathbb{S}^2 \backslash \Omega$, we may assume E is a compact plane set. We begin by recalling a classical result [1] to set stage for the kind of problems under discussion.

Theorem 1 (Painlevé)**.** *Assume that for every $\varepsilon > 0$, the set E can be covered by discs the sum of whose radii does not exceed ε. Then $H^\infty(\Omega)$ consists only of constants.*

Theorem 2. *If E has positive area, then $H^\infty(\Omega)$ has non-constant functions.*

Observe that the hypothesis of these theorems is measure theoretic, while the conclusion deals with the *capacity* of Ω to support nontrivial bounded analytic functions. Furthermore, the proof of both these theorems involves convolving the singular kernel $\frac{1}{z}$ with some Borel measure. Ahlfors made this observation precise by proving that the set E is removable for bounded analytic functions if and only if its analytic capacity $\gamma(E) = 0$; the relevant definitions are as follows- A compact set $E \subset \mathbb{C}$ is said to be removable for bounded analytic functions if for any open set Ω containing E, every bounded analytic function on $\Omega \backslash E$ has an analytic extension to Ω. Let as before E be a plane compact set and Ω its complement, then

$$\gamma(E) := \sup_{\mathcal{A}} |f'(\infty)|, \qquad \mathcal{A} = \{f : \Omega \to \mathbb{C}, f \in H^\infty(\Omega), \|f\|_\infty \leq 1\}. \tag{1}$$

Here, $f'(\infty) := \lim_{z\to\infty} z(f(z) - f(\infty))$. For an arbitrary set $A \subset \mathbb{C}$, one defines $\gamma(A) = \sup\{\gamma(E), E \subset A \text{ is compact }\}$.

Painlevé's problem consists of characterizing removable sets for bounded analytic functions in terms of metric/geometric properties of these sets. By Ahlfors' theorem above, this amounts to characterizing the foregoing purely analytic definition of analytic capacity in geometric terms. Vitushkin in the 50's raised the question of semi-additivity of analytic capacity, namely, does there exist a universal constant $C > 0$ such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F)).$$

X. Tolsa [2] proved that the answer to Vitushkin's question is in the affirmative, and in the process provided a solution to Painlevé's problem. These accomplishments of Tolsa are the subject of this short note. More precisely, the theorems proven in [2] are

Date: August 3, 2014.

Theorem 3 (Semiadditivity of Analytic Capacity)**.** *Let $E \subset \mathbb{C}$ be compact. Let $E_i, i \geq 1$ be Borel sets such that $E = \cup_{i=1}^{\infty} E_i$. Then,*

$$\gamma(E) \leq C \sum_{i=1}^{\infty} \gamma(E_i),$$

where C is an absolute constant.

Theorem 4 (Painlevé's Problem)**.** *A compact set $E \subset \mathbb{C}$ is non-removable for bounded analytic functions if and only if it supports a positive Radon measure with linear growth and finite curvature.*

See below for definitions of curvature of measures and its geometric implications. This note is organized as follows. This section, providing some background borrows heavily from X.Tolsa's recent delightful book [3] on the subject. Part two makes some noises on the proof of the main theorems.

A. Analytic Capacity. Recall that for a compact set E, we define

$$\gamma(E) = \sup_{\mathcal{A}} |f'(\infty)|.$$

If $u \in \mathcal{A}$, we say that u is admissible for E. For example, the capacity of a point is zero since any bounded analytic function on the punctured plane is a constant. On the other hand, if E is a closed disc centered at the origin and radius $r > 0$, the function $u(z) := r/z$ is admissible for E, and so $\gamma(E) > 0$. Furthermore, by a normal families argument, it follows that the supremum in (1) is achieved. In fact, it is not hard to see that the admissible function satisfying $f'(\infty) = \gamma(E)$ is unique; this is called the *Ahlfors function* of E. Moreover, any function u that achieves this supremum in fact satisfies $u(\infty) = 0$. For such a function, $\lim_{z\to\infty} zu(az+b) = au'(\infty)$, so that $\gamma(aE+b) = |a|\gamma(E)$ for any pair of complex numbers a, b.
It is not hard to see that the analytic capacity is a set monotone function, i.e. if $E \subset F$, then $\gamma(E) \leq \gamma(F)$, and $\gamma(E) = \gamma(\partial_o E)$, where $\partial_o E$ is the boundary of the unbounded component of $\mathbb{S}^2 \backslash E$. When $E \subset \mathbb{C}$ is a compact connected set different from a single point, and f is a conformal map between the unbounded connected component of $\mathbb{S}^2 \backslash E$ to the disc vanishing at ∞, then $\gamma(E) = |f'(\infty)|$. This provides us a recipe to compute the analytic capacity of some sets, for instance, the analytic capacity of a closed disc $\overline{B}(0, r)$ is r, while that of a closed line segment of length ℓ is $\ell/4$. In fact, from the Koebe-Beiberbach 1/4 theorem, it follows that $\mathrm{diam}(E)/4 \leq \gamma(E) \leq \mathrm{diam}(E)$ for any compact connected set $E \subset \mathbb{C}$. Consequently, if $\gamma(E) = 0$, then E is totally disconnected.
Concerning the relationship between analytic capacity and rectifiability, we only make a few comments. If $\dim_H(E) > 1$, then $\gamma(E) > 0$. Moreover if $\dim_H(E) < 1$ then $\gamma(E) = 0$. However, it is not true that $\gamma(E) > 0$ if and only if $H^1(E) > 0$. The counter example of a set with positive length and vanishing analytic capacity is given by the corner quarters Cantor set. Finally, in connection with Painlevé problem and Vitushkin's conjecture, David proved that $E \subset \mathbb{C}$ satisfies $\gamma(E) = 0$ if and only if E is purely unrectifiable.

B. The Cauchy Transform and Vitushkin's Localization Operator. The *Cauchy transform* of a complex finite measure ν on $\mathbb{C}$ is defined by

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi - z}\, d\nu(\xi). \tag{2}$$

By Fubini's theorem, taking into account the fact that $z \mapsto \frac{1}{|z|}$ is locally integrable with respect to planar Lebesgue measure, the integral is absolutely convergent for (Lebesgue) a.e. $z \in \mathbb{C}$.
The primordial role of the Cauchy transform in analysis arises from the fact that $z \mapsto \frac{1}{\pi z}$ is the fundamental solution to the Cauchy Riemann operator. In the context of analytic capacity, if ν is any compactly supported distribution, then $\mathcal{C}\nu$ is analytic outside the support of the distribution ν, and verifies $\mathcal{C}\nu(\infty) = 0$ and $(\mathcal{C}\nu)'(\infty) = -\langle \nu, 1 \rangle = -\nu(\mathbb{C})$.
Given $f \in L^1_{loc}(\mathbb{C})$ and $\phi \in C^\infty$ is compactly supported, we define the *Vitushkin localization operator* associated to ϕ by

$$V_\phi f = \phi f + \frac{1}{\pi}\mathcal{C}(f\overline{\partial}f). \tag{3}$$

It is not hard to see that we have

$$V_\phi(f) = -\frac{1}{\pi}\mathcal{C}(\phi\overline{\partial}f). \tag{4}$$

This in turn implies that when $f = \mathcal{C}\nu$, where ν is a compactly supported measure or distribution, it verifies

$$V_\phi(\mathcal{C}\nu) = \mathcal{C}(\pi\nu).$$

This identity justifies the name for this operator: indeed, the Vitushkin operator $V_\phi(\mathcal{C}\nu)$ is analytic in the larger set $\mathbb{C}\backslash supp(\phi\nu)$. Hence the singularities are now localized to $supp(\phi) \cap supp(\nu)$. The Vitushkin localization operator posses many nice regularity properties, see [3] for details. More over, it can be used to prove the following special case of the semi-additivity problem.

Theorem 5. *For a compact set $E \subset \mathbb{C}$ and a closed disc or square $D \subset \mathbb{C}$ we have*

$$\gamma(E \cup D) \le c(\gamma(E) + \gamma(D)).$$

This result also holds with E, D replaced by closed rectangles R and S in the complex plane (without the assumption that their sides are required to be parallel to the axes).

C. Menger Curvature and curvature of a measure. Given three points $x, y, z \in \mathbb{C}$ their Menger curvature is $c(x, y, z) = \frac{1}{R(x,y,z)}$, where $R(x, y, z)$ is the radius of the circumcircle of triangle x, y, z. In the degenrate cases that the three points are collinear or that two of them coincide, we set $c(x, y, z) = 0$. The law of sines provides a useful identities for the Menger curvature of three given points in $\mathbb{C}$ in terms of the side lengths of Δxyz and for instance its area. From these identities, it is not hard to prove what has come to be known as Melnikov's miracle:

Proposition 6. *Let $z_1, z_2, z_3 \in \mathbb{C}$ be pairwise different. Then*

$$c(z_1, z_2, z_3)^2 = \sum_{\sigma \in S_3} \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(\overline{z_{\sigma_3} - z_{\sigma_4}})}.$$

For a positive Radon measure μ, we write

$$c^2_\mu(x) = \int\int c(x, y, z)^2 \, d\mu(y)\, d\mu(z),$$

and the curvature of the measure μ as

$$c^2(\mu) = \int c^2_\mu(x)\, d\mu(x).$$

For example, it is not hard to see that the curvature of the arc length measure on a circle of radius r is $8\pi^3 r$. The deep David-Leger theorem justifies why the curvature of a measure is indeed a geometric quantity: it states that for a set E of finite length and $\mu = H^1\lfloor E$, finiteness of $c^2(\mu)$ implies that E is rectifiable. By integrating thrice the identity in the last proposition, one obtains the following elementary yet deep connection between the Cauchy transform of a measure and its curvature. The theorem we are talking about is the following.

Theorem 7. *Let μ be a finite Radon measure on $\mathbb{C}$ with c_0 linear growth. We have,*

$$\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{1}{6}c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})),$$

with

$$|O(\mu(\mathbb{C}))| \leq cc_0^2\mu(\mathbb{C}),$$

where c is some absolute constant.

D. The Capacity γ_+. The capacity γ_+ of a compact set $E \subset \mathbb{C}$ is defined by

$$\gamma_+(E) := \sup\{\mu(E) : supp(\mu) \subset E, \|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1\}.$$

The capacity γ_+ has a definition similar to that of γ, except we now require $f = \mathcal{C}\nu$ be the Cauchy transform of some Radon measure (indeed, $(\mathcal{C}\mu)'(\infty) = -\mu(E)$). The importance of this capacity lies in the fact that semi-additivity of γ_+ follows from its characterization in terms of its curvature. It is trivial that $\gamma_+(E) \leq \gamma(E)$ for any set E. Consequently, the crux of the matter in proving 3 is the following theorem asserting the comparability of γ and γ_+.

Theorem 8. *There exists an absolute constant A such that*

$$\gamma(E) \leq A\gamma_+(E),$$

for any compact set E.

The characterization of γ_+ in terms of curvature is contained in the following theorem. Let $\Sigma(E)$ be the set of Radon measures supported on E that have $1-$ linear growth.

Theorem 9. *For any compact set $E \subset \mathbb{C}$, we have*

$$\begin{aligned}\gamma_+(E) \approx &\sup\{\mu(E) : \mu \in \sigma(E), \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1 \forall\varepsilon > 0\},\\ &\sup\{\mu(E) : \mu \in \sigma(E), \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \leq \mu(E) \forall\varepsilon > 0\},\\ &\sup\{\mu(E) : \mu \in \sigma(E), c^2(\mu) \leq \mu(E)\},\\ &\sup\{\mu(E) : \mu \in \sigma(E), \|\mathcal{C}_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1\}.\end{aligned} \tag{5}$$

Since the last term in the train of approximate equalities is semi-additive, the semi-additivity of γ_+ follows.

2. Ideas involved in proofs of theorems 3 and 4

A. A *Tb* theorem of Nazarov, Treil and Volberg. Let $\mathcal{D}_0$ be the lattice of dyadic squares from $\mathbb{C}$, and denote by $\mathcal{D}(w) = w + \mathcal{D}_0$. Then the following Tb theorem was proved in [4] (see also ([3],chapter 5)).

Theorem 10. *Let μ be a finite measure supported on a compact set $F \subset \mathbb{C}$. Suppose that there exist a complex measure ν and for each $w \in \mathbb{C}$ two subsets $H_{\mathcal{D}(w)}, T_{\mathcal{D}(w)} \subset \mathbb{C}$ made up of dyadic squares from $\mathcal{D}(w)$ such that*

(1) *Every ball B_r of radius r such that $\mu(B_r) > c_0 r$ is contained in $\cap_{w\in\mathbb{C}} H_{\mathcal{D}(w)}$.*

(2) $\nu = b\mu$ *where* b *is some function such that* $\|b\|_\infty \leq c_b$.
(3) $\int_{\mathbb{C}\setminus H_{\mathcal{D}(w)}} \mathcal{C}_*\nu \, d\mu \leq c_*\mu(F)$ *for all* $w \in \mathbb{C}$.
(4) *If* $Q \in \mathcal{D}(w)$ *is such that* $Q \not\subset T_{\mathcal{D}(w)}$ *then* $\mu(Q) \leq c_{acc}|\nu(Q)|$ *(i.e.* Q *is an accretive square).*
(5) $\mu(H_{\mathcal{D}(w)} \cup T_{\mathcal{D}(w)}) \leq \delta_0\mu(F)$ *for all* $w \in \mathbb{C}$ *and some* $\delta_0 < 1$.

Then, there exists a subset $G \subset F\setminus \cap_{w\in\mathbb{C}} \left(H_{\mathcal{D}(w)} \cup T_{\mathcal{D}(w)}\right)$ *such that*
(i) μ *satisfies* $\mu(G) \geq c_1^{-1}\mu(F)$,
(ii) $\mu\lfloor G$ *has* c_0- *linear growth.*
(iii) the Cauchy transform is bounded in $L^2(\mu\lfloor G)$. *The constant* c_1 *and the bound for the* $L^2(\mu\lfloor G)$ *norm depend only on* c_0, c_b, c_*, c_{acc} *and* δ_0.

Condition (1) in the theorem implies that $\mu\lfloor\mathbb{C}\setminus\cap_{w\in\mathbb{C}}H_{\mathcal{D}(w)}$ has linear growth, while condition (3) ensures that $C_*\nu$ is not too big in $\mathcal{C}\setminus H_{\mathcal{D}(w)}$. The statement (4) is an accrevity condition for ν, while the last condition (5) asserts that the bad sets $H_{\mathcal{D}(w)}$ and $T_{\mathcal{D}(w)}$ are not too big. While the theorem has been stated for the Cauchy transform, it is true for singular integral operators that are more general. Consequently, the proof in [4], [3] do not exploit the connection between the Cauchy transform and the curvature.

Comparability of γ_+ and γ.. An immediate consequence of the Tb theorem of Nazarov, Treil and Volberg is the following (easier) statement regarding the comparability of the capacities γ and γ_+. This result is due to David.

Theorem 11. *Let* $E \subset \mathbb{C}$ *be compact with* $H^1(E) < \infty$ *and* $\gamma(E) > 0$. *Then* $\gamma_+(E) > 0$.

A direct application of the Tb theorem turns out to be insufficient to prove theorem 8. The steps involved first include construction of a measure ν, we begin by approximating E at a certain *intermediate scale* (call this approximation F) with disjoint finitely many cubes Q_i, construct suitable measures ν_i supported on these squares Q_i and consider the measure $\nu = \sum_i \nu_i$. The hope is that if the squares are constructed sufficiently largely, the variation $|\nu|$ will be sufficiently small; there is a competition to win: if the squares are too big, then we might lose $\gamma_+(F) \leq c\gamma_+(E)$. The result of this construction is a complex measure ν supported on F satisfying $|\nu(F)| \approx |\nu|(F) = \gamma(E)$. Taking a suitable measure μ with $\mu(F) \approx \gamma(E)$ and $supp(\mu) \supset supp(\nu)$, prepares us well for an application of the $T(b)$ theorem of [4]. The Tb theorem in turn implies that $\gamma_+(F) \geq c^{-1}\mu(E)$, and we will be done. An induction argument on the size of a rectangle R is used to show that $\gamma(E \cap R) \approx \gamma_+(E \cap R)$.

References

[1] Garnett, John. *Analytic capacity and measure.* Springer, 1972.
[2] Tolsa, Xavier. *Painlev's problem and the semiadditivity of analytic capacity.* Acta Mathematica 190.1 (2003): 105-149.
[3] Tolsa, Xavier. *Analytic capacity.* Springer International Publishing, 2014.
[4] Nazarov F.,Treil S, and Volberg A., *The $T(b)$ theorem on non-homogeneous spaces that proves a conjecture of Vitushkin* Preprint 519 CRM Barcelona 2002.

Department of Mathematics, Indiana University, Bloomington, Indiana 47405
E-mail address: rvenkatr@indiana.edu