PROCEEDINGS OF THE INTERNET ANALYSIS SEMINAR ON MULTIPARAMETER HARMONIC ANALYSIS

Contents

Overview of the Workshop

This workshop was part of the Internet Analysis Seminar that is the education component of the National Science Foundation – DMS # 0955432 held by Brett D. Wick. The Internet Analysis Seminar consists of three phases that run over the course of a standard academic year. Each year, a topic in complex analysis, function theory, harmonic analysis, or operator theory is chosen and an internet seminar will be developed with corresponding lectures. The course will introduce advanced graduate students and post-doctoral researchers to various topics in those areas and, in particular, their interaction.

This was a workshop that focused on multiparameter harmonic analysis. Each of the participants was assigned one of the following papers to read:

[1] Carlos Cabrelli, Michael T. Lacey, Ursula Molter, and Jill C. Pipher, *Variations on the theme of Journé's lemma*, Houston J. Math. **32** (2006), no. 3, 833–861.

[2] Sun-Yung A. Chang and Robert Fefferman, *The Calderón-Zygmund decomposition on product domains*, Amer. J. Math. **104** (1982), no. 3, 455–468.

[3] ______, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), no. 1, 179–201.

[4] Yongsheng Han, Ji Li, and Guozhen Lu, *Duality of multiparameter Hardy spaces H^p on spaces of homogeneous type*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), no. 4, 645–685.

[5] Michael Lacey and Jason Metcalfe, *Paraproducts in one and several parameters*, Forum Math. **19** (2007), no. 2, 325–351.

[6] Michael T. Lacey, Stefanie Petermichl, Jill C. Pipher, and Brett D. Wick, *Multiparameter Riesz commutators*, Amer. J. Math. **131** (2009), no. 3, 731–769.

[7] Henri Martikainen, *Representation of Bi-Parameter Singular Integrals by Dyadic Operators*, preprint (2011), available at `http://arxiv.org/abs/1110.1890`.

[8] Sandra Pott and Paco Villarroya, *A $T(1)$ theorem on product spaces*, preprint (2011), available at `http://arxiv.org/abs/1105.2516`.

[9] Sergei Treil, *H^1 and dyadic H^1*, Linear and complex analysis, Amer. Math. Soc. Transl. Ser. 2, vol. 226, Amer. Math. Soc., Providence, RI, 2009, pp. 179–193.

They were then responsible to prepare two one-hour lectures based on the paper and an extended abstract based on the paper. This proceeding is the collection of the extended abstract prepared by each participant. The following people participated in the workshop:

Gagik Amirkhanyan	Georgia Institute of Technolgoy
Theresa C. Andersson	Brown University
Kelly Bickel	Washington University - St. Louis
Amalia Culiuc	Brown University
Tim Feguson	Vanderbilt University
Jingguo Lai	Brown University
Mishko Mitkovski	Georgia Institute of Technology
Eyvindur Palsson	University of Rochestor
Zachary J. Smith	University of Tennessee, Knoxville
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VARIATIONS ON THE THEME OF JOURNÉ'S LEMMA

CARLOS CABRELLI, MICHAEL LACEY, URSULA MOLTER, JILL PIPHER

presented by Amalia Culiuc

Abstract. Journé's lemma, proved in 1987 [3], is an important tool in the product BMO theory and the control of Carleson measures. Previous work by Carberry and Seeger [1], Fefferman and Pipher [2], Muscalu, Pipher, Tao, and Thiele [4][5], Pipher [6], and others has provided various extensions and applications of the lemma, although its precise role in the product theory is still not well defined. The following results provide an overview of what is known about Journé's lemma and its possible refinements.

1. Overview of Results

We begin by defining the concepts present in the lemma. Let $\mathcal{I}$ be a collection of intervals in $\mathbb{R}$. $\mathcal{I}$ is said to be a *grid* if any two intervals in this collection are either nested or disjoint, and any collection of intervals in $\mathbb{R}$ with this property is said to have *the grid property.* Note that in particular $\mathcal{D}$, the set of *dyadic intervals* in $\mathbb{R}$, defined by

$$\mathcal{D} = \{[j2^k, (j+1)2^k) : \ j, k \in \mathbb{Z}\},$$

has the grid property, since if $I, I' \in \mathcal{D}$, then $I \cap I' \in \{I, I', \emptyset\}$.

Based on the definition of dyadic intervals, in d dimensions we can define the collection of *dyadic rectangles*:

$$\mathcal{D}^d = \left\{R = \prod_{j=1}^{d} R_j : R_j \in \mathcal{D}\right\}.$$

Now let $\mathcal{U}$ be a subset of $\mathcal{D}^d$. The *shadow* of $\mathcal{U}$ is the set

$$\mathrm{sh}(\mathcal{U}) = \bigcup_{R \in \mathcal{U}} R.$$

Journé's lemma also uses the concept of embeddedness of dyadic rectangles $R \in \mathcal{U}$ into $\mathrm{sh}(\mathcal{U})$. Intuitively, the embeddedness is a measure of how "close" to the boundary of $\mathrm{sh}(U)$ a rectangle is. To give a more precise definition, we introduce the notion of dilations.

In one dimension, for an interval R and $\lambda > 0$, the *dilation* of R by λ, denoted by λR, is the interval with the same center as R, but length equal to $\lambda|R|$. In d dimensions, for any $R \in \mathcal{D}^d$ and $\overrightarrow{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_d)$, the dilation of R by $\overrightarrow{\lambda}$ is the rectangle

$$\mathrm{Dil}_{\overrightarrow{\lambda}} R = \otimes_{j=1}^{d} \lambda_j R_j.$$

For any collection of dyadic rectangles $\mathcal{U}$ such that $\mathrm{sh}(\mathcal{U})$ has finite measure, we also define $\mathrm{Enl}(\mathcal{U})$, the *enlarged set* of $\mathcal{U}$ by

$$\mathrm{Enl}(\mathcal{U} = \left\{M \mathbb{1}_{\mathrm{sh}(\mathcal{U})} > \frac{1}{2}\right\},$$

Here M represents the strong maximal function,

$$Mf(x) = \sup_{x \in R} \frac{1}{|R|} \int_R \|f(y)\| dy.$$

Finally, we define two parameter *embeddedness*: if $\mathcal{U}$ is a set of two dimensional rectangles $R = R_1 \times R_2$, then for any R, its embeddedness in $\mathcal{U}$ is

$$\mathrm{emb}(R;\mathcal{U}) = \sup \big\{ \mu > 1 : (\mu R_1) \times R_2 \subset \mathrm{Enl}(\mathcal{U}) \big\}.$$

Having introduced the necessary concepts, we are now ready to state Journé's lemma in its original formulation:

Lemma 1. *Let $\varepsilon > 0$ be fixed and let $\mathcal{U}' \subset \mathcal{U}$ be any subcollection of pairwise incomparable dyadic rectangles in $\mathcal{U}$. Then the following inequality holds:*

$$\sum_{R \in \mathcal{U}'} emb(R, \mathcal{U})^{-\varepsilon} |R| \lesssim |sh(\mathcal{U}')|,$$

where the implied constant depends only upon ε.

As an application of this lemma, for any map $\alpha : \mathcal{D}^d \to \mathbb{R}_+$, define

$$\|\alpha\|_{CM} = \sup_{\mathcal{U}} |\mathrm{sh}(\mathcal{U})|^{-1} \sum_{R \in \mathcal{U}} \alpha(R).$$

Although it is not immediately obvious, CM stands for "Carleson measure". The standard definition of a Carleson measure μ_α on $\mathbb{R}^d \times \mathbb{R}^d_+$ is

$$\mu_\alpha = \sum_{R \in \mathcal{D}^d} \alpha(R) \delta_{R \times \|R\|},$$

where $\|R\| = (|R_1|, |R_2|, ..., |R_d|)$. If for any set $U \subset \mathbb{R}^d$ we define the tent over U , $\mathrm{Tent}(U) \subset \mathbb{R}^d \times \mathbb{R}^d_+$ by

$$\mathrm{Tent}(U) = \bigcup_{\substack{R \in \mathcal{D}^d \\ R \subset U}} R \times [0, |R_1|] \times ... \times [0, |R_d|],$$

then we can state the following inequality for all sets of finite measures $U \subset \mathbb{R}^d$:

$$\mu_\alpha(\mathrm{Tent}(U)) \leq \|\alpha\|_{CM} |U|.$$

Furthermore, we can state a discrete form of the Carleson Embedding Theorem.

For a map $\alpha : \mathcal{D}^d \to [0, \infty)$, consider the operator

$$T_\alpha f = \sum_{I \in \mathcal{D}^d} \alpha(R) \mathbb{1}_R \int_R f(y) dy.$$

Theorem 2. *For all $1 < p < \infty$,*

$$\|T_\alpha\|_p \simeq \|\alpha\|_{CM}.$$

This explains the relationship between the definition above and Carleson measures.

Notice now that in the expression for $\|\alpha\|_{CM}$, the supremum is taken over general subsets $U \subset \mathbb{R}^d$ that have finite measure. In one dimension, by an additional argument, it is possible to restrict the supremum to intervals. In higher dimensions, one would expect to be able to restrict the definition to the supremum over rectangles. However, this is not the case. Denote the supremum over rectangles by $\|\alpha\|_{CM(\mathrm{rec})}$. In 2 or more dimensions, it is, for instance, possible to define α such that $\|\alpha\|_{CM} = 1$, but $\|\alpha\|_{CM(\mathrm{rec})} < \varepsilon$. The importance of Journé's lemma in this setting is that in two dimensions, its application leads to a relation between the CM norm and the rectangular norm, thus providing a way to control $\|\alpha\|_{CM}$.

The following is a corollary of Lemma 1:

Corollary 3. *Let $\varepsilon > 0$ and $\mu > 1$ be fixed, and let $\mathcal{U}$ be any collection of rectangles in the plane whose shadow has finite area. Let $\mathcal{U}_\mu \subset \mathcal{U}$ be a collection of rectangles such that for all $R \in \mathcal{U}_\mu$, $emb(R,\mathcal{U}) \simeq \mu$. Then*

$$\|\alpha|_{\mathcal{U}_\mu}\|_{CM} \lesssim \mu^{\varepsilon}\|\alpha\|_{CM(rec)}.$$

In dimension 3 and above, a similar result would require a generalization of Journé's lemma. In fact, the lemma admits multiple types of refinement. In what follows we describe these possible refinements and state some related results.

To begin with, we can obtain a variant of the lemma by redefining $\mathrm{emb}(R,\mathcal{U})$, the measure of embeddedness used in the original form. Consider, for instance, the notion of embededness obtained by expanding all sides of a rectangle simultaneously.

Define the enlarged sets inductively by

$$\mathrm{Enl}_2(\mathcal{U}) = \left\{ M\mathbb{1}_{\mathrm{sh}(\mathcal{U})} > \frac{1}{16} \right\}$$
$$\mathrm{Enl}_{j+1}(\mathcal{U}) = \{\mathrm{Enl}_2(\mathrm{Enl}_j(\mathcal{U})) \text{ for } j > 2\}$$

Then for a dyadic rectangle $R \in \mathcal{U}$, the embeddedness is given by

$$\mathrm{emb}(R, \mathrm{Enl}_j(\mathcal{U})) = \sup\{\mu \geq 1 : \ \mu R \subset \mathrm{Enl}_j(\mathcal{U})\}, \ j \geq 2.$$

Note that this new measure of embeddedness is essentially smaller than the original. This case leads to the following version of Journé's lemma:

Lemma 4. *Let $\varepsilon > 0$ be fixed and let $\mathcal{U}$ be a collection of rectangles whose shadow has finite measure in the plane. Then for all subcollections $\mathcal{U}' \subset \mathcal{U}$,*

$$\sum_{R \in \mathcal{U}'} emb(R, Enl_2(\mathcal{U}))^{-\varepsilon}|R| \lesssim |sh(\mathcal{U}')|,$$

and the implied constant depends only on ε.

An even more general version of the statement is obtained by allowing the dilation of both sides of a rectangle by different constants. For a vector (μ_1, μ_2), set

$$\mathrm{emb}(R,\mathcal{U}) = \sup\{\mu_1\mu_2 : \mathrm{Dil}_{(\mu_1,\mu_2)}R \subset \mathrm{Enl}_2(\mathcal{U}), \mu_1, \mu_2 \geq 1\}.$$

Then we have:

Lemma 5. *Let $\varepsilon > 0$ be fixed and let $\mathcal{U}$ be a collection of rectangles in the plane whose shadow has finite measure. Then for all collections $\mathcal{U}' \subset \mathcal{U}$ of rectangles which are maximal (with respect to inclusion),*

$$\sum_{R \in \mathcal{U}'} emb(R,\mathcal{U}')^{-\varepsilon}|R| \lesssim |sh(\mathcal{U}')|,$$

where the implied constant depends only on ε.

A second type of refinement of Journé's lemma relies on redefining the enlarged set, $\mathrm{Enl}(\mathcal{U})$. Notice that in the original statement we have $|\mathrm{Enl}(\mathcal{U})| \le K|\mathrm{sh}(\mathcal{U})|$, where K is a constant with $K > 1$. It is possible, however, to redefine $\mathrm{Enl}(\mathcal{U})$ for any arbitrary $\delta > 0$ such that, with this new definition, $|\mathrm{Enl}(\mathcal{U})| \le (1+\delta)|\mathrm{sh}(\mathcal{U})|$. The case of interest is, of course, $\delta > 0$ being arbitrarily small. For a set V, take

$$\mathrm{emb}(R, V) = \sup\{\mu \ge 1 : \ \mu R \subset V\}, \ R \in \mathcal{U}.$$

With this definition, we have the following result:

Lemma 6. *For any $0 < \delta, \varepsilon < 1$, there exists a constant $K_{\delta,\epsilon}$ such that for all collections of two dimensional rectangles $\mathcal{U}$ whose shadow has finite measure, there exists a set $V \subset sh(\mathcal{U})$ such that $|V| < (1+\delta)|sh(\mathcal{U})|$, and for any subcollection $\mathcal{U}' \subset \mathcal{U}$,*

$$\sum_{R \in \mathcal{U}'} emb(R, V)^{-\varepsilon}|R| \lesssim |sh(\mathcal{U}')|,$$

Finally, a third possible refinement is the extension to three or more parameters. The advantage of the two parameter case is the simplicity of the relation between two intersecting rectangles: if R and R' are distinct two dimensional rectangles that intersect and are not comparable, then the two sides of the rectangles must be in reverse order with respect to inclusion. In three or more dimensions, the intersection of rectangles implies more complicated relations. There are, however, methods to pass to higher numbers parameters and also to replace rectangles by other sets in those constructions. These extensions are particularly important, as they apply to the construction of the three parameter BMO space from the two parameter BMO space.

To introduce a variant of Journé's lemma in three or more parameters, we can measure the embeddedness by considering dilations in only one coordinate, as in the original statement. However, in this case, it is necessary to form sums over more general sets than rectangles. Define U to be a subset of $\mathbb{R}^d$ and let $\mathcal{U}$ be a set of maximal dyadic rectangles contained in U. Then define

$$\begin{aligned} \mathrm{Enl}(\mathcal{U}) &= \{M\mathbb{1}_{\mathrm{sh}(\mathcal{U})} > 1/2\} \\ \mathrm{emb}(R,\mathcal{U}) &= \sup\{\mu \ge 1 : \mathrm{Dil}_{(\mu,1,\dots,1)}R \subset \mathrm{Enl}(\mathcal{U})\}. \end{aligned}$$

Also, for a subcollection $\mathcal{U}' \subset \mathcal{U}$, a fixed $j \in \mathbb{N}$, and a dyadic interval $I \in \mathcal{D}$, let

$$F(I, j,\mathcal{U}') = \bigcup\{I \times R' : I \times R' \in \mathcal{U}', 2^{j-1} \le \mathrm{emb}(I \times R',\mathcal{U}) < 2^j\}.$$

Then we can state the following lemma:

Lemma 7. *Let $\varepsilon > 0$ be fixed. Then for all collections of rectangles $\mathcal{U}$ whose shadow has finite measure and all subcollections $\mathcal{U}' \subset \mathcal{U}$,*

$$\sum_{j=1}^{\infty}\sum_{I \in \mathcal{D}} 2^{-\varepsilon j}|F(I, j,\mathcal{U}')| \lesssim |sh(\mathcal{U}')|.$$

Furthermore, for any integer $n > 1$ and $1 < p < \infty$,

$$\|\sum_{j=1}^{\infty}\sum_{I \in \mathcal{D}} 2^{-\varepsilon j}(M\mathbb{1}_{F(I,j,\mathcal{U}')})^n\|_p \lesssim |sh(\mathcal{U}')|^{1/p}.$$

It is also possible to state a version of the lemma with summation over rectangles by introducing a different notation. In d dimensions, let

$$\mathrm{Enl}(\mathcal{U}) = \{M\mathbb{1}_{\mathrm{sh}(\mathcal{U})} > 1/2d\}.$$

Then for any integer $1 \leq j \leq d$, set

$$\mathrm{emb}(j, R) = \sup\{\mu \geq 1 : R_1 \times ... \times \mu R_j \times ... \times R_d \subset \mathrm{Enl}(\mathcal{U})\}.$$

With this notation, we have a version of the lemma, that restricts to rectangles and weighs each $|R|$ by the largest embeddedness. The advantage of such a formulation is that rectangles are simpler objects to work it. Nevertheles, in this case, the expression for embeddedness becomes more complicated than before, since it is now defined as a product. This result can be stated as follows:

Lemma 8. *For any $d \geq 3$, any $0 < \varepsilon < 1$, all collections $\mathcal{U}$ of pairwise incomparable dyadic rectangles R in $\mathbb{R}^d$, and all subcollections $\mathcal{U}' \subset \mathcal{U}$, we have*

$$\sum_{R \in \mathcal{U}'} |R| \prod_{j=1}^{d-1} emb(j, R)^{-\varepsilon} \lesssim |sh(\mathcal{U}')|.$$

References

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THE CALDERON-ZYGMUND DECOMPOSITION ON PRODUCT DOMAINS

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presented by Gagik Amirkhanyan

ABSTRACT. In section 1 the atomic decomposition of H^1 of the product of upper half space is studied. In section 2 the atomic decomposition is used to get the Calderon-Zygmund decomposition of H^1, in section 3 it is demonstrated how the result of section 1 can be modified to give H^p-atomic decomposition for all $0 < p \leq 1$.

1. Atomic decomposition of H^1

In what follows we work exclusively with the domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ with its boundary $\mathbf{R}^2$ A point in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ will be denoted (t, y) where $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2),\ y_i \geq 0$. We will use the following notations:

$\psi(t) \in C^1(\mathbf{R})$ is an even function supported on $[-1, 1]$ and $\int_{-1}^{1} \psi(t)dt = 0$

$\psi_y(t) = (1/y)\psi(t/y)$ for $y > 0$

$\psi_y(t) = \psi_{y_1}(t_1)\psi_{y_2}(t_2)$ for $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2),\ y_i \geq 0.$

If f is a function defined on $\mathbf{R}^2$ then $f(t, y)$ will, by definition, mean

$$f(t, y) = f * \psi_y(t).$$

Further, if $x = (x_1, x_2) \in \mathbf{R}^2$, $\Gamma(x)$ will denote the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$ where

$$\Gamma(x_i) = \{(t_i, y_i) \in \mathbf{R}^2_+ : |x_i - t_i| < y_i\},\ i = 1, 2$$

Given a function f on $\mathbf{R}^2$ we define its double S-function by

$$S^2(f)(x) = \int\int_{\Gamma(x)} |f(t, y)|^2 \frac{dtdy}{y_1^2 y_2^2}.$$

For $1 < p < \infty$ it's know that

$$\|S(f)\|_p \leq c_p \|f\|_p$$

We define functions in $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, $0 < p < \infty$ as those functions f with $S(f) \in L^p(\mathbf{R}^2)$ and we set

$$\|f\|_{H^p} = \|S(f)\|_p.$$

This definition of H^p spaces is equivalent to the one defined via boundary values of functions of bi-holomorphic functions on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$.

Definition (on $\mathbf{R}$) An *atom* is a function $a(x)$ supported on an interval I such that

$$\int_I a(x)dx = 0 \quad \text{and} \quad \|a(x)\|_\infty \leq \frac{1}{|I|}$$

Theorem (R. Coifman) $f \in H^1(\mathbf{R})$ if and only if f can be written as $f = \sum \lambda_k a_k$ where a_k are atoms and $\lambda_k \geq 0$ satisfy $\sum |\lambda_k| \leq A\|f\|_{H^1}$.

Definition (on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$) An *atom* is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set Ω of finite measure such that

(1) $\|a\|_2 \leq \frac{1}{|\Omega|^{1/2}}$

(2) a can be further decomposed into *elementary particles* a_R as follows:

(i) $a_R = \sum_R a_R$ where a_R is supported in the triple of distinct dyadic rectangles $R \subset \Omega$ (say $R = I \times J$)

(ii) $\int_I a(x_1, x_2')dx_1 = \int_J a(x_1', x_2)dx_2 = 0$ for each $x_1' \in I,\ x_2' \in J$

(iii) a_R is C^1 with $\|a_R\|_\infty \le d_R$,

$$\left\|\frac{\partial a_R}{\partial x_1}\right\|_\infty \le \frac{d_R}{|I|}, \quad \left\|\frac{\partial a_R}{\partial x_2}\right\|_\infty \le \frac{d_R}{|J|}$$

with $\sum d_R^2 |R| \le A/|\Omega|$.

Theorem 1. $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ if and only if f can be written as $f = \sum \lambda_k a_k$ where a_k are atoms and $\lambda_k \ge 0$ satisfy $\sum \lambda_k \le A\|f\|_{H^1}$.

2. Calderon-Zygmund Decomposition

First we formulate the classical Calderon-Zygund Decomposition for $L^1(\mathbf{R}^n)$ functions.

Theorem (Calderon-Zygmund Decomposition). $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$ then there exists a disjoint collection of dyadic cubes $\{Q_i : i = 1, 2, ...\}$ such that

$$\alpha < \frac{1}{|Q_i|}\int_{Q_i} |f(x)|dx \le 2^n\alpha, \quad i = 1, 2, ...,$$

and

$$f(x) \le \alpha \quad \text{for a.e. } x \in \mathbf{R}^n \setminus \cup_{i=1}^\infty Q_i.$$

Given f as above, we can write f as the sum of a "good" function g and a "bad" function b, $f = g + b$, where $g \le 2^n\alpha$ and b is supported on $\cup_{i=1}^\infty Q_i$ with

$$\int_{Q_i} |b(x)|dx \le 2^n\alpha|Q_i| \quad \text{and} \quad \int_{Q_i} b(x)dx = 0.$$

Next we show Calderon-Zygmund Decomposition on product domains.

Calderon-Zygmund Lemma. Let $\alpha > 0$ be given and $f \in L^p(\mathbf{R}^2)$, $1 < p < \infty$. Then we may write $f = g + b$ where $g \in L^2(\mathbf{R}^2)$ and $b \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ with

$$\|g\|_2^2 \le \alpha^{2-p}\|f\|_p^p \quad \text{and} \quad \|b\|_{H^1} \le c\alpha^{1-p}\|f\|_p^p,$$

where c is a universal constant.

Remark It is shown that there exist constants λ_k and atoms b_k with $\sum |\lambda_k| \le \alpha^{1-p}\|f\|_p^p$ and $f = g + \sum \lambda_k b_k$. Theorem 1 implies that $b = \sum \lambda_k b_k$ is in H^1.

Using the Calderon-Zygund decomposition, we obtain the following:

Theorem 2. Let T be a linear operator which is bounded from $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ to $L^1(\mathbf{R}^2)$ and bounded on $L^2(\mathbf{R}^2)$. Then T is bounded on $L^p(\mathbf{R}^2)$ for all $1 < p < 2$.

Proof. Let $f \in L^p(\mathbf{R}^2)$ and $\alpha > 0$. According to the Calderon-Zygmund Lemma, we may write $f = g + b$ where

$$\|g\|_2^2 \le \alpha^{2-p}\|f\|_p^p \quad \text{and} \quad \|b\|_{H^1} \le c\alpha^{1-p}\|f\|_p^p$$

$$
\begin{aligned}
m\{|Tf| > \alpha\} &\le m\{|Tg| > \alpha/2\} + m\{|Tb| > \alpha/2\} \\
&\le c\left(\frac{1}{\alpha^2}\|Tg\|_2^2 + \frac{1}{\alpha}\|Tb\|_1\right) \\
&\le c\left(\frac{1}{\alpha^2}\|g\|_2^2 + \frac{1}{\alpha}\|b\|_{H^1}\right) \\
&\le c\frac{1}{\alpha^p}\|f\|_p^p.
\end{aligned}
$$

T is therefore weak-type (p,p) for $1 < p < 2$ and according to the Marcinkiewicz Theorem T is bounded on L^p in the same range of p.

3. Atomic Decomposition on $H^p(\mathbf{R}),\ 0 < p < 1$

We extend the same method to give the atomic decomposition for $H^p(\mathbf{R}),\ 0 < p < 1$. Note that for the same reasons as in the classical upper half plane, we need some higher orders vanishing property of a *p-atom* for H^p-decomposition. First we recall the classical case.

Definition (on $\mathbf{R}$) An *p-atom* is a function $a(x)$ supported on an interval I such that

$$
\|a(x)\|_\infty \le \frac{1}{|I|^{1/p}} \quad \text{and} \quad \int_I a(x)x^k dx = 0
$$

for all $0 \le k \le \frac{1}{p} - 1$.

Theorem (R. Coifman) $f \in H^p(\mathbf{R})$ if and only if f can be written as $f = \sum \lambda_k a_k$ where a_k are p-atoms and $\lambda_k \ge 0$ satisfy

$$
A\|f\|_{H^p}^p \le \sum |\lambda_k|^p \le B\|f\|_{H^p}^p.
$$

In the case of the product domain we have the following definition.

Definition (on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$) An *p-atom* is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set Ω of finite measure such that

(1) $\|a\|_2^2 \le |\Omega|^{1-2/p}$

(2) a can be further decomposed into *elementary particles* a_R as follows:

  (i) $a_R = \sum_R a_R$ where a_R is supported in the triple of distinct dyadic rectangles $R \subset \Omega$ (say $R = I \times J$)

  (ii) $\int_I a(x_1, x_2')x_1^k dx_1 = \int_J a(x_1', x_2)x_2^k dx_2 = 0$ for each $x_1' \in I,\ x_2' \in J$ and $0 \le k \le k(p)$, where $k(p) = 2/p - 3/2$

  (iii) a_R is C^m with $\|a_R\|_\infty \le d_R$,

$$
\left\|\frac{\partial^m a_R}{\partial x_1^m}\right\|_\infty \le \frac{d_R}{|I|^m}, \quad \left\|\frac{\partial^m a_R}{\partial x_2^m}\right\|_\infty \le \frac{d_R}{|J|^m},\ 0 < m \le k(p) + 1
$$

  with $\sum d_R^2 |R| \le A|\Omega|^{1-2/p}$.

With this definition of p-*atoms*, we can state the parallel result of Theorem 1 for H^p atomic decomposition.

Theorem 3. $f \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ then we may write $f = \sum \lambda_k a_k$ where a_k are p-atoms and $\lambda_k \geq 0$ satisfy $\sum \lambda_k^p \leq c_p \|f\|^p_{H^p}$.

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A CONTINUOUS VERSION OF THE H^1-BMO DUALITY

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presented by Zachary J. Smith

1. Introduction and 1-dimensional Background

The question of duality is a natural one: what are the linear functionals acting on a space. For certain spaces, namely topological vector spaces, one can ask a related question that takes advantage of the further structure given by a topology. This is, what are the continuous linear functionals?

The question of the dual of the real Hardy space $H^1(\mathbb{R})$ is well-known to be the space of functions of bounded mean oscillation, BMO.

Theorem 1. *The space BMO is the dual space of the Hardy space $H^1(\mathbb{R})$. The pairing is what you expect: $\langle f, g\rangle = \int\limits_{-\infty}^{\infty} f(x)g(x)dx$, defined for $g \in BMO$ and $f \in C^\infty$, a dense subset of H^1.*

The natural question that follows is does this result carry over into higher dimensional spaces, for example what is the dual of $H^1(\mathbb{D}^2)$? The answer was given by Chang and Fefferman in 1980. Their theorem is as follows: the dual space of $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the space product BMO.

We note that product BMO does not have the mean oscillation property you would naively expect. We will detail below precisely what "product" BMO means.

In the following, we will consider the obvious questions:

1: What do we mean by H^1, and what is the proper analogy on the bidisc?
2: What is BMO, and what its higher-dimensional analog?
3: Does this theorem lift to the bidisc?

Note in everything we work on the upper half-space rather than the bidiisc. This allows the arguments, which are mostly geometric in nature, to carry through naturally and easily.

2. Hardy Space and BMO Preliminaries

Definition 1. *The Hardy space $H^1(\mathbb{R})$ is defined to be the boundary values of $H^1(\mathbb{R}^2_+)$. This latter space is defined as functions $f \in Hol(\mathbb{R}^2_+)$ that satisfy $\sup_{y>0} \int f(x+iy)dx < \infty$.*

In extending to higher dimensions, the definition is a bit more complicated. Note that a function is biharmonic if it is harmonic in each variable separately.

Definition 2. *The space $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ to be defined to be biharmonic functions u having their nontangential maximal functions $N(u)$ in $L^1(\mathbb{R}^2)$.*

We note that this is one of several equivalent definitons of this space; a good reference for this is Gundy and Stein.

To work in higher dimensions we will first need a few preliminaries. Let $\psi \in C^1(\mathbb{R})$ be an even function supported on $[-1,1]$ with mean 0. Then for $x = (x_1, x_2)$ in $\mathbb{R}^2$ and $y = (y_1, y_2)$ in $(\mathbb{R}_+)^2$, define

$$\psi_y(x) = \frac{1}{y_1 y_2}\psi\left(\frac{x_1}{y_1}\right)\psi\left(\frac{x_2}{y_2}\right)$$

The function ψ will be fixed, and normalized so that $\int\limits_0^\infty \left|\hat{\psi}(\xi)\right|^2 \frac{d\xi}{\xi} = 1$. We define the extension of a function f defined on the boundary to be $f(x,y) = f * \psi_y$. We hence have for $f \in H^1$

$$f(x,y) = \iint\limits_{(t,y)\in\mathbb{R}^2_+\times\mathbb{R}^2_+} f(t,y)\psi_y(x-t)\frac{dtdy}{y_1 y_2}$$

In one dimension the sweep (or balayage) of a measure is useful: the sweep of a the absolute value of a measure is in BMO if the measure itself in Carleson. In higher dimensions we will have the following important analogue, which Chang and Fefferman call the "double-S" function.

$$S^2(f)(x_1,x_2) = \iint_{(t,y)\in\Gamma(x_1)\times\Gamma(x_2)} |f(t,y)|^2 \frac{dtdy}{y_1^2 y_2^2}$$

Here $\Gamma(x_i)$ is the usual nontangential approach region. This function will play a vital role in our computations.

The last preliminary we shall need is the 'rectangle' function. For $f \in H^1$ and R a rectangle , define

$$f_R(x,y) = \iint\limits_{(t,y)\in R_+} f(t,y)\psi_y(x-t)\frac{dtdy}{y_1 y_2}$$

The region R_+ is part of a Carleson box. This function will be key in writing down an atomic decomposition.

3. Atomic Decompositions

One useful property of $H^1(\mathbb{R})$ used in the proof is that it has an atomic decomposition, as follows:

Definition 3. *An H^1 atom is a function $a(x)$ that satisfies:*

- *The support of a lies in a bounded interval I*
- $\int_I a(x)dx = 0$
- $||a||_\infty \le \frac{1}{|I|}$

Given this definition, the useful fact is that for $f \in H^1$ we have $f = \sum \lambda_k a_k(x)$. Moreover $\sum |\lambda_k| \le C||f||_{H^1}$.

As we have named the higher dimensional space H^1, we ask if some of the properties carries over. It takes some work, but given the right definition of atom, we shall have this result again.

Definition 4. *An atom on $\mathbb{R}^2$ is a function $a(x_1, x_2)$ satisfying:*

- *The support of a is contained in an open set Ω*
- *$\int_I a(x_1, x_2)dx_1 = 0$ where I is any component interval of any x_1 cross-section of Ω.*
- *$\int_J a(x_1, x_2)dx_2 = 0$ where J is any component interval of any x_2 cross-section of Ω.*
- *$a = \sum_R a_R$, where each a_R is supported on a rectangle $R \subset \Omega$.*

Technical details left for the talk: The rectangles are a collection of maximal dyadic rectangles, and the functions a_R also have some smoothness requirements.

With this in place, we get the following:

Theorem 2. *(Chang-Fefferman): Let $f \in H^1$. Then f can be written as $f = \sum \lambda_k a_k$ where the a_k are atoms and $\lambda_k \geq 0$ satisfy $\sum \lambda_k \leq C||f||_{H^1}$.*

Sketch of Proof: The proof of this theorem contains much of the geometric content used in the larger duality theorem. The key ingredient is finding a collection of maximal dyadic rectangles, and using the 'rectangle' function as defined above to be the atom.

4. Candidates for the dual: Product BMO

We again begin in 1 dimension. For a function $\phi \in L^1_{loc}(\mathbb{R})$, we say ϕ is of bounded mean oscillation (BMO) if

$$\sup_I \frac{1}{|I|} \int_I |\phi - \phi_I|^2 dx = ||\phi||_*^2 < \infty$$

. Here the supremum is taken over all finite intervals I, and ϕ_I is the average value of ϕ over such an interval.

By replacing intervals with rectangles, one would hope to get an analogy for BMO. Unfortunately such functions in "rectangular" BMO may not act continuously on H^1 of the bidisc, as was shown by Carleson. It was the work of Chang and Fefferman to find the right analogous space.

Following Chang and Fefferman, we define a few candidates for the dual of H^1. Note we will continue to use the name BMO (commonly called product BMO), though it no longer has the mean oscillation property one would naively expect.

There are a few approaches to coming up with candidates to the dual space. The first is not surprising: we know we can characterize 1-dimensional BMO in terms of some Carleson-type condition. We shall see that the following extends this idea to the bidisk.

Definition 5. *The space $BMO_{(a)}$ is the space of locally integrable functions ϕ such that*

$$\sup_\Omega \frac{1}{|\Omega|} || \sum_{R \subset \Omega} \phi_R = ||\phi||_*^2 < \infty$$

Here the supremum ranges over all open sets of finite measure, and the rectangles form a maximal dyadic decomposition of Ω.

The second approach to finding a dual space comes from studying the atomic decomposition; by coming up with a pairing for each atom one can determine the entire dual space.

Definition 6. *The space* $BMO_{(b)}$ *is the space of locally integrable functions* ϕ *such that given any open set* $\Omega \subset \mathbb{R}^2$ *there exists a function* $\tilde{\phi_\Omega}$ *satisfying the following:*

$$\frac{1}{|\Omega|}\int_\Omega |\phi(t) - \tilde{\phi_\Omega}(t)|^2 dt \leq M$$

for some constant M *independent of* Ω.

Furthermore, $\tilde{\phi_\Omega}$ *satisfies regularity conditions similar to those asked of the* H^1 *atoms.*

5. The Theorem

Theorem 3. *(Chang, Fefferman '80):*

Let $\phi \in L^2(\mathbb{R}^2)$ *satisfy*

$\int \phi(x_1, x_2)dx_1 = \int \phi(x_1, x_2)dx_2 = 0$ *for all* $(x_1, x_2) \in \mathbb{R}^2$.

Then the following are equivalent:

(i) $\phi \in BMO_{(a)}$

(ii) $\phi \in BMO_{(b)}$

(iii) $\frac{1}{|\Omega|}\sum_{R\subset\Omega} S_R^2(\phi) < \infty$, *where the supremum ranges over all the finite open sets* Ω, *and for each dyadic rectangle* R

$$S_R^2(\phi) = \iint_{R_+} |\phi(t,y)|^2 \frac{dtdy}{y_1 y_2}$$

This is in some sense the double-S function of the Carleson region.

(iv) ϕ *is in the (continuous) dual of* H^1

Note the third condition is equivalent to the Carleson condition

Sketch of Proof: From our definitions and remarks above, it seems that proving $(i) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv)$ will be straightforward. The easiest is perhaps showing $(ii) \Rightarrow (iv)$, as one can check this on atoms. The flavor of the rest of these proofs is highly geometric, again relying on the dyadic rectangle decomposition.

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DUALITY OF MULTIPARAMETER HARDY SPACES H^p ON SPACES OF HOMOGENEOUS TYPE

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presented by Tim Ferguson

ABSTRACT. Let $\mathcal{X}$ be a space of homogeneous type. Let $p_0 = \frac{2}{2+\theta}$, where θ, to be defined later, is a parameter appearing in the definition of spaces of homogeneous type. Let $H^p(\mathcal{X} \times \mathcal{X})$ be the two parameter Hardy space for $p_0 < p \le 1$. Let $\mathrm{CMO}^\mathrm{p}(\mathcal{X}\times\mathcal{X})$ denote the Carleson measure space with exponent p, which is defined in this paper. The authors show that the dual space of $H^p(\mathcal{X} \times \mathcal{X})$ is $\mathrm{CMO}^\mathrm{p}(\mathcal{X} \times \mathcal{X})$, at least for $p_0 < p \le 1$.

A quasi-metric ρ on a set $\mathcal{X}$ is a function from $\mathcal{X} \times \mathcal{X}$ to $[0, \infty)$ such that the following three conditions hold:

(1) $\rho(x, y) = 0$ if and only if $x = y$.
(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{X}$
(3) There exists a constant $A \ge 1$ such that for all $x, y, z \in \mathcal{X}$, we have

$$\rho(x, y) \le A[\rho(x, z) + \rho(z, y)].$$

Thus, a quasi-metric is like a metric, except that the normal triangle inequality is replaced by a "triangle inequality" that holds up to a multiplicative constant.

Now we define spaces of homogeneous type. Let $0 < \theta \le 1$. A space of homogeneous type is a collection $(\mathcal{X}, \rho, \mu)_\theta$, where $\mathcal{X}$ is a set, ρ is a quasi-metric on $\mathcal{X}$, and μ is a nonnegative Borel regular measure on $\mathcal{X}$. We require that for all r such that $0 < r < \mathrm{diam}(\mathcal{X})$, we have that $\mu(B(x, r)) \sim r$, where $B(x, r)$ is the ball of radius r centered at $x \in \mathcal{X}$. Lastly, we require that there is a constant $C_0 > 0$ such that for all $x, x', y \in \mathcal{X}$, we have

$$|\rho(x, y) - \rho(x', y)| \le C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta}.$$

Note that if $C_0 = 1$ and $\theta = 1$, this last condition is just the triangle inequality.

Instead of assuming that $\mu(B(x, r)) \sim r$, we could have assumed that $\mu(B(x, r)) \sim r^d$, where $d > 0$. However, in [4], Macias and Segovia proved that for $d > 0$, one can always find a quasi-metric $\overline{\rho}$ giving the same topology as ρ such that $\mu(B(x, r)) \sim r$, so throughout the paper the authors assume that $d = 1$.

From the definition, it is clear that $\mathbb{R}$ is a space of homogeneous type. By the result of [4], it is clear that $\mathbb{R}^d$ is a space of homogeneous type for any natural number d.

The main difficulty of working on spaces of homogeneous type is that there are no translations or dilations, and no Fourier transform. Thus, the challenge is to find methods to prove results without using these basic tools. The paper in question relies heavily on Littlewood-Paley theory and a discrete Calderón reproducing formula for spaces of homogeneous type.

To understand the methods of the paper, we first need to define a dyadic decomposition on a space of homogeneous type. In [1] and [5], constructions are given which provide an analogue of the dyadic decomposition on Euclidean space. The statement used by the paper is the following:

Lemma (2.2). *There exists a collection $\{Q_a^k \subset \mathcal{X} : k \in \mathbb{Z}, a \in I_k\}$ of open subsets, called "cubes", where I_k is some index set, as well as constants $C_1, C_2 > 0$ such that*

(1) *For each fixed k, we have $\mu(\mathcal{X} \setminus \cup_a Q_a^k) = 0$, and $Q_a^k \cap Q_b^k = \emptyset$ if $a \neq b$,*
(2) *For any a, b, k, l with $l \geq k$, either $Q_b^l \subset Q_a^k$ or Q_b^l and Q_a^k are disjoint,*
(3) *For each pair of k and a, there is a unique b such that $Q_a^k \subset Q_b^l$,*
(4) $\operatorname{diam}(Q_a^k) \leq C_1(1/2)^k$
(5) *each Q_a^k contains some ball of radius $C_2(1/2)^k$.*

Fix some large positive integer J. Let $k \in \mathbb{Z}$ and $\tau \in I_k$. We denote by $N(k,\tau)$ the number of cubes Q_α^{k+J} that are contained in Q_τ^k, and we denote such a cube by $Q_\tau^{k,v}$, where $1 \leq v \leq N(k,\tau)$.

In order to discuss Littlewood-Paley theory, we first need to define an approximation to the identity on spaces of homogeneous type.

Definition (2.3). *A sequence $\{S_k\}_{k\in\mathbb{Z}}$ is said to be an approximation to the identity of order ϵ, where $0 < \epsilon \leq \theta$, if there is some constant C such that the kernel $S_k(x,y)$ of S_k satisfies*

(1)
$$|S_k(x,y)| \leq C \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}},$$

(2) *For $\rho(x,x') \leq (1/2A)(2^{-k} + \rho(x,y))$, we have*
$$|S_k(x,y) - S_k(x',y)| \leq C \left(\frac{\rho(x,x')}{2^{-k} + \rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}}$$

(3) *For $\rho(y,y') \leq (1/2A)(2^{-k} + \rho(x,y))$, we have*
$$|S_k(x,y) - S_k(x,y')| \leq C \left(\frac{\rho(y,y')}{2^{-k} + \rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}}$$

(4) *For $\rho(x,x'), \rho(y,y') \leq (1/2A)(2^{-k} + \rho(x,y))$ we have*
$$|S_k(x,y) - S_k(x,y') - S_k(x',y) + S_k(x',y')| \leq$$
$$C \left(\frac{\rho(x,x')}{2^{-k} + \rho(x,y)}\right)^\epsilon \left(\frac{\rho(y,y')}{2^{-k} + \rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}}$$

(5)
$$\int_{\mathcal{X}} S_k(x,y)\, d\mu(y) = \int_{\mathcal{X}} S_k(x,y)\, d\mu(x) = 1$$

As k increases, the kernels $S_k(x,y)$ become more concentrated around the diagonal $x = y$. The approximation to the identity $S_k(x,y)$ is analogous to the Poisson kernel $P_{2^{-k}}(x-y)$ on the upper half plane.

We now need to introduce the space of test functions on $\mathcal{X} \times \mathcal{X}$. They can be thought of as analogues of functions in the Schwartz class. Just as H^p is traditionally defined to be a subspace of the tempered distributions, we will define H^p to be a subspace of the dual space of the test functions.

First we define the space of test functions on $\mathcal{X}$.

Definition (2.4). *Fix $\beta, \gamma, r > 0$. A function f defined on $\mathcal{X}$ is said to be a test function of type (β,γ) centered at $x_0 \in \mathcal{X}$ with width r if there is a constant $C > 0$ such that f satisfies the following conditions:*

(1)

$$|f(x)| \le C \frac{r^\gamma}{(r+\rho(x,x_0))^{1+\gamma}}$$

(2) *If* $\rho(x,x') \le \frac{1}{2A}(r+\rho(x,x_0))$ *then*

$$|f(x)-f(x')| \le C\left(\frac{\rho(x,x')}{r+\rho(x,x_0)}\right)^\beta \frac{r^\gamma}{(r+\rho(x,x_0))^{1+\gamma}}$$

(3)

$$\int_{\mathcal{X}} f(x)\, d\mu(x) = 0.$$

If f is such a test function, we say $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ and we define its norm to be the infimum over the set of all C for which conditions 1 and 2 hold.

Definition (2.5)**.** *For* $i = 1, 2$, *fix* $\gamma_i, \beta_i, r_i > 0$ *and let* $(x_0, y_0) \in \mathcal{X} \times \mathcal{X}$. *We say that a function* f *defined on* $\mathcal{X} \times \mathcal{X}$ *is a test function of type* $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ *centered at* (x_0, y_0) *and with widths* (r_1, r_2) *if there is a constant* $C > 0$ *such that*

(1)

$$\|f(\cdot,y)\|_{\mathcal{G}(x_0,r_1,\beta_1,\gamma_1)} \le C \frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}},$$

(2) *For* $\rho(y,y') \le \frac{1}{2A}(r_2+\rho(y,y_0))$ *we have*

$$\|f(\cdot,y)-f(\cdot,y')\|_{\mathcal{G}(x_0,r_1,\beta_1,\gamma_1)} \le C\left(\frac{\rho(y,y')}{r_2+\rho(y,y_0)}\right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}},$$

(3) *Condition 1 should hold with the roles of* x *and* y *reversed.*
(4) *Condition 2 should hold with the roles of* x *and* y *reversed.*

We denote this space of test functions by $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and define the norm in this space to be the smallest C such that the above definition holds.

It is not difficult to see that no matter which x_0 and y_0 we choose, we get the same space of functions with equivalent norm. We choose some fixed (x_0, y_0) and let $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$. If $0 < \beta_1, \beta_2, \gamma_1, \gamma_2 < \theta$, we define the space $\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to be the completion of $\mathcal{G}(\theta, \theta; \theta, \theta)$ in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$. Note that $\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space.

We define the Littlewood-Paley operators D_k as $S_k - S_{k-1}$, where S is an approximation of the identity. We can now define the *Littlewood-Paley-Stein* square function by

$$g(f)(x_1,x_2) = \left[\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |D_j D_k(f)(x_1,x_2)|^2\right]^{1/2}.$$

It can be show that, for $f \in L^p$ for $1 < p < \infty$, we have $\|g(f)\|_p \approx \|f\|_p$.

We now define the space $H^p(\mathcal{X} \times \mathcal{X})$ for certain values of $p < 1$. In what follows, we let Y' denote the dual space of Y.

Definition (2.6). *Let $\{S_k\}$ be an approximation to the identity of order θ. Suppose that $\frac{1}{1+\theta} < p \leq 1$ and $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$. Then we define the Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ to be the set of all $f \in \left(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\right)'$ such that $\|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})} < \infty$. We define*

$$\|f\|_{H^p(\mathcal{X}\times\mathcal{X})} = \|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})}$$

An extremely important lemma is the following, which the authors call the Min-Max comparison principle.

Lemma (2.8). *Let all notation be the same as in definition 2.6. Let $\{P_k\}$ be another approximation to the identity of order θ, and let E_k denote the corresponding Littlewood-Paley operators. Suppose that $\frac{1}{1+\theta} < p \leq 1$ and $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$. Then there is a constant $C > 0$ such that for all $f \in \left(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\right)'$, we have*

$$\left\| \left(\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \sup_{z \in Q_{\tau_1}^{k_1,v_1}, w \in Q_{\tau_2}^{k_2,v_2}} |D_{k_1} D_{k_2}(f)(z,w)|^2 \right.\right.$$
$$\left.\left. \times \chi_{Q_{\tau_1}^{k_1,v_1}}(x) \chi_{Q_{\tau_2}^{k_2,v_2}}(y) \right)^{1/2} \right\|_{L^p}$$
$$\leq C \left\| \left(\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \inf_{z \in Q_{\tau_1}^{k_1,v_1}, w \in Q_{\tau_2}^{k_2,v_2}} |E_{k_1} E_{k_2}(f)(z,w)|^2 \right.\right.$$
$$\left.\left. \times \chi_{Q_{\tau_1}^{k_1,v_1}}(x) \chi_{Q_{\tau_2}^{k_2,v_2}}(y) \right)^{1/2} \right\|_{L^p}$$

where the integration for the L^p norm is respect to $d\mu(x)d\mu(y)$.

For the proof of this lemma, the authors reference [3]. This lemma is very useful for several reasons. First, it lets one show that the definition of H^p given does not depend on the choice of approximation to the identity. To see this, note that the left hand side is greater than the H^p norm with operators D_k, and the right side is less than C times the H^p norm with operators E_k.

Also, the lemma in question allows one to approximate the integration defining the L^p norm of the g function by a sort of "Riemann sum". The lemma states that it does not matter whether we take the smallest or largest possible "Riemann sum", we still get an equivalent norm. Of course, the sums in the lemma are infinite. As the coefficient k gets larger, we need to sum over smaller and smaller dyadic boxes $Q_\tau^{k,v}$. This lemma allows us to relate Hardy space functions to sequences, which is crucial in proving duality.

We now can finally define the Calderon measure spaces, which we will see are dual to H^p.

Definition (3.1). *Let β_i, γ_i, etc. be as before. The Carleson measure space $\mathrm{CMO^p}(\mathcal{X} \times \mathcal{X})$ is the set of all $f \in (\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2))'$ such that*

$$\sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega} |D_{k_1} D_{k_2}(f)(x,y)|^2 \chi_{Q_{\tau_1}^{k_1,v_1}}(x) \chi_{Q_{\tau_2}^{k_2,v_2}}(y)\, d\mu(x) d\mu(y) \right)^{1/2}$$

is finite. The above expression defines the norm in $\mathrm{CMO^p}$.

The above definition can be though of as analogous to the single parameter case on $\mathbb{R}$, where $\phi \in \mathrm{BMO}$ if and only if $|\nabla u|^2 y dx\, dy$ is a Carleson measure, where u is the harmonic extension of ϕ to the upper half plane.

A very important theorem is the Min-Max comparison principle for $\mathrm{CMO^p}$. For ease of notation, let R denote an arbitrary dyadic rectangle $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$.

Theorem (3.2)**.** *If* $2/(2+\theta) < p \leq 1$*, then there is some constant* $C > 0$ *such that*

$$\sup_{\Omega}\left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \mu(R) \sup_{(x,y)\in R} |D_{k_1}D_{k_2}(f)(x,y)|^2\right)^{1/2}$$
$$\leq C \sup_{\Omega}\left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \mu(R) \inf_{(x,y)\in R} |D_{k_1}D_{k_2}(f)(x,y)|^2\right)^{1/2}$$

In fact, one can use different approximations to the identity on the left and right side of the inequality, which allows us to see that the CMO spaces are well defined. As before, another reason this theorem is important is because it allows us to consider a sort of "Riemann sum" instead of integration in the definition of CMO. This allows us to relate the CMO spaces to sequences, which again is key to proving the duality.

To prove this formula, the authors use the following discreet Calderon reproducing formula. Again, we let R denote an arbitrary dyadic rectangle of the form $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$.

Lemma (2.8)**.** *Let the notation be the same as above. For each* R*, choose a point* $(x', y') \in R$*. Then there are families of linear operators* $\{\widetilde{D}_k\}$ *and* $\{\overline{D}_k\}$ *such that*

$$\begin{aligned} f(x,y) &= \sum_R \mu(R) \widetilde{D}_{k_1}\widetilde{D}_{k_2}(x,y,x',y') D_{k_1}D_{k_2}(f)(x',y') \\ &= \sum_R \mu(R) D_{k_1}D_{k_2}(x,y,x',y') \overline{D}_{k_1}\overline{D}_{k_2}(f)(x',y'), \end{aligned}$$

By applying this lemma to $D_{k_1}D_{k_2}f$ and choosing (x', y') to be the point where $|D_{k_1}D_{k_2}(f)(x,y)|$ is minimized in R, we obtain an expression that can be used to relate the left side of the inequality in Theorem 3.2 to something resembling the right side. Then by changing the order of summation and a careful consideration of various geometric quantities involved, we can prove Theorem 3.2.

The authors also introduce two spaces of sequences. The first is called s^p, and is defined to be the space of all complex valued sequences $\{\lambda_{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}}\}$ such that

$$\|\lambda\|_{s^p} = \left\| \left[\sum_R \left(|\lambda_R| \widetilde{\chi_R}(\cdot) \right)^2 \right]^{1/2} \right\|_{L^p} < \infty.$$

Here $\widetilde{\chi_R} = \mu(R)^{-1/2}\chi_R$.

Similarly, we define the space c^p of sequences to be all sequences $\{t_{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}}\}$ such that

$$\|t\|_{c^p} = \sup_{\Omega} \left(\int_{\Omega} \sum_{R\subset\Omega} \left(|t_R| \widehat{\chi}(x,y) \right)^2 \, d\mu(x) d\mu(y) \right)^{1/2} < \infty.$$

The authors show these two sequence spaces are dual to each other. To do this, they use inequalities involving the Hardy-Littlewood maximal function and also use the duality properties of certain weighted ℓ^2 spaces related to both s^p and c^p.

Now using the Min-Max inequalities, and methods very similar to those used in the proofs of the Min-Max inequalities, the authors relate the space s^p to H^p and c^p to $\mathrm{CMO}^{\mathrm{p}}$. They then show that H^p has $\mathrm{CMO}^{\mathrm{p}}$ as its dual space for the case where $\frac{2}{2+\theta} = p_0 < p \leq 1$.

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PARAPRODUCTS IN ONE AND SEVERAL VARIABLES

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presented by Kelly Bickel

ABSTRACT. This paper presents a leisurely proof of the discrete Coifman-Meyer Theorem in both the one and multi-parameter settings.

1. Introduction

Let us first consider the general single and bi-parameter Coifman-Meyer Theorems, which we combine into one statement below.

For the one parameter case, let m be a bounded function on $\mathbb{R}^2$, smooth away from the origin and satisfying

$$|\partial^\alpha m(\zeta)| \lesssim \frac{1}{|\zeta|^{|\alpha|}}, \tag{1}$$

for sufficiently many multi-indices α and define the bilinear operator $T_m^{(1)}$ by

$$T_m^{(1)}(f,g) = \int_{\mathbb{R}^2} m(\zeta)\hat{f}(\zeta_1)\hat{g}(\zeta_2)e^{2\pi i x(\zeta_1+\zeta_2)}d\zeta,$$

for Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$. We can generalize this by allowing m to be defined on $\mathbb{R}^{2n}$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. For the bi-parameter case, let $m(\zeta,\eta)$ be a bounded function on $\mathbb{R}^4$, smooth away from $\{(\zeta_1,\eta_1)=0\} \cup \{(\zeta_2,\eta_2)=0\}$ and satisfying the estimate

$$|\partial_\zeta^\alpha \partial_\eta^\beta m(\zeta,\eta)| \lesssim \frac{1}{|(\zeta_1,\eta_1)|^{\alpha_1+\beta_1}} \frac{1}{|(\zeta_2,\eta_2)|^{\alpha_2+\beta_2}}, \tag{2}$$

for sufficiently many multi-indices α and β. Then we can define the bilinear operator $T_m^{(2)}$ as follows:

$$T_m^{(2)}(f,g) = \int_{\mathbb{R}^4} m(\zeta,\eta)\hat{f}(\zeta)\hat{g}(\eta)e^{2\pi i x(\zeta+\eta)}d\zeta d\eta,$$

where $f, g \in \mathcal{S}(\mathbb{R}^2)$. The Coifman-Meyer Theorem [4] states

Theorem 1. *If m is a symbol satisfying the above estimates, then the bilinear operator $T_m^{(j)}$ maps $L^p \times L^q \to L^r$ whenever $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$.*

There is a deep connection between such operators T_m and single or bi-parameter paraproducts. Indeed, for particular symbols m, T_m is a paraproduct, and the proof of Theorem 1 can be reduced to the case of considering certain model paraproducts. See [4] for the bi-parameter reduction argument. Lacey and Metcalfe concern themselves only with the model paraproduct (discrete) case of Theorem 1, which we will soon discuss in detail.

Before discussing the particulars, it is worth noting the importance of Theorem 1, and specifically, that the theorem is a key step in deriving fractional Leibniz inequalities, as detailed in [4]. For simplicity, we discuss the one parameter result first. Let $f, g \in \mathcal{S}(\mathbb{R}^2)$ and for $\alpha > 0$, define the fractional derivative $\mathcal{D}^\alpha$ by $\widehat{\mathcal{D}^\alpha f}(\zeta) = |\zeta|^\alpha \hat{f}(\zeta)$.

Then, one can define paraproducts Π_j for $j = 0, \ldots, 3$ such that the Coifman-Meyer Theorem applies to each Π_j and

$$fg = \sum_{j=0}^{3} \Pi_j(f, g).$$

Using the structure of these paraproducts, one can find Coifman-Meyer paraproducts Π_1' and Π_2' with

$$\mathcal{D}^\alpha\big(\Pi_1(f,g)\big) = \Pi_1'(f, \mathcal{D}^\alpha g) \text{ and } \mathcal{D}^\alpha\big(\Pi_2(f,g)\big) = \Pi_2'(\mathcal{D}^\alpha f, g),$$

and similar Π_0' and Π_3' paraproducts. Then, using the Coifman-Meyer Theorem, we can derive the following Leibniz rule (called the Kato-Ponce inequality [2]) as follows:

$$\begin{aligned} ||\mathcal{D}^\alpha(f,g)\|_r &\leq \sum_{j=0}^{3} \|\mathcal{D}^\alpha(\Pi_j(f,g))\|_r \\ &\lesssim \|\mathcal{D}^\alpha f\|_p \|g\|_q + \|f\|_p \|\mathcal{D}^\alpha g\|_q, \end{aligned}$$

for $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$. Now, for $f, g \in \mathcal{S}(\mathbb{R}^2)$ and for $\alpha, \beta > 0$, define the partial differential operator $\mathcal{D}_1^\alpha \mathcal{D}_2^\beta$ by $\widehat{\mathcal{D}_1^\alpha \mathcal{D}_2^\beta f}(\zeta) = |\zeta_1|^\alpha |\zeta_2|^\beta \hat{f}(\zeta)$. Then, using the bi-parameter Coifman-Meyer Theorem and analogous manipulations of paraproducts, one can deduce

$$\|\mathcal{D}_1^\alpha \mathcal{D}_2^\beta (fg)\|_r \lesssim \|\mathcal{D}_1^\alpha \mathcal{D}_2^\beta f\|_p \|g\|_q + \|f\|_p \|\mathcal{D}_1^\alpha \mathcal{D}_2^\beta g\|_q + \|\mathcal{D}_1^\alpha f\|_p \|\mathcal{D}_2^\beta g\|_q + \|\mathcal{D}_2^\beta f\|_p \|\mathcal{D}_1^\alpha g\|_q,$$

for $1 < p, q \leq \infty$, $1/r = 1/p + 1/q$, and $0 < r < \infty$.

2. One-Parameter Paraproducts

Lacey and Metcalfe first provide a detailed discussion of the one-parameter discrete Coifman-Meyer Theorem and its proof, thus motivating much of the proof of the multi-parameter case.

Let I be an interval. Then ϕ_I is a *bump function adapted to* I iff $\|\phi_I\|_2 = 1$ and

$$|D^n \phi_I(x)| \lesssim |I|^{-n-1/2} \left(1 + \frac{|x - c(I)|}{|I|}\right)^{-N}, \qquad n = 1, 2,$$

where $c(I)$ is the center of I and N is sufficiently large. Let $\mathscr{D}$ be the set of dyadic intervals. Then the model paraproducts are bilinear operators of the form:

$$B(f_1, f_2) = \sum_{I \in \mathscr{D}} |I|^{-1/2} \phi_{3,I} \prod_{j=1}^{2} \langle f_j, \phi_{j,I} \rangle,$$

where, for each I, each $\phi_{j,I}$ is adapted to I and two of the $\phi_{j,I}$ have integral zero. The precise theorem proven by Lacey and Metcalfe is:

Theorem 2. *The bilinear operator B maps $L^{p_1} \times L^{p_2} \to L^r$ whenever $1 < p_1, p_2 \leq \infty$, $1/r = 1/p_1 + 1/p_2$, and $0 < r < \infty$.*

The arguments generalize immediately to multi-linear model paraproducts as well. Also, we can assume without loss of generality that, for each I, the $\phi_{I,2}$ and $\phi_{I,3}$ have integral zero. In the proof, Lacey and Metcalfe make use of variations of the maximal function and square function, defined as follows:

$$\begin{aligned} M_B g &:= \sup_{I \in \mathscr{D}} \mathbf{1}_I \frac{|\langle g, \phi_{1,I}\rangle|}{\sqrt{|I|}} \\ S_j g &:= \left[\sum_{I \in \mathscr{D}} \frac{|\langle g, \phi_{j,I}\rangle|^2}{|I|} \mathbf{1}_I \right]^{1/2}, \end{aligned}$$

for $j = 2, 3$. They also use the following well-known estimates:

$$\begin{aligned} M_B(g)(x) &\leq M(g)(x) \\ \|S_j g\|_p &\lesssim \|g\|_p, \end{aligned}$$

where M denotes the typical maximal function, $1 < p < \infty$, and $j = 2, 3$.

The proof of Theorem 2 now splits into two cases. Lacey and Metcalfe first consider the cases when $1 < r < \infty$ because for these ranges, one can use a duality argument. Specifically, let r' be dual to r, fix $f_3 \in L^{r'}$ with $\|f_3\|_{r'} = 1$, and calculate:

$$\begin{aligned} \langle B(f_1, f_2), f_3\rangle &\leq \int \sum_{I \in \mathscr{D}} |I|^{-3/2} \prod_{j=1}^{3} |\langle f_j, \phi_{j,I}\rangle| \mathbf{1}_I \\ &\leq \int (M f_1)(S_2 f_2)(S_3 f_3) \\ &\leq \|M f_1\|_{p_1} \|S_2 f_2\|_{p_2} \|S_3 f_3\|_{r'} \\ &\lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}, \end{aligned}$$

where Hölder's inequality is used twice, once for sums and once for integrals. For situations where $1/2 < r < 1$, one must first consider the multi-linear form

$$\Lambda(f_1, f_2, f_3) := \sum_{I \in \mathscr{D}} |I|^{-1/2} \prod_{j=1}^{3} \langle f_j, \phi_{j,I}\rangle,$$

and show that Λ is (almost) of generalized restricted type (p_1, p_2, p_3), where $1/p_3 = 1 - 1/r$ and hence, can be negative. Specifically, for each f_1, f_2, and set E, one must find a set $E' \subseteq E$ with $|E'| \geq 1/2|E|$ such that

$$\Lambda(f_1, f_2, f_3) \lesssim |E|^{1/p_3} \|f_1\|_{p_1} \|f_2\|_{p_2}, \tag{3}$$

for all f_3 supported in E' and bounded by 1. By multi-linearity, we can assume $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$. As the class of the multi-linear forms Λ is invariant under dilations by powers of two, we can assume $|E| = 1$.

It then follows easily (for instance, using Lemma 5.4 in [1]) that B satisfies

$$\lambda |\{B(f_1, f_2) > \lambda\}|^{1/r} \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2},$$

and multi-linear Marcinkiewicz interpolation will yield the desired strong estimates.

Then the problem is reduced to finding, for each pair (f_1, f_2, E), a subset E' such that (3) holds. Metcalfe and Lacey do not consider this general problem for one-parameter paraproducts. Rather, they only examine the Haar paraproduct defined by

$$B(f_1, f_2) = \sum_{I \in \mathscr{D}} |I|^{-1/2} h_I \langle f_1, |h_I| \rangle \langle f_2, h_I \rangle,$$

where the Haar functions h_I are defined as $h_I = |I|^{-1/2}(\mathbf{1}_{I_l} - \mathbf{1}_{I_r})$. Using the structure of the Haar functions, they obtain the weak estimates directly and do not explicitly prove (3). Nevertheless, the proof is similar to the general proof, insofar as they remove exceptional sets where the f_1, f_2 are large. Specifically, they disregard the set F defined by

$$E := \bigcup_{j=1}^{2} \{Mf_j > 1\}, \qquad F := \{M\mathbf{1}_E > \tfrac{1}{2}\}.$$

It should be noted that there are also endpoint results. As numerous methods are used to procure the endpoint results, we omit them as to maintain a coherent flow of ideas.

3. Multi-Parameter Paraproducts

Many definitions and results in the multi-parameter case generalize nicely from the one-parameter case. In this summary, we restrict to the bi-parameter, two-dimensional case for simplicity.

Let $\mathscr{R}$ denote the set of dyadic rectangles in $\mathbb{R}^2$. A function ϕ_R is *adapted to the rectangle* R, where $R = R_1 \times R_2$, if $\phi_R(x) = \phi_{R_1}(x_1)\phi_{R_2}(x_2)$, where ϕ_{R_k} is adapted to R_k. We consider model paraproducts of the form:

$$B(f_1, f_2) = \sum_{R \in \mathscr{R}} |R|^{-1/2} \phi_{3,R} \prod_{j=1}^{2} \langle f_j, \phi_{j,R} \rangle,$$

where for each coordinate x_k, $k = 1, 2$, there are two positions in $j = 1, 2, 3$ such that

$$\int_{\mathbb{R}} \phi_{j,R}(x_1, x_2) dx_k = 0 \text{ for all } x_i \neq x_k \text{ and all } R \in \mathscr{R}.$$

Then we say B has x_k zeros in the j^{th} position (or $\{\phi_{j,R}\}$ has x_k zeros.) Notice that this condition is a clear generalization of the condition in one parameter that, for two of $j = 1, 2, 3$, $\phi_{j,I}$ has integral zero for each I. Under those conditions, Lacey and Metcalfe prove the following result:

Theorem 3. *The bilinear operator B maps $L^{p_1} \times L^{p_2} \to L^r$ whenever $1 < p_1, p_2 \leq \infty$, $1/r = 1/p_1 + 1/p_2$, and $0 < r < \infty$.*

Again, they utilize variants of square and maximal functions, adapted to the specific bump functions appearing in the given paraproduct. Specifically, they use the following iterates of one-variable square and maximal functions:

$$\begin{aligned}
MM(f) &:= \sup_{R\in\mathscr{R}} \frac{|\langle f,\phi_R\rangle|}{\sqrt{|R|}}\mathbf{1}_R \\
S_1M_2(f) &:= \left[\sum_{R_1\in\mathscr{D}} \sup_{R_2\in\mathscr{D}} \frac{|\langle f,\phi_{R_1\times R_2}\rangle|^2}{|R|}\mathbf{1}_R\right]^{1/2}, \qquad R = R_1\times R_2 \\
SS(f) &:= \left[\sum_{R\in\mathscr{R}} \frac{|\langle f,\phi_R\rangle|^2}{|R|}\mathbf{1}_R\right]^{1/2},
\end{aligned}$$

where we can similarly define S_2M_1, M_1S_2, and M_2S_1. As it can be show that $M_iS_j \le S_jM_i$ point-wise, we only consider iterates where the maximal function is applied first. Also, if a square function is applied to the set $\{\phi_R\}$ in the x_k coordinate, we require the functions $\{\phi_R\}$ to have x_k zeros.

As before, these operators are bounded from L^p to L^p for $1 < p < \infty$. (However, the arguments are more extensive, particularly for the S_1M_2 and S_2M_1 operators.) Metcalfe and Lacey then use the flexibility in the definition of *adapted functions* to prove a refinement of the L^2 bound. In particular, if $\mathscr{O}$ is a family of dyadic rectangles, f is a function, and $\mu > 1$ is a constant such that

$$supp(f)\cap \mu R = \emptyset\ \forall\ R\in\mathscr{O} \quad \text{then} \quad \|T_{\mathscr{O}}\|_2 \lesssim \mu^{-N'}\|f\|_2, \tag{4}$$

where N' only depends on the value N in the definition of adapted and $T_{\mathscr{O}}$ is an iterated operator T restricted to a sum and/or suprema over the class $\mathscr{O}$.

As in the one-parameter proof, the theorem breaks into two cases. Metcalfe and Lacey first consider $1 < r < \infty$, where one can use a duality argument. To illustrate the ideas in the proof, assume B has x_1 zeros in the $j = 1,2$ positions and x_2 zeros in the $j = 1,3$ positions. Let $f_3 \in L^{r'}$ with $\|f_3\|_{r'} = 1$ and r' dual to r. Then we can calculate:

$$\begin{aligned}
\langle B(f_1,f_2),f_3\rangle &\le \int \sum_{R\in\mathscr{R}} |R|^{-3/2}\prod_{j=1}^{3}|\langle f_j,\phi_{j,R}\rangle|\mathbf{1}_R \\
&\le \int (SSf_1)(S_1M_2f_2)(S_2M_1f_3) \\
&\le \|SSf_1\|_{p_1}\|S_1M_2f_2\|_{p_2}\|S_2M_1f_3\|_{r'} \\
&\lesssim \|f_1\|_{p_1}\|f_2\|_{p_2},
\end{aligned} \tag{5}$$

where Hölder's inequality is again used twice, once for sums and once for integrals. The case for $1/2 < r < 1$ also proceeds as in the one -parameter case. Define

$$\Lambda(f_1,f_2,f_3) = \sum_{R\in\mathscr{R}} |R|^{-1/2}\prod_{j=1}^{3}|\langle f_j,\phi_{j,R}\rangle| = \sum_{R\in\mathscr{R}}|R|\prod_{j=1}^{3}\frac{|\langle f_j,\phi_{j,R}\rangle|}{\sqrt{|R|}}. \tag{6}$$

As before, it suffices to find, for each tuple (f_1, f_2, E) with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$ and $|E| = 1$, a set $E' \subset E$ with $|E'| \ge \frac{1}{2}|E|$ such that for all f_3 supported in E' and bounded by 1,

$$\Lambda(f_1,f_2,f_3) \lesssim 1. \tag{7}$$

4. The Set E'

In this section, we discuss the technicalities of establishing (7) for a soon-to-be defined E'. The summary becomes a bit detailed here, but the arguments in this part are particularly important, as this is where the Lacey/Metcalfe paper differs some from related papers such as [4].

One can assume f_1 and f_2 are smooth and compactly supported. Observe, in analogy with (5), that

$$\langle \mathbb{B}(f_1, f_2), f_3\rangle \leq \int (T_1 f_1)(T_2 f_2)(T_3 f_3),$$

for some T_j, where each is an iterated square and/or maximal operator. Define $4\nu = \min(p_1, p_2)$ and let T_0 be the strong maximal function (in two parameters). Define

$$\begin{aligned}
\Omega_{j,l} &:= \Big\{T_j f_j > C2^l\Big\}, \qquad l \in \mathbb{Z},\ j = 1, 2, \\
\Omega_l &:= \bigcup_{j=1}^{2} \Omega_{j,l}, \\
\Omega &:= \bigcup_{l\in\mathbb{N}} \Big\{T_0 \mathbf{1}_{\Omega_l} > \frac{1}{100} 2^{-\nu l}\Big\}, \\
\tilde{\Omega} &:= \Big\{T_0 \mathbf{1}_{\Omega} > \frac{1}{2}\Big\}.
\end{aligned}$$

Then one can choose C such that $|\tilde{\Omega}| < 1/2$ and define $E' = E \cap \tilde{\Omega}^c$. At this point, Metcalfe and Lacey decompose the sum (6) into sums over several classes of rectangles, defined based on where T_j's are sufficiently small/large. For such a class $\mathscr{O}$, define the restricted sum:

$$\mathrm{Sum}(\mathscr{O}) := \sum_{R\in\mathscr{O}} |R| \prod_{j=1}^{3} \frac{|\langle f_j, \phi_{j,R}\rangle|}{\sqrt{|R|}}.$$

We split the rectangles into classes as follows: R is in class $\mathscr{O}_{j,l}$ iff l is the greatest integer so that

$$|R \cap \Omega_{j,l}| = |R \cap \{T_j f_j > C2^l\}| \geq \frac{1}{100}|R|.$$

As the $T_j f_j$ are bounded, every rectangle R is in precisely one $\mathscr{O}_{j,l}$ for each j and so we can associate to R a tuple $\vec{l} = (l_1, l_2, l_3)$ of integers. This characterization is important because Lacey and Metcalfe establish the following technical lemma:

Lemma 4. *Let c_1, c_2, c_3 be positive constants, and let $\mathscr{O}$ be a collection of rectangles. Then if*

$$|R \cap \{T_j f_j > c_j\}| \leq \frac{1}{100}|R|, \qquad \forall\ R \in \mathscr{O}$$

holds for $j = 1, 2, 3$, we have

$$\mathit{Sum}(\mathscr{O}) \lesssim |\mathit{sh}(\mathscr{O})| \prod_{j=1}^{3} c_j,$$

and if it hold for $j = 1, 2$ we have

$$\mathit{Sum}(\mathscr{O}) \lesssim c_1 c_2 |\mathit{sh}(\mathscr{O})|^{1/2} \|T_{3,\mathscr{O}}\|_2,$$

where $sh(\mathscr{O})$ *is the shadow of* $\mathscr{O}$*, and* $T_{3,\mathscr{O}}$ *is the operator* T_3 *restricted to a sum and/or suprema over the class* $\mathscr{O}$*.*

Returning to the proof of (7), first consider $\vec{l}$ with $l_1, l_2, l_3 \le 0$ and for each such $\vec{l}$, define

$$\mathscr{O}_{\vec{l}} = \bigcap_{j=1}^{3} \mathscr{O}_{j,l_j}.$$

Using the technical lemma and weak L^p bounds for T_j, one can show

$$\mathrm{Sum}(\mathscr{O}_{\vec{l}}) \lesssim 2^{l_1+l_2+l_3}|\mathrm{sh}(\mathscr{O}_{\vec{l}})| \lesssim 2^{l_1+l_2+l_3}2^{-\theta_1 p_1 l_1 - \theta_2 p_2 l_2 - \theta_3 p_3 l_3},$$

where $\theta_1 + \theta_2 + \theta_3 = 1$ and $p_3 > 1$. Then

$$\sum_{l_1,l_2,l_3 \le 0} \mathrm{Sum}(\mathscr{O}_{\vec{l}}) \lesssim \sum_{l_1,l_2,l_3 \le 0} 2^{l_1(1-p_1\theta_1)+l_2(1-p_2\theta_2)+l_3(1-p_3\theta_3)},$$

which sums to a constant as long as the θ_j's and p_3 are chosen so that each $1 - p_j\theta_j > 0$. A similar argument works for $\vec{l}$ with $l_1, l_2 \le 0$ and $l_3 > 0$. Now consider R where at least one of $l_1, l_2 > 0$. For $\vec{l}$ with l_1 or $l_2 > 0$, define

$$\mathscr{P}_{\vec{l}} = \bigcap_{j=1}^{2} \mathscr{O}_{j,l_j}.$$

For simplicity, assume $l_1 > 0$. Then $T_1 f_1$ is large on each R in $\mathscr{P}_{\vec{l}}$ and it can be shown that

$$2^{\nu l_1/2} R \cap E' = \emptyset.$$

Applying the second half of the technical lemma and (4) gives:

$$\mathrm{Sum}(\mathscr{P}_{\vec{l}}) \lesssim 2^{l_1+l_2} \min(2^{-10 l_1}, 2^{-10 l_2}), \tag{8}$$

where N' was chosen to be sufficiently large. As (8) is clearly summable over all such $\vec{l}$, this establishes the main result. It should be noted that there are various endpoint results that are not discussed in this summary.

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MULTPARAMETER RIESZ COMMUTATORS

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presented by Mishko Mitkovski

Abstract. We discuss the result of M. Lacey, S. Petermichel, J. Phipher and B. Wick which characterizes the multiparameter BMO of S. Y. A. Chang and R. Fefferman, by the multiparameter commutators of Riesz transforms. Their result generalizes the one-parameter result of R. Coifman, R. Rochberg, and G. Weiss, and at the same time extends the work of M. Lacey and S. Ferguson and M. Lacey and E. Terwilleger, on multiparameter commutators with Hilbert transforms.

1. Introduction

The result of this paper is a generalization of the classical one-parameter result of R. Coifman, R. Rochberg, and G. Weiss which characterizes the real variable H^1 (which we will denote by H^2_{Re}) in terms of certain commutators. This is a final step in a series of extensions obtained previously by M. Lacey, S. Ferguson and E. Terwilleger.

The main motivation and application of all these results is the extension of certain well known factorization theorems from the classical Hardy spaces to the Hardy spaces in several variables. These factorization theorems are important because they play a crucial role in the proof of Nehari-type results for various multidimensional Hardy spaces. The possibility to prove factorization theorems using real-variable methods are consequence of the close and interesting interplay between the following three results: The weak factorization theorem in H^1, the $H^1_{Re} - BMO$ duality, and the L^2 boundedness of the commutator $[M_b, H] = M_bH - HM_b$, where M_b is a multiplication operator and H is the Hilbert transform.

1.1. One-parameter case.

To make things more clear we first consider the classical one-parameter case. The strong factorization theorem in the classical (one-dimensional) case says:

Theorem 1.1. *Every $F \in H^1(\mathbb{R}) = H^1(\mathbb{C}_+)$ can be factored as $F = GH$ with $G, H \in H^2(\mathbb{R})$. Moreover, $\|F\|_1 = \inf \|G\|_2 \|H\|_2$, where the infimum is taken over all factorizations $F = GH$ with $\|G\|_2 = \|H\|_2 = 1$.*

This result can be very easily proved using techniques from complex analysis. It is well known however that this beautiful factorization property is false for Hardy spaces in several dimensions. Still a weaker factorization property continues to hold.

Theorem 1.2. *Every $F \in H^1(\mathbb{R}^n)$ can be represented as $F = \sum G_iH_i$ with $G_i, H_i \in H^2(\mathbb{R}^n)$. Moreover, $\|F\|_1 = \inf \sum \|G_i\|_2 \|H_i\|_2$, where the infimum is taken over all representations $F = \sum G_iH_i$ with $\sum_i \|G_i\|_2 \|H_i\|_2 \leq 1$.*

Even though the factorization property was extremely easy to prove (even in the strong form) still its generalization to several variables showed to be very difficult. As of now, there is no proof which uses complex analytic techniques and the only proof available is based on real variable techniques.

The second result is the famous C. Fefferman duality theorem. It says that the real variable Hardy space $H^1_{Re}(\mathbb{R})$ is dual to the space $BMO(\mathbb{R})$.

Theorem 1.3. *Let $b \in BMO(\mathbb{R}^n)$. Then*

$$|\int_{\mathbb{R}^n} b(x)f(x)dx| = \|b\|_{BMO},$$

where the supremum is taken over all $f \in H^1_{Re}(\mathbb{R}^n)$ of norm no greater than 1.

Using the factorization result it easy to see that we can rewrite the previous equality in the following form

$$\sup|\int_{\mathbb{R}^n} b(x)(g_1(x)\tilde{g}_2(x) + \tilde{g}_1(x)g_2(x))dx| = \|b\|_{BMO}$$

where the supremum is taken over all $g_1, g_2 \in L^2(\mathbb{R}^n)$ of norm no greater than 1. Using some basic properties of the Hilbert transform is is easy to see that the last result is equivalent to the L^2 boundedness of the commutator operator $[B, H] := BH - HB$, where $M_b f = bf$ is the multiplication operator and $Hf = \tilde{f}$ is the usual Hilbert transform. This is actually the third important result.

Theorem 1.4. *Let $b \in BMO$. Then the commutator operator $[M_b, H]$ is bounded on $L^2(\mathbb{R})$ with norm equal to $\|b\|_{BMO}$.*

The multidimensional version of this result is given by the following theorem. As usual, R_i denotes the i-th Riesz transform.

Theorem 1.5. *If $b \in BMO$ then each of the commutator operators $[M_b, R_i]$ is bounded on $L^2(\mathbb{R}^n)$ and*

$$\sup_i \|[M_b, R_i]\| = \|b\|_{BMO}.$$

Conversely, if all these commutators are bounded then $b \in BMO$.

As seen above the factorization result combined with the C. Fefferman duality theorem easily imply the commutator result. It is not hard to see that if we replace the strong factorization with the weak factorization property any two of these three results easily imply the remaining one. This simple observation turns out to be crucial because it allows the use of real-variable techniques to prove the desired weak factorization. Namely, one can use real variable techniques to prove both the Fefferman duality theorem and the boundedness of the commutator not just in several variables but also in the multi parameter setting. As a result one can prove a weak factorization and consequently Nehari-type theorems both for a wide variety of Hardy spaces including the Hardy space on the unit ball $\mathbb{B}^n$ and on the polydics $\mathbb{T}^n$.

1.2. **Multi-parameter case.** The same line of ideas continues to hold in the multi-parameter setting. Namely, if there is an $H^1_{Re} - BMO$-type duality result and a commutator result available one can combine them as before and obtain weak factorization results for the multi-parameter H^1 space. As a consequence, one in particular obtains a Nehari theorem for the polydisc. This is especially striking since the polydisc is not a pseudo-convex domain and hence the classical techniques are very limited in this case.

The $H^1_{Re} - BMO$-type duality results are supplied by the product theory of A. Chang and R. Fefferman. The commutator theorem in its full generality on the other hand is the main result of the paper that we are discussing. It was obtained in several steps. It was first proved for by S. Ferguson and M. Lacey for commutators on $L^2(\mathbb{R} \otimes \mathbb{R})$ and then extended to by M. Lacey and E. Terwilleger to $L^2(\mathbb{R} \otimes \cdots \otimes \mathbb{R})$. As a consequence, Nehari theorems for the bidisc and the polydisc were obtained. The general version that we state below does not seem to have immediate application to Nehari-type results, but represents a natural extension of the previously mentioned results.

The main theorem is concerned with commutators acting on a product space $\mathbb{R}^{\vec{d}} := \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2} \otimes \ldots \mathbb{R}^{d_t}$, where $\vec{d} = (d_1, d_2, \ldots, d_t) \in \mathbb{N}^t$. For $f, b \in \mathcal{S}$ the family of commutators is defined by

$$C_{\vec{j}}(b, f) := [[\ldots[[M_b, R_{1,j_1}], R_{2,j_2}]\ldots], R_{t,j_t}]f,$$

indexed by $\vec{j} = (j_1, j_2, \ldots, j_t) \in \mathbb{N}^t$, with $1 \le j_k \le d_k$. Here R_{k,d_k} denotes the kth Riesz transform acting on $\mathbb{R}^{d_k}$.

Theorem 1.6. *The following estimate holds*

$$\sup_{\vec{j}} C_{\vec{j}}(b, f) \simeq \|b\|_{BMO},$$

where the last norm is taken in the Chung-Fefferman multi parameter BMO.

2. Tools and Notation

2.1. **Wavelet basis.** The wavelet basis for $L^2(\mathbb{R}^{\vec{d}})$ which is used in the proof is defined in the following usual way. Let $\mathcal{D}_{\vec{d}} := \otimes_{k=1}^t \mathcal{D}_{d_k}$ be the dyadic grid on the tensor product $\mathbb{R}^{\vec{d}}$, where

$$\mathcal{D}_{d_k} := \{j + [0, 2^s)^{d_k} : j \in \mathbb{Z}^{d_k}, s \in \mathbb{Z}\}$$

is the standard d_k-dimensional dyadic grid. Starting with a scaling function w^0 or a father wavelet w^1, for each $\epsilon \in Sig_{\vec{d}} := \{\vec{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_t) : \epsilon_k \in \{0,1\}^{d_k} - \{\vec{1}\}\}$. and $R = Q_1 \times Q_2 \ldots Q_t \in \mathcal{D}_{\vec{d}}$ define

$$w_R^{\epsilon}(x_1, x_2, \ldots, x_t) := \prod_{k=1}^t Tr_{c(Q_k)} Dil_{|Q_k|} w^{\epsilon_k}(x_k).$$

Here the standard translation $Tr_a f(x) = f(x-a)$ and dilation $D_a f(x) = a^{-d/2} f(x/a)$ operators on $L^2(\mathbb{R}^d)$ are used, and as usual $c(Q)$ denotes the center of Q and $|Q|$ denotes the Lebesque measure of Q. The collection

$$\{w_R^{\epsilon} : \epsilon \in Sig_{\vec{d}}, R \in \mathcal{D}_{\vec{d}}\}$$

forms a wavelet orthonormal basis for $L^2(\mathbb{R}^{\vec{d}})$. The usual choices for the scaling function and the father wavelet are $w^0 = -1_{[0,1/2]} + 1_{[1/2,1]}$, $w^1 = 1_[0,1]$. They give a rise to the Haar wavelet basis. Since some smoothness and localization properties of the basis are needed in their proof the authors use the wavelet basis generated by the Meyer wavelet. This wavelet, found by Y. Meyer, arises from a Schwartz function w^1, whose Fourier transform is supported on $1/3 \leq |\xi| \leq 8/3$. Furthermore, the Fourier transform of w^1 is identically equal to 1 on the intervals $1 \leq |\xi| \leq 2$. One of the reasons this is such a useful wavelet basis is the following fact which is exploited several times in the proof. If $8|I| < |I'|$ then the Fourier transform of $w^1_I w^1_{I'}$ is supported on $(4|I|)^{-1} < |\xi| < 3|I|^{-1}$. From the Meyer father wavelet w^1 there is a standard way to define a scaling function w^0 and then the whole wavelet basis $Wave_M := \{w^\epsilon_R : \epsilon \in Sig_{\vec{d}}, R \in \mathcal{D}_{\vec{d}}\}$ is obtained as described above. From now on we will exclusively use this wavelet basis.

2.2. BMO **and** BMO_{-1} **spaces.** It is well known that the wavelet basis $Wave_M$ can be used to characterize the multi parameter Chang-Fefferman $BMO(\mathbb{R}^{\vec{d}})$. Namely,

$$\|b\|^2_{BMO(\mathbb{R}^{\vec{d}})} = \sup_{U \subset \mathbb{R}^{\vec{d}}} \frac{1}{|U|} \sum_{R \subset U} \sum_{\vec{\epsilon} \in Sig_{\vec{d}}} \left|\left\langle b, w^{\vec{\epsilon}}_R \right\rangle\right|^2 .$$

Here the supremeum is taken over all open sets in $\mathbb{R}^{\vec{d}}$ of finite measure. This definition is quite difficult to work with since the supremum that appears is taken over a very large class of sets.

The following version of the BMO norm turns out to be crucial in the proof of the main result.

$$\|b\|^2_{BMO_{-1}} = \sup_{\mathcal{U}} \frac{1}{|\cup\{R : R \in \mathcal{U}\}|} \sum_{R \in \mathcal{U}} \sum_{\vec{\epsilon} \in Sig_{\vec{d}}} \left|\left\langle b, w^{\vec{\epsilon}}_R \right\rangle\right|^2 ,$$

where the supremum is taken over all collections $\mathcal{U} \subset \mathcal{D}_{\vec{d}}$ with the property that there exists a coordinate s such that any two rectangles in $\mathcal{U}$ have the same s-th coordinate (which is some cube of course). L. Carlesson produced examples of functions with BMO norm 1 and arbitrary small BMO_{-1}. Still, Journes Lemma permits us, with certain restrictions, to dominate the BMO norm by the BMO_{-1} norm. This lemma is crucial ingredient in the proof of the main result.

Lemma 2.1. *Let $\mathcal{U}$ be a collection of rectangles such that $\cup\{R : R \in \mathcal{U}\}$ has finite measure. For any $\eta > 0$, we can construct $V \supset \cup\{R : R \in \mathcal{U}\}$ with $|V| < (1+\eta)\,|\cup\{R : R \in \mathcal{U}\}|$ and a function $Emb : \mathcal{U} \to [1, \infty)$ so that*

- *$Emb(R)R \subset V$ for all $R \in \mathcal{U}$*
- $$\left\| \sum_{\vec{\epsilon} \in Sig_{\vec{d}}} \sum_{R \in \mathcal{U}} Emb(R)^{-C} \left\langle f, w^{\vec{\epsilon}}_R \right\rangle w^{\vec{\epsilon}}_R \right\|_{BMO} \leq K_\eta \, \|f\|_{BMO_{-1}} .$$

The constant C appearing in the lemma can be quite big. Still, the Carlesson example shows that the term involving $Emb(R)$ must appear.

2.3. **Paraproducts.** The paraproducts that arise in the proof are of a somewhat general nature, and the authors had to make some definitions which will permit a reasonably general definition of a paraproduct.

For $j = 1, 2, 3$ let $\{\varphi_{j,R}(x) : R \in \mathcal{D}_{\vec{d}}\}$ be three families of functions with the property that

$$|D^m \varphi_{j,R}(x)| \lesssim |Q|^{-m/d} \left[Tr_{c(Q)} Dil_{|Q|} (1 + |x|^2)^{-1}\right]^N ,$$

where N is arbitrary integer and $m \leq d+1$. Define a bilinear operator B by

$$B(f,g) := \sum_{R \in \mathcal{D}_{\vec{d}}} \frac{\langle f, \varphi_{1,R} \rangle}{\sqrt{|R|}} \langle g, \varphi_{2,R} \rangle \, \varphi_{3,R}.$$

Such bilinear operators are known as paraproducts. A well known result of Journe says that if all the functions $\varphi_{1,R}$ have mean zero in all the coordinates and in addition for each coordinate s all the elements in one of the families $\varphi_{2,R}, \varphi_{3,R}$ have mean zero in the s-th coordinate then the bilinear operator $B(f,g)$ must be bounded as an operator from $BMO \times L^p$ into L^p. Particularly relevant for us is the reformulation of this theorem which says that the tensor product of two bounded paraproducts must be a bounded operator.

2.4. **Cone transforms.** For a cone in $C \subset R^d$ we fix a CalderonZygmund kernel K_C which satisfies the usual size and smoothness conditions, and in addition,

$$1_C \leq \hat{K}_C \leq 1_{(1+\kappa)C}.$$

Such operators are called cone transforms. It is useful to define a norm using the cone transforms in the place of the Riesz transforms. Namely. define

$$\|b\|_{Cone} := \sup \|[[\ldots[[M_b, T_{C_1}], T_{C_2}]\ldots], T_{C_t}]\|,$$

where the supremum is taken over all cone transform with a fixe cone aperture. As a byproduct of the proof of the main theorem one obtains that the cone norm is also equivalent to the BMO norm of b.

3. Proof of the main result

3.1. **Proof of the upper bound.** The authors prove a more general statement for the upper bound. Namely, they show that one can replace the Riesz projections by more general Calderon-Zigmund operators. The main idea is to obtain a decomposition of an one parameter commutator into a sum of paraproducts. Each of these paraproducts is bounded. Therefore, to obtain the multiparameter statement one only need to show the tensor product of the elements in the decomposition are themselves bounded operators. Note that one of the essential difficulties in the multiparameter setting is that the tensor product of bounded operators need not be bounded. So, this statement is not obvious. However, a well known result of Journe says exactly that the tensor product of two bounded paraproducts is bounded. Therefore, the upper bound would be proved if one can show that the one-parameter commutator is decomposable into a sum of paraproducts.

To accomplish this they consider the wavelet projections

$$F_j := \sum_{\epsilon \in Sig_d} \sum_{|Q| \geq 2^{jd}} w_Q^\epsilon \otimes w_Q^\epsilon,$$

and

$$\Delta F_j := \sum_{\epsilon \in Sig_d} \sum_{|Q| = 2^{jd}} w_Q^\epsilon \otimes w_Q^\epsilon,$$

and expand the commutator in these wavelet projections.

$$[T_K, M_b]f = \sum_{j,j'} [T_K, M_{\Delta F_j b}] \Delta F_{j'} f.$$

Then they split the last sum in two parts. The principal contribution comes from the part of the sum where the indices satisfy $j < j' + 3$ (this is the part of the sum with no cancellations expected). It is not hard to show that this part of the sum can be written as a sum of two paraproducts that can be estimated using the general results for paraproducts mentioned above.

The authors also prove a multi parameter version of this result that they later use in the proof of the lower bound. The strategy of the proof is very similar but the details are expectedly more involved. In this case one considers more general wavelet projections

$$F_{\vec{l},J} := \sum_{\vec{\epsilon} \in Sig_{\vec{d}}} \sum_{|Q_s|=2^{k_s}} w_R^{\vec{\epsilon}} \otimes w_R^{\vec{\epsilon}},$$

and

$$\Delta F_{\vec{k}} := \sum_{\vec{\epsilon} \in Sig_{\vec{d}}} \sum_{|Q_s| \geq 2^{k_s}, s \notin J} w_R^{\vec{\epsilon}} \otimes w_R^{\vec{\epsilon}}.$$

The second sum in the expression for $F_{\vec{l},J}$ is taken over all $R = Q_! \otimes \cdots \otimes Q_t \in \mathcal{D}_{\vec{d}}$ such that $|Q_s| \geq 2^{l_s}$ for all $s \notin J$. The following notation is also used in the result below. Write $R' \lesssim_J R$ if $|Q'_s| \leq |Q_s|$ for $s \notin J$ and $|Q'_s| = |Q_s|$ otherwise.

Theorem 3.1. *Let T_K be a product Calderon-Zygmund operator on $L^2(\mathbb{R}^{\vec{d}})$. For all $J \subset \{1, \ldots, t\}$ and $\vec{k} \in \mathbb{Z}^t$ with $k_s \in [3, 8]$ for $s \in J$ and $k_s \in [-8, 8]$ for $s \notin J$ we have*

$$\left\| \sum_{\vec{l} \in \mathbb{Z}^t} (\Delta F_{\vec{l}} b) T_K F_{\vec{l}+\vec{k},J} f \right\|_2 \lesssim \|b\|_{BMO} \|f\|_2 .$$

Moreover, for a fixed integer A suppose that whenever $\left\langle b, w_{R'}^{\vec{\epsilon}'} \right\rangle \neq 0$ and $\left\langle f, w_R^{\vec{\epsilon}} \right\rangle \neq 0$ with $R' \lesssim R$ this always implies that $AR \cap R' = \emptyset$. Then we also have the following more precise estimate

$$\left\| \sum_{\vec{l} \in \mathbb{Z}^t} (\Delta F_{\vec{l}} b) T_K F_{\vec{l}+\vec{k},J} f \right\|_2 \lesssim A^{-100t} \|b\|_{BMO} \|f\|_2 .$$

Let us return to the proof of the upper bound. The remaining part of the sum (when the indices satisfy $j \geq j' + 3$) consists of considerably smaller terms due to cancellation. The principal point in estimating this part lies in showing the inequality

$$\left| [T_K, M_{w_Q^{\epsilon}}] w_{Q'}^{\epsilon'}(x) \right| \lesssim \left| \frac{Q}{Q'} \right|^{1+1/2d} \left(1 + \frac{dist(Q, Q')}{|Q|^{1/d}} \right)^{-N} |Q|^{-1/2} \chi_{Q'}(x)^N,$$

where $\chi_{Q'}(x) := Tr_{c(Q')} Dil_{|Q'|}(1 + x^2)^{-1}$ is the Poisson kernel adapted to the cube Q'.

Very recently the same group of authors found a simpler proof of the upper bound. There, they use the classical Haar basis in the place of the Meyer wavelets,

3.2. **Proof of the lower bound.** This is the most difficult part of the proof. The proof uses induction on the number of parameters. The base case is $t = 1$. As mentioned in the beginning Coifman, Rochberg and Weiss gave a concise proof of this result. However, for the induction proof to work it turns out to be necessary to prove the base step for the cone transforms as well. Luckily, this is a consequence of a deep line of investigation begun by Uchiyama, who extended both directions of the Coifman, Rochberg, and Weiss result to more general Calderon–Zygmund operators. In particular, a result of Song-Ying Li gives as a Corollary to his Theorem, this essential result, which also covers the base case $t = 1$.

In the inductive stage of the proof, the authors use the induction hypothesis to derive a lower bound on the commutator norms in terms of the BMO_{-1}-norm. Here they use the induction hypothesis in its equivalent weak factorization form. They only give a proof of this part for commutators involving Riesz transforms, but the proof for cone transforms is similar.

The next step in the proof is to bootstrap from this weaker inequality to the full inequality. This is by far the most intricate part.

At this stage it is essential to first prove the result for cone transforms. Elements of this proof in this case are essential to address the Riesz norm case.

First, fix a symbol b with BMO norm one, but with small BMO_{-1} norm. With such b fixed, choose an open set U for which the supremum is achieved in the definition of the BMO norm of b. Let $\mathcal{U}$ be the collection of $R \in \mathcal{D}_{\vec{d}}$ such that $R \subset U$. Define $\beta := P_{\mathcal{U}} b$, where

$$P_{\mathcal{U}} b := \sum_{\vec{\epsilon} \in Sig_{\vec{d}}} \sum_{R \in \mathcal{U}} \left\langle b, w_R^{\vec{\epsilon}} \right\rangle w_R^{\vec{\epsilon}}.$$

This function β is used to build a test function to demonstrate a lower bound on the Cone norm. Choose an open set V which satisfies all the conditions in Journe's lemma, and form the collection $\mathcal{V}$ of $R \in \mathcal{D}_{\vec{d}}$ such that $R \subset V$ but which are not elements of$\mathcal{U}$. Finally, define $\mathcal{W}$ as the collection of all remaining dyadic rectangles. The final and the most difficult step is to construct two sequences of cones C_s and D_s such that the test function $\gamma := T_{D_1} T_{D_2} \dots T_{D_t} \beta$ satisfies $\|\gamma\| \geq 4^{-t}$ (is bounded away from zero) and the iterated commutator formed by M_b and $T_{C_1}, T_{C_2} \dots T_{C_t}$ applied to this γ has a norm bounded from below. This last requirement is accomplished by showing that the iterated commutators formed by $M_{P_{\mathcal{V}} b}$ (or $M_{P_{\mathcal{W}} b}$) and $T_{C_1}, T_{C_2} \dots T_{C_t}$ applied to γ have arbitrary small norm, and the one involving $M_{P_{\mathcal{U}} b}$ has a norm bounded away from zero. This finishes the prove of the lower bound.

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REPRESENTATION OF BI-PARAMETER SINGULAR INTEGRALS BY DYADIC OPERATORS

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Abstract. To extend a singular integral operator to the multiparameter case, we indeed look to define $\langle Tf, g\rangle$. Until recently, there were bascially two "extremes" to how to define this: using tensor products or Journe's operator-valued kernels. In this note we give the highlights of the new representation theorem for the desired $\langle Tf, g\rangle$ in terms of simplier shift operators. We note that this represtntation is for the biparameter case.

1. Introduction

In multiparameter harmonic analysis, the dichotomy between multiple variables and multiple parameters becomes glowingly apparent. Most every result in harmonic analysis has a clean statement for $\mathbb{R}^n$ that is essentially the same as for $\mathbb{R}$; at its essence there is no fundamental difference between a cube in $\mathbb{R}$ as a cube in $\mathbb{R}^n$. A cube is measured by one parameter, a side length. However, in $\mathbb{R}^n$, $n \geq 1$, we have a choice. Cubes are no longer the sole ubiquitous objects, we can also look at all the fun objects such as the maximal function with respect to rectangles which have two parameters. Starting with the theme of the maximal function, we can see how the strong maximal function defined with respect to rectangles is different from the usual Hardy-Littlewood maximal function.

The standard example of a singular integral operator is the Hilbert transform. We can create the biparameter version with respect to rectangles, which is much more complicated. Though we can write it as the tensor product $H_1 H_2 = *\frac{1}{x_1 x_2}$, the singularity now encompasses the coordinate axes instead of a single point.

The underlying geometry of $\mathbb{R}$ versus $\mathbb{R}^n, n > 1$ lies at the heart of this difficulty. Every open set in $\mathbb{R}$ can be decomposed as a disjoint union of open intervals. This aids us immensely in decomposition, and we do not have a comparable way of decomposing open sets for $n > 1$.

To extend a singular integral operator to the multiparameter case, we indeed look to define $\langle Tf, g\rangle$. Until recently, there were bascially two "extremes" to how to define this: using tensor products or Journe's operator-valued kernels.

What this paper does is prove a new representation theorem for the desired $\langle Tf, g\rangle$ in terms of simplier shift operators. We note that this represtntation is for the biparameter case. Many times, generalizing even farther is much more difficult since underlying the biparameter case is the fact that "slices" are one-parameter, and hence we can use the open set decomposition in $\mathbb{R}$. The interesting part is how this representation is achieved: through a new characterization of Calderon-Zygmund operators, inspired by [3], without resorting to vector valued techniques or a priori boundedness estimates. This representation allows us to get our hands on how T operates on different spaces. The main theorem of this paper is

Theorem 1.1. *We have*

$$\langle Tf, g\rangle = C_T E_{w_n} E_{w_m} \sum_{i,j\in\mathbb{Z}} 2^{-max(i_1,i_2)\delta/2} 2^{-max(j_1,j_2)\delta/2} \langle S^{i,j} f, g\rangle \tag{1.2}$$

where the shifts S are taken with respect to the pair of dyadic grdis $(\mathcal{D}_n, \mathcal{D}_m)$ and $w_n \in \{\{0,1\}^n\}^{\mathbb{Z}}$ corresponds to a random shift.

A recent application of the importance of this representation is the famous A_2 conjecture, proved by Hytonen and Lerner, though with contributions by many, many others [4],[1]. Both the techniques leading to the proof of 1.1 and the applications of the result are fascinating! The type of representation in 1.1 and the lemmas used to prove it are in the flavor of [4]. And there are many other open questions that can possibly be attacked using such a representation, as it translates singular intergral operator questions to dyadic questions.

2. Assumptions and notations

We begin by introducing this new Calderon-Zygmund structure of our operators.

Definition 2.1. We have the representation

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x,y) f(y) g(x) dx dy \tag{2.2}$$

Some notations and assumptions include: $f = f_1 f_2 = f_1 \otimes f_2$ to indicate the splitting of f into the two parameters, $supp f_1 \cap g_1 = \emptyset$ and similarily for f_2, g_2. We stress that $x, y \in \mathbb{R}^{n+m}$ whereas we use subscripts for each parameter: $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$.

And here are the kernel assumptions. Notice how they differ from the one-parameter case.

Definition 2.3. The many assumptions include the single decay condition:

$$|K(x,y)| \le C \frac{1}{|x_1 - y_1|^n} \frac{1}{|x_2 - y_2|^m},$$

the Holder smoothness in x and y – notice that for these the numerators change to include the 2^2 choices (eiterh x_1 or y_1 for the first estimate and similarity for the second).

$$|K(x,y) - K(x,(y_1,y_2')) - K(x,(y_1',y_2)) + K(x,y')| \le C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|y_1 - y_1'| \le |x_1 - y_1|/2$ and $|y_2 - y_2'| \le |x_2 - y_2|/2$,

$$|K(x,y) - K((x_1,x_2'),y) - K((x_1',x_2),y) + K(x',y)| \le C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|x_1 - x_1'| \le |x_1 - y_1|/2$ and $|x_2 - x_2'| \le |x_2 - y_2|/2$,

the mixed smoothness

$$|K(x,y) - K((x_1,x_2'),y) - K(x,(y_1',y_2)) + K((x_1',x_2),(y_1',y_2))| \le C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|y_1 - y_1'| \le |x_1 - y_1|/2$ and $|x_2 - x_2'| \le |x_2 - y_2|/2$,

$$|K(x,y) - K(x,(y_1,y_2')) - K((x_1',x_2),y) + K((x_1,x_2),(y_1,y_2'))| \le C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|x_1 - x_1'| \le |x_1 - y_1|/2$ and $|y_2 - y_2'| \le |x_2 - y_2|/2$.

Also, we have the combined decay/smoothness discussion – conditions provided to account for decay in one variable and Holder in the other variable (four additional conditions).

We assume the Calderon-Zygmund structure in each parameter, where we enforce the "sliced" kernel representation:

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{f_1,g_1}(x_1,y_1)f_1(y_1)g_1(x_1)dx_1dy_1$$

with decay

$$|K_1(x_1,y_1)| \leq C(f_2,g_2)\frac{1}{|x_1-y_1|^n}$$

and smoothness in x and y

$$|K_1(x_1,y_1) - K(x_1',y_1)| \leq C(f_2,g_2)\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}$$

$$|K(x_1,y_1) - K(x_1,y_1')| \leq C(f_2,g_2)\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}$$

when $|x_1-x_1'| << |x_1-y_1|/2$ and $|y_1-y_1'| < |x_1-y_1|/2$. Here $C(f,g)$ is a constant with small control over the diagonal: $C(\chi_v,\chi_v)+C(\chi_v,u_v)+C(u_v,\chi_v) \leq C|V|$, where u_v is V *adapted with zero mean*, that is $supp(u_v) \subset V, |u_v| \leq 1$ and $\int u_v = 0$. We also require the corresponding conditions for a K_2 representation.

Note that besides the constants' dependence, these are the same as for the single parameter case. These sliced conditions are what you would expect for a tensor product generalization of singular integrals – that the kernel estimates are required for each variable separately. Here, however, we see that these are just some of the requirements, as boundedness is much more complex. This corresponds to the size of singularities and our earlier discussion.

Journe dealt with multiparameter operators using vector valued inequalities. He thought of the kernel as an operator in one parameter, with the other parameter fixed. This allowed him the much simpler definition statement of a Calderon-Zygmund kernel, however, this definition encodes much complexity. Additionally, we have a priori boundedness assumptions mentnioned earlier.

Definition 2.4. Let B is a Banach space and $0 < \delta < 1$. Journe's vector valued kernerl is a continous function $K : \mathbb{R}^2/\Delta \to B$ such that:

$$\|K(x,t)\|_B \leq C\frac{1}{|x-t|^\delta}$$

and

$$\|K(x,t) - K(x',t')\|_B \leq C\frac{(|x-x'|+|t-t'|)^\delta}{|x-t|^{1+\delta}}$$

when $(|x-x'|+|t-t'| < |x-t|)/2$.

An essential point to make is that a Calderon-Zygmund operator is one with a Calderon-Zygmund kernel but is additionally bounded on L^2. Once we have this, we get many other boundedness conditions, such as boundedness on L^p, for free; hence showing the original L^2 bound for an operator with Calderson-Zygmund kernel is quite important. The original celebrated $T(1)$ theorem of David and Journe gave us simple criteria to ensure a L^2 bound using only the boundedness of $T(1), T^*(1)$ and the easily satisfied weak boundedness property (WBP). To get the L^2 bound here, we have a $T(1)$ type theorem assumptions:

$T(1), T^*(1), T_1(1), T_1^*(1) \in BMO_p$ and a WBP: $|\langle T(\chi_K \otimes \chi_V), \chi_K \otimes \chi_V\rangle| \leq C|K||V|$ for all cubes $K \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ where T_1 is the partial adjoint.

The BMO_p is the product BMO space, which is not so easy to generalize in more than one parameter. There are several canddidate spaces, but the right one should ensure the boundedness of $T : L^\infty \to BMO$ for operators $T = T_1T_2$. In this paper, we use the dual H^1 via boundedness of a square funtion to define BMO_p, but here is the direct definition in the dyadic setting (using the Haar basis). We stick with the dyadic case, since it is simpler to state and still illustrates what this "correct" BMO_p space is. For the continuous version using continuous wavelet basis, see [2].

Definition 2.5. We say $f \in BMO_p$ if

$$\sup_{\Omega} \sum_{I \times J \in \Omega} \langle f, h_I \otimes h_J\rangle^2 \leq C$$

where Ω is any open set. The $\{h_I\}$ are Haar functions $h_I = |I|^{-1/2}(\chi_l - \chi_r)$, where χ_l is the characteristic function of the left half of a dyadic interval, and χ_r is the right half. The h_I form a basis of L^2 as well as many other Banach spaces, as long as we add the noncancellative constant function 1.

The Haar functions form a localized basis, which naturally fit with the dyadic structure of a space. We can then define the sqaure function via Haar:

Definition 2.6. The square function is

$$Sq(f) = [\sum_{K \in D_n} \sum_{V \in D_m} |\langle f, h_K \otimes u_V\rangle|^2 \frac{\chi_K \otimes \chi_V}{|K||V|}]^{1/2}$$

Then f is in the product Hardy space H^1 if and only if $\|S_q(f)\|_{L^1} < \infty$.

We also need a few "diagonal" BMO conditions (using characteristic functions and some adapted functions).

3. Main results

A key concept to the proof of 1.1 is the use of random dyadic grids. A basic averaging property with regards to these grids will be proved, allowing us to rewrite the desired decomposition.

Definition 3.1. Let D_n be the standard dyadic grid and w_n is as defined before. Then we define the shifted dyadic grid $D_n + w = \{I + w : I \in D_n\}$ where, $w = \sum_{2^{-i} < l(I)} 2^{-1} w_n^i$.

Now we define the fundamental notions of good and bad cubes.

Definition 3.2. We call a dyadic cube $I \in \mathbb{R}^n$ bad if there is another cube J such that both $l(I) \geq 2^r l(J)$ and $d(I, \partial(J) \leq 2l(I)^\gamma l(J)^{1-\gamma}$ where $\gamma = \delta/(2n + 2\delta)$ (so since δ is a small parameter, γ is about $\delta/2$, also small and r is a fixed parameter set to make the probability of having a good situation positive. We stress that this is always possible by lemma 2.3 of [4].

Note that there is a more general definition of bad cubes, from [4], where one defines a radially decreasing function, ϕ, with certain properties, and requiring $d(I, \partial(J) \leq \phi(\frac{l(I)}{l(J})l(J)$. Martikainen's choice corresponds to the classical $\phi(t) = t^\gamma$.

There are a few key facts to note about good and bad cubes. First of all, intuitively, the definition of badness indicates that there is a much bigger cube very close to our given cube I. Second, the position $I+w$ is independent to the badness (since this involves relative position to the larger cubes $J+w$, hence only dependent on w_j for $2^{-j} \geq l(I)$).

We now define the objects needed to represent $\langle Tf, g\rangle$ that are easy to bound and analyze – the biparameter shifts – which are always tied to a dyadic grid $D_a \otimes D_b$ on $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 3.3. Given nonnegative integers $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}^2$, define

$$S^{(i,j)} f = \sum_{K \in D_a} \sum_{V \in D_b} A_{KV} f \tag{3.4}$$

where

$$A_{KV} f = \sum_{I_1, I_2 \subseteq K} \sum_{J_1, J_2 \subseteq V} a_{I,K,J,K} \langle f, h_{I_1} \otimes u_{J_1}\rangle h_{I_2} \otimes u_{J_2}$$

and $l(I_1) = 2^{-i_1} l(K)$ and similarity for i_2, j_1 and j_2. Moreover, we have that $a \leq \frac{|I_1||I_2||J_1||J_2|}{|K||V|}$. and the subshifts (where $K \in A, V \in B$) are L^2 bounded with a maximum norm of 1.

Note that we can rewrite the shift S in the handy kernel representation:
(3.5)

$$Sf(x) = \sum_{K,V} A_{KV} f(x) = \sum_{K,V} \frac{1}{|K \times V|} \int_{K \times V} K_{AV}(x,y) f(y) dy = \int_{\mathbb{R}^{n+m}} K_S(x,y) f(y) dy$$

where K is constant on dyadic rectangles smaller than those in the summands of 3.4, has norm ≤ 1 and bounded support in $(K \times V) \times (K \times V)$. As an example, we can consider shifts where $i_1 = i_2 = j_1 = j_2 = 1$, which give rise to kernels constant on quarters of "Haar rectangles".

The following lemma allows us to decompose using only good cubes. This is one of the most important observations and this is the only time where the probabilistic structure of random dyadic grids is used. The original is from [4].

Lemma 3.6. *We have*

$$\langle Tf, g\rangle = C \mathbb{E}_{w_n} \mathbb{E}_{w_m} \sum_{I_1, I_2 \in D_n} \sum_{J_1, J_2 \in D_m} \chi_{good}(smaller(I)) \chi_{good}(smaller(J) \langle T(h_{I_1} \otimes u_{J_1}), h_{I_2} \otimes u_{J_2}\rangle \langle f, h_{I_1} \otimes u_{J_1}\rangle \langle g, h_{I_2} \otimes u_{J_2}\rangle$$

where $C = 1/(\pi^n_{good} \pi^m_{good})$ *and the summation over all the* $2^n - 1$ *or* $2^m - 1$ *cancellative Haar functions is suppressed.*

Thanks to 3.6, to prove 1.1 we now separate the sums over good cubes only. First we fix the random variables, which fixes the dyadic grids (hence, we do not need the probability any more)! We need to examine

$$\sum_{l(I_1) \leq l(I_2)} \sum_{l(J_1) \leq l(J_2)} \langle T(h_{I_1} \otimes u_{J_1}, h_{I_2} \otimes u_{J_2}\rangle \langle f, h_{I_1} \otimes u_{J_1}\rangle \langle g, h_{I_2} \otimes u_{J_2}\rangle$$

and divide the sum that appears as follows

$$\sum_{l(I_1) \leq l(I_2)} = \sum_{d(I_1, I_2) > l(I_1)^\gamma l(I_2)^{1-\gamma}} + \sum_{I_1 \subseteq I_2} + \sum_{I_1 = I_2} + \sum_{d(I_1, I_2) \leq l(I_1)^\gamma l(I_2)^{1-\gamma}, I_1 \cap I_2 = \emptyset}$$

where the four sums are denoted separate, in, equal, and near. We must do this for when $l(I_1) \leq l(I_2)$ and $l(J_1) \leq l(J_2)$ resulting the mixed types (ie: separate for $\mathbb{R}^n$ and near for $\mathbb{R}^m$, etc., a total of 4+3+2+1 cases). We also consider the symmetric case $l(I_1) \geq l(I_2)$

and $l(J_1) \geq l(J_2)$, along with some partial symmetric cases that are handled differently: $l(I_1) \leq l(I_2)$ and $l(J_1) > l(J_2)$ or $l(J_1) \leq l(J_2)$ and $l(I_1) > l(I_2)$. Remeber that we have suppressed the critical fact that all cubes are good!

The proof of 1.1 is a case by case analysis of each scenario mentioned above. By extimation, each piece can be estimated by a sum of the simple shifts with good decay factors, allowing us the representation 1.2 and hence the following corollary.

Corollary 3.7. *The singular integral operator defined in 1.2 is L^2 bounded.*

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A $T(1)$ THEOREM ON PRODUCT SPACES

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ABSTRACT. Only a year after publishing his celebrated classical $T(1)$ theorem with G. David [2], J. L. Journé established an extension to product spaces [3]. The conditions in his extension require that operator bounds be established, in stark contrast to the classical theorem. In their paper [5], S. Pott and P. Villarroya, establish a new $T(1)$ theorem for product spaces which not only is more in the spirit of the classical one, but also applies to operators that do not fall under the scope of Journé's theorem.

1. The classical $T(1)$ theorem

Calderón-Zygmund theory is concerned with the study of singular integral operators that roughly speaking are of the type

$$T(f)(x) = \int K(x,y)f(y)\ dy.$$

The main interest is in establishing L^p estimates for such operators. In order to hope for such estimates conditions need to be imposed on the kernel $K(x,y)$. A classical choice are Lipschitz type regularity conditions. A kernel is said to be a standard Calderón-Zygmund kernel if there exist $\delta > 0$, $C < \infty$ such that for $x, y \in \mathbb{R}^n$ and all $z \in \mathbb{R}^n$ such that $|x-z| < \frac{|x-y|}{2}$ we have:

(i) $|K(x,y)| \leq C|x-y|^{-n}$

(ii) $|K(x,y) - K(z,y)| \leq C\left(\dfrac{|x-z|}{|x-y|}\right)^{\delta} |x-y|^{-n}$

(iii) $|K(y,x) - K(y,z)| \leq C\left(\dfrac{|x-z|}{|x-y|}\right)^{\delta} |x-y|^{-n}$

To be more precise with the definition of singular integral operators then we say that an operator T is associated to a standard Calderón-Zygmund kernel K if, whenever $f, g \in C_0^\infty$ have disjoint supports,

$$\langle T(f), g\rangle = \iint K(x,y)f(y)g(x)dydx.$$

Further one can associate a bilinear form Λ to a standard Calderón-Zygmund kernel K if, whenever $f, g \in C_0^\infty$ have disjoint supports,

$$\Lambda(f,g) = \iint K(x,y)f(y)g(x)dydx.$$

The dual operator, T^*, of T is then defined by

$$\langle f, T^*(g)\rangle = \langle T(f), g\rangle = \Lambda(f,g).$$

In order to state the classical $T(1)$ theorem we first need a couple of definitions.

Definition 1. *A Schwartz function $\phi : \mathbb{R}^m \to \mathbb{C}$ is said to be an L^p-normalized bump function adapted of order N to a box $I := I_1 \times \ldots \times I_m$ if there exists a constant C such that*

$$|\partial^\alpha \phi(x)| \le C \prod_{k=1}^{m} |I_k|^{-\frac{1}{p}-\alpha_k} \chi_I^N(x)$$

for each $0 \le |\alpha| \le N$, where

$$\chi_I(x) = \left(1 + \| \left(\frac{x_1 - c(I_1)}{|I_1|}, \ldots, \frac{x_m - c(I_m)}{|I_m|}\right) \|^2\right)^{-1/2}$$

where $c(I_j)$ denotes the center of the interval I_j.

Definition 2. *A singular integral operator T is said to be weakly bounded if there exist $N, C < \infty$ such that for any box I and any L^2-adapted bump functions ϕ_I, ψ_I of order N we have*

$$|\langle T(\phi_I), \psi_I \rangle| \le C$$

Note that L^2 boundedness of the operator T implies weak boundedness.

Definition 3. $\mathsf{BMO}_{\mathsf{rect}}$ *is the set of equivalence classes of locally integrable functions f modulo additive constants for which the following supremum, taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes, is finite*

$$\sup_Q |Q|^{-1} \int_Q |f - f_Q| = \|f\|_{\mathsf{BMO}_{\mathsf{rect}}}.$$

where $f_Q = |Q|^{-1} \int_Q f$.

We are now ready to state the classical $T(1)$ theorem.

Theorem 4. [2] *Suppose T is a singular integral operator. Then $T : L^2 \to L^2$ if and only if*

- *T is weakly bounded*
- $T(1) \in \mathsf{BMO}_{\mathsf{rect}}$
- $T^*(1) \in \mathsf{BMO}_{\mathsf{rect}}$

Note that $T(1)$ and $T^*(1)$ both have to be defined through distribution theory.

2. Journé's extension to product spaces

Journé was interested in obtaining a similar $T(1)$ theorem for multiparameter singular integral operators. These are operators whose class of kernels is homogeneous with respect to non-isotropic dilations of the form $\rho_{\delta_1,\ldots,\delta_n}(x_1, \ldots, x_n) = (\delta_1 x_1, \ldots, \delta_n x_n)$ for $x_i \in \mathbb{R}^{d_i}$ and $\delta_i > 0$, where the number of the parameters of the problem coincides with the quantity of independent dilations. A simple example of such an operator is the multiple Hilbert transform, defined in $\mathbb{R}^n$ by

$$H_1 \cdots H_n(f) = \text{ p.v. } f * \frac{1}{x_1 \cdots x_n}.$$

We see that the kernel is not a standard Calderón-Zygmund kernel so the classical $T(1)$ theorem does not apply. Of course a direct application of Fubini's theorem shows that the operator is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, however the situation is in general not so simple.

One immediate observation is that the kernels for multiparameter singular integral operators can be far more singular than standard Calderón-Zygmund kernels. As a consequence, these operators do not generally map $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ and the strong maximal operator does not control their boundedness properties. Hence many of the standard techniques are not available in the multiparameter setting. To overcome this difficulty Journé chose to use vector valued Calderón-Zygmund theory, which thus requires us to go through some definitions.

Definition 5. *Let Δ be the diagonal in $\mathbb{R}^2$ and B be a Banach space. A continuous function $K : \mathbb{R}^2 \setminus \Delta \to B$ is called a vector valued standard Calderón-Zygmund kernel, if for some $0 < \delta \leq 1$ and some constant $C > 0$ we have*

$$\|K(x,t)\|_B \leq C|x-t|^{-1}$$

$$\|K(x,t) - K(x',t')\|_B \leq C(|x-x'| + |t-t'|)^{\delta}|x-t|^{-1-\delta}$$

whenever $|x-x'| + |t-t'| \leq |x-t|/2$. In this context $|K|$ usually denotes the best constant in both inequalities.

Definition 6. *A continuous linear mapping T from $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ into its algebraic dual is called a singular integral operator if there are $K^1, K^2 : \mathbb{R}^2 \setminus \Delta \to \mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$ vector valued Calderón-Zygmund kernels such that for $f_1, f_2, g_1, g_2 \in C_0^\infty(\mathbb{R})$ we have*

$$\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle \iint_{\mathbb{R}^2} f_1(t_1) g_1(x_1) \langle K^1(x_1,t_1) f_2, g_2 \rangle dt_1 dx_1$$

whenever supp $f_1 \cap$ supp $g_1 = \emptyset$ and symmetrically for K^2.

Observe that K^1 being a vector valued Calderón-Zygmund kernel implies that $K^1(x_1,t_1)$ is a Calderón-Zygmund operator bounded on $L^2(\mathbb{R}^2)$ and that its Calderón-Zygmund norm, defined by $\|K^1(x_1,t_1)\|_{CZ} := \|K^1(x_1,t_1)\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} + |K^1|$ satisfies the bounds from definition 5. Similar for K^2. Thus we have already encountered operator norms in contrast to the classical $T(1)$ theorem.

Definition 7. *We define restricted operators T^i, $i = 1, 2$, in the following way. Given $f_i, g_i \in C_0^\infty(\mathbb{R})$ for $i = 1, 2$, let $\langle T^1(f_2), g_2 \rangle, \langle T^2(f_1), g_1 \rangle : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})'$ be defined by*

$$\langle \langle T^1(f_2), g_2 \rangle f_1, g_1 \rangle = \langle \langle T^2(f_1), g_1 \rangle f_2, g_2 \rangle = \langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle$$

Notice that the kernel of T^1 for example is precisely $\langle K^1(x_1,t_1) f_2, g_2 \rangle$.

Definition 8. *A singular integral operator T is said to satisfy the weak boundedness property if for any bounded subset A of $C_0^\infty(\mathbb{R})$ there is a constant $C > 0$, that may depend on A, such that for any $f, g \in A$ we have that*

$$\|\langle T^i(f_{x,R}), g_{x,R} \rangle\|_{CZ} := \|\langle T^i(f_{x,R}), g_{x,R} \rangle\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} + |K^i| \leq C$$

where $f_{x,R} = R^{-1/2} f(R^{-1}(y-x))$ and the same for $g_{x,R}$.

Observe that here we have encountered more operator norms.

Definition 9. *Associated to a singular integral operator T we can define its partial adjoints T_1 and T_2 by*

$$\langle T_1(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \langle T(g_1 \otimes f_2), f_1 \otimes g_2 \rangle$$

and similarly for T_2.

Notice that $T_2 = T_1^*$.

Definition 10. *Define $\mathsf{BMO}_{\mathsf{prod}}(\mathbb{R}^n)$ to be the dual of the Hardy space $H^1(\mathbb{R}^n)$.*

Note that $\mathsf{BMO}_{\mathsf{rect}}(\mathbb{R})$ coincides with $\mathsf{BMO}_{\mathsf{prod}}(\mathbb{R})$. However for $n \geq 2$ then $\mathsf{BMO}_{\mathsf{rect}}(\mathbb{R}^n) \subsetneq \mathsf{BMO}_{\mathsf{prod}}(\mathbb{R}^n)$ as shown in an example by Carleson. Thus one can say that $\mathsf{BMO}_{\mathsf{prod}}(\mathbb{R}^n)$ is more complicated in general than $\mathsf{BMO}_{\mathsf{rect}}(\mathbb{R}^n)$ from the classical $T(1)$ theorem.

We are now ready to state Journé's theorem.

Theorem 11. [3] *Let T be a singular integral operator as described in definition 6 satisfying the weak boundedness property and $T(1), T^*(1), T_1(1), T_1^*(1) \in \mathsf{BMO}_{\mathsf{prod}}(\mathbb{R}^2)$. Then T extends boundedly on $L^2(\mathbb{R}^2)$.*

3. Main theorem

The use of vector valued Calderón-Zygmund theory was also adopted by other authors, such as R. Fefferman, in later developments of singular integral operators in product spaces. S. Pott and P. Villarroya wanted to get rid of any hypothesis that required a priori boundedness of operators and wanted a theorem more in the spirit of the classical $T(1)$ theorem. This meant dropping the vector valued Calderón-Zygmund theory. Thus we need to go through a new set of definitions before we can state the main theorem.

Definition 12. *Let Δ be the diagonal in $\mathbb{R}^2$. A function $K : (\mathbb{R}^2 \setminus \Delta) \times (\mathbb{R}^2 \setminus \Delta) \to \mathbb{R}$ is called a product Calerón-Zygmund kernel, if for some $0 < \delta \leq 1$ and some constant $C > 0$ we have*

$$|K(x,t)| \leq C \prod_{i=1,2} \frac{1}{|x_i - t_i|}$$

$$|K(x,t) - K((x_1, x_2'), (t_1, t_2')) - K((x_1', x_2), (t_1', t_2)) + K(x', t')| \leq C \prod_{i=1,2} \frac{(|x_i - x_i'| + |t_i - t_i'|)^\delta}{|x_i - t_i|^{1+\delta}}$$

whenever $2(|x_i - x_i'| + |t_i - t_i'|) \leq |x_i - t_i|$.

Instead of now focusing on an operator associated to such a kernel we focus rather on a bilinear form associated to it. We start with a couple of definitions.

Definition 13. *Given a bilinear form Λ, we define linear operators T, T^* through duality:*

$$\langle T(f), g \rangle = \langle f, T^*(g) \rangle = \Lambda(f, g).$$

Definition 14. *We define the restricted bilinear forms by*

$$\langle \Lambda^1(f_2, g_2) f_1, g_1 \rangle = \langle \Lambda^2(f_1, g_1) f_2, g_2 \rangle = \Lambda(f_1 \otimes f_2, g_1 \otimes g_2)$$

We will call restricted operators T^i to the linear operators associated with the restricted bilinear form Λ^i through duality $\Lambda^i(f_j, g_j) = \langle T^i(f_j), g_j \rangle$.

Notice that the kernels of the forms Λ^i depend on the variables of the functions f_j, g_j and so we will often write $\Lambda^i_{t_j, x_j}$.

Definition 15. *A bilinear form $\Lambda : \mathcal{S}(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ is said to be associated with a product Calderón-Zygmund kernel K if it satisfies the following three integral representations:*

1. *for all Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^2)$ such that $f(\cdot, t_2), g(\cdot, x_2)$ and $f(t_1, \cdot), g(x_1, \cdot)$ have respectively disjoint supports, we have*
$$\Lambda(f,g) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t)g(x)K(x,t)dxdt$$
2. *for all Schwartz functions $f_1, f_2, g_1, g_2 \in \mathcal{S}(\mathbb{R})$ such that f_1 and g_1 have disjoint supports, we have*
$$\Lambda(f,g) = \int_{\mathbb{R}}\int_{\mathbb{R}} f_1(t_1)g_1(x_1)\Lambda^1(f_2,g_2)dx_1dt_1$$
3. *analogous representation for Λ^2.*

If the form is continuous on $\mathcal{S}(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)$ then it will be called a bilinear Calderón-Zygmund form.

We define the weak boundedness condition the same way as in definition 2. One of the novelties from S. Pott and P. Villarroya is the following new condition we need.

Definition 16. *We say that a bilinear form Λ satisfies the mixed weak boundedness Calderón-Zygmund condition, if there exist $N, C < \infty$ such that for any interval I and any L^2-adapted bump functions ϕ_I, ψ_I of order N we have*
$$|\Lambda^i_{t_j,x_j}(\phi_I,\psi_I)| \le C|t_j - x_j|^{-1}$$
$$|(\Lambda^i_{t_j,x_j} - \Lambda^i_{t'_j,x'_j})(\phi_I,\psi_I)| \le C(|x_j - x'_j| + |t_j - t'_j|)^{\delta}|t_j - x_j|^{-(1+\delta)}$$
whenever $2(|x_j - x'_j| + |t_j - t'_j|) < |t_j - x_j|$ for all $i, j \in \{1,2\}$.

Definition 17. *We define the adjoint bilinear forms Λ_i such that for $f = f_1\otimes f_2$, $g = g_1\otimes g_2$ functions of tensor product type, we have*
$$\Lambda_1(f,g) = \Lambda(g_1\otimes f_2, f_1\otimes g_2),\ \ \Lambda_2(f,g) = \Lambda(f_1\otimes g_2, g_1\otimes f_2)$$
and then extended by linearity and continuity. We will also denote $\Lambda_0 = \Lambda$.

These new bilinear forms are also associated with linear operators T_1, T_2 via duality
$$\langle T_i(f), g\rangle = \langle f, T_i^*(g)\rangle = \Lambda_i(f,g).$$

Note that in the case of tensor products $f = f_1\otimes f_2, g = g_1\otimes g_2$ then
$$\langle T_1(f_1\otimes f_2), g_1\otimes g_2\rangle = \Lambda_1(f,g) = \Lambda(g_1\otimes f_2, f_1\otimes g_2) = \langle T(g_1\otimes f_2), f_1\otimes g_2\rangle$$
which matches with the adjoint bilinear forms in definition 9.

We are now finally ready to state the bi-parameter $T(1)$ theorem by S. Pott and P. Villarroya.

Theorem 18. [5] *Let Λ be a bilinear Calderón-Zygmund form satisfying the mixed weak boundedness Calderón-Zygmund condition. Then the following are equivalent:*

1. *Λ_i are bounded bilinear forms on $L^2(\mathbb{R}^2)$ for all $i = 0, 1, 2$. (Recall $\Lambda_0 = \Lambda$.)*
2. *Λ satisfies the weak boundedness condition and the special cancellation conditions:*
   (a) $T(1), T^*(1), T_1(1), T_1^*(1) \in \mathsf{BMO}_{\mathsf{prod}}(\mathbb{R}^2)$

(b) $\langle T(\phi_I \otimes 1), \psi_I \otimes \cdot\rangle, \langle T(1 \otimes \phi_I), \cdot \otimes \psi_I\rangle, \langle T^*(\phi_I \otimes 1), \psi_I \otimes \cdot\rangle, \langle T^*(1 \otimes \phi_I), \cdot \otimes \psi_I\rangle \in \mathsf{BMO}(\mathbb{R})$ *for all* ϕ_I, ψ_I *bump functions adapted to* I *with norms uniformly bounded in* I.

As with the classical theory then care has to be taken in defining all the objects in the theorem carefully through distribution theory. We also note that the authors state a general multiparameter version of the above theorem.

A weak point of the result, that is in common with Journé's result, is that the stated sufficient conditions are not necessary. The conditions imply not only boundedness of T but also of T_1. Journé constructed a counterexample for which T is bounded but not T_1.

The proof of the theorem follows the same general ideas as the proof of the classical $T(1)$ theorem. First one shows that the theorem holds true for bilinear forms Λ that fulfill a certain special cancellation property. Then one shows that the special cancellation property can always be obtained by subtracting certain paraproducts from a general bilinear form. The proof is of course much more complex than in the classical case and the authors bump among other things into new bi-parameter modified square functions.

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H^1 AND DYADIC H^1

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Abstract. We summarize a simple proof of the fact that the average over all dyadic lattices of the dyadic H^1-norm of a function gives an equivalent H^1-norm. This proof works for both one-parameter and multi-parameter Hardy spaces.

1. Introduction

It has long been considered that:

How can we relate H^1 to dyadic H^1 and BMO to dyadic BMO?

A first result of such type on H^1 for the one-parameter case is proved in [2]. By duality, such result is equivalent to the statement on BMO for the one-parameter case in [3]. Also, we have result on BMO for the two-parameter case in [4]. In this note, we will summarize a proof given by S. Treil on H^1 in [1] which works for both one-parameter and milti-parameter cases. Most recently, J. Phiper and her coauthors in [5] have even characterized H^1 and BMO by using only finitely many well-chosen dyadic lattices.

2. Preliminaries

We first introduce some improtant notations and terminologies:

2.1. Cubes and dyadic lattices.

Definition 2.1. A *cube* in $\mathbb{R}^N$ is an object obtained from the *standard cube* $[0,1)^N$ by dilations and shifts.
We use $l(Q)$ to denote the side-length of cube Q. Given a cube Q, one can split it by dividing each side in halves into 2^N cubes Q_k of side-length $l(Q)/2$: Q_k are called the *children* of Q.

Definition 2.2. The *standard dyadic lattice* $\mathcal{D}_0$ on $\mathbb{R}^N$ is

$$\mathcal{D}_0 := \{([0,1)^N + j) \cdot 2^k : j \in \mathbb{Z}^N, k \in \mathbb{Z}\}.$$

A *dyadic lattice* $\mathcal{D}$ is a shift of the standard dyadic lattice $\mathcal{D}_0$.

2.2. Random dyadic lattice. Our random lattice will contain the dyadic cubes of standard size 2^k ($k \in \mathbb{Z}$), but will be "randomly shifted" with respect to the standard dyadic lattice $\mathcal{D}_0$.

First construct a random lattice of dyadic intervals on the real line $\mathbb{R}$:

Let $(\Omega, \mathbb{P})$ be some probability space and let $x(\omega)$ be a radom variable uniformly distributed over the interval [0,1).

Let $\xi_j(\omega)$ be random variables satisfying $\mathbb{P}\{\xi_j = +1\} = \mathbb{P}\{\xi_j = -1\} = 1/2$. Assume also that $x(\omega), \xi_j(\omega), j \in \mathbb{N}$ are independent. Define the random lattice $\mathcal{D}(\omega)$ as follows:

(i) We require that $I_0(\omega) := [x(\omega) - 1, x(\omega)] \in \mathcal{D}(\omega)$; this gives us all intervals in $\mathcal{D}(\omega)$ of length 2^k, $k < 0$.

(ii) To determine the rest of the intervals, it is enough to know dyadic intervals $I_k(\omega) \supset I_0(\omega)$, of length 2^k, $k \geq 0$. The intervals $I_k(\omega)$ are determind inductively: if $I_{k-1}(\omega)$ is already known (and thus all intervals of length 2^{k-1} in $\mathcal{D}(\omega)$), then
- $I_k(\omega)$ is the union of $I_{k-1}(\omega)$ and its right neighbour if $\xi_k(\omega) = +1$, and
- $I_k(\omega)$ is the union of $I_{k-1}(\omega)$ and its right neighbour if $\xi_k(\omega) = -1$.

To get a random dyadic lattice in $\mathbb{R}^N$ we just take N independent random dyadic lattices $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_N$ in $\mathbb{R}$ and consider all cubes $Q = I_1 \times I_2 \times ... \times I_N$, $I_k \in \mathcal{D}_k$.

3. Statement of the Main Theorems

3.1. **One-parameter case.** Let $H^1 = H^1(\mathbb{R}^N)$ be the usual *real variable* Hardy space on $\mathbb{R}^N$, and let $H^1_{\mathcal{D}}$ be its dyadic counterpart, defined as follows.

Consider a dyadic lattice $\mathcal{D}$ in $\mathbb{R}^N$. Let $E_k = E_k^{\mathcal{D}}$ be the *averaging operator* over cubes $Q \in \mathcal{D}$ of size 2^k, $E_k f(x) := |Q|^{-1} \int_Q f$, where $Q \in \mathcal{D}$ is the cube in of size 2^k containing x.

Define the *difference operator* by $\Delta_k = \Delta_k^{\mathcal{D}} := E_{k-1}^{\mathcal{D}} - E_k^{\mathcal{D}}$, and define the *dyadic square function* $S = S_{\mathcal{D}}$ by

$$(S_{\mathcal{D}} f)(x) := \left(\sum_{k \in \mathbb{Z}} |\Delta_k^{\mathcal{D}} f(x)|^2 \right)^{1/2}.$$

Definition 3.1. A function $f \in L^1_{\text{loc}}$ is in the *dyadic Hardy space* $H^1_{\mathcal{D}}$ (with respect to the dyadic lattice $\mathcal{D}$) if $||f||_{H^1_{\mathcal{D}}} := ||S_{\mathcal{D}} f||_1 < \infty$.

Now let $\mathcal{D}(\omega)$, $\omega \in \Omega$ be the random dyadic lattice, as described above, and let $\mathbb{E} = \mathbb{E}_\omega$ denotes the expectation with respect to ω, we have

Theorem 3.2. *A function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ belongs to H^1 if and only if*

$$\int_{\mathbb{R}^N} [\mathbb{E}(|S_{\mathcal{D}(\omega)} f(x)|^2)]^{1/2} dx < \infty.$$

Moreover, the latter quality gives an equivalent norm on H^1.

3.2. **Multi-parameter case.** The above results can be generalized to the case of multi-parameter Hardy spaces. Let $H^1(X_1 \times X_2 \times ... \times X_n)$, where $X_k = \mathbb{R}^{N_k}$, for $k = 1, 2, ..., n$ be the n-parameter *real variable* Hardy space, see Section 4.2.1 for the precise definition.

Define its dyadic counterpart as follows. Let $\mathcal{D}_k$ be a dyadic lattice on X_k, $k = 1, 2, ..., n$ and let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D} \times ... \times \mathcal{D}_n$ be the product dyadic lattice on $X = X_1 \times X_2 \times ... \times X_n$; the elements on $\mathcal{D}$ are the "rectangles" $R = Q_1 \times Q_2 \times ... \times Q_n$, $Q_k \in \mathcal{D}_k$.

For a multi-index $\mathbf{k} = (k_1, k_2, ..., k_n)$ define on $X = X_1 \times X_2 \times ... \times X_n$ the *averaging operator* $\mathbb{E}_{\mathbf{k}} := \mathbb{E}^1_{k_1} \mathbb{E}^2_{k_2} ... \mathbb{E}^n_{k_n}$ and the *difference operator* $\Delta_{\mathbf{k}} := \Delta^1_{k_1} \Delta^2_{k_2} ... \Delta^n_{k_n}$, where $\mathbb{E}^j_{k_j}$ and $\Delta^j_{k_j}$ are the "one variable" averages and differences described as above.

Define the multi-parameter *dyadic square function* $S = S_{\mathcal{D}}$ by

$$(S_{\mathcal{D}} f)(x) := \left(\sum_{\mathbf{k} \in \mathbb{Z}^n} |\Delta_{\mathbf{k}}^{\mathcal{D}} f(x)|^2 \right)^{1/2}.$$

Definition 3.3. Let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D} \times ... \times \mathcal{D}_n$ be a product dyadic lattice on $X = X_1 \times X_2 \times ... \times X_n$. We say that a function $f \in L^1_{\text{loc}}(X)$ belongs to the *dyadic Hardy space* $H^1_{\mathcal{D}}(X)$ if $||f||_{H_{\mathcal{D}^1}} := ||S_{\mathcal{D}} f||_1 < \infty$.

Now let $\mathcal{D}(\omega) = \mathcal{D}_1(\omega) \times \mathcal{D}(\omega) \times ... \times \mathcal{D}_n(\omega)$ be the multi-parameter random dyadic lattice, we have

Theorem 3.4. *A function $f \in L^1_{\text{loc}}(X)$ belongs to $H^1(X)$ if and only if*

$$\int_X [\mathbb{E}(|S_{\mathcal{D}(\omega)} f(x)|^2)]^{1/2} dx < \infty.$$

Moreover, the latter quality gives an equivalent norm on H^1.

4. Outline of the proof

4.1. **One-parameter case.** The sufficiency follows easily from the well known H^1-BMO duality ($(H^1)^*$=BMO, $(H^1_{\mathcal{D}})^*$=BMO$_{\mathcal{D}}$) and the trivial inclusion BMO$\subset$ BMO$_{\mathcal{D}}$, which imply the $H^1_{\mathcal{D}} \subset H^1$, and so $||f||_{H^1} \leq C||S_{\mathcal{D}} f||_1 = C\int_{\mathbb{R}^N} |S_{\mathcal{D}} f| dx$, hence by Tonelli theorem and Hölder inequality we have

$$||f||_{H^1} \leq C\mathbb{E}\left(\int_{\mathbb{R}^N} |S_{\mathcal{D}(\omega)} f| dx\right) \leq C\int_{\mathbb{R}^N} [\mathbb{E}(|S_{\mathcal{D}(\omega)} f(x)|^2)]^{1/2} dx < \infty.$$

The proof of the necessity is a little bit more involved. We start with a general definition and then introduce an elegant idea which allows us to employ all the Calderón-Zygmund operator theory to prove the necessity.

Definition 4.1. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, consider first their tensor product $\mathcal{H} \otimes \mathcal{K}$ as vector spaces, and then introduce an inner product by

$$\langle \phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2 \rangle_{\mathcal{H}\otimes\mathcal{K}} := \langle \phi_1, \phi_2 \rangle_{\mathcal{H}} \langle \psi_1, \psi_2 \rangle_{\mathcal{K}} \text{ for all } \phi_1, \phi_2 \in \mathcal{H} \text{ and } \psi_1, \psi_2 \in \mathcal{K},$$

finally, take the completion under this inner product. The resulting Hilbert space is the tensor product of $\mathcal{H}$ and $\mathcal{K}$.

If one of the spaces is a function space, for example if $\mathcal{H} = L^2$, then $\mathcal{H} \otimes \mathcal{K}$ can be intepreted as L^2 with values in $\mathcal{K}$.

4.1.1. *"Vectorization" of the dyadic square function.* There is a standard way of making the nonlinear operator $S_{\mathcal{D}}$ into a linear one by treating $S_{\mathcal{D}} f$ as a vector-valued function.

Define the vector-valued square function $\mathbf{S}_{\mathcal{D}} f$ by

$$\mathbf{S}_{\mathcal{D}} f(k, x) := \Delta_k^{\mathcal{D}} f(x), \ k \in \mathbb{Z}, \ x \in \mathbb{R}^N.$$

We will treat $\mathbf{S}_{\mathcal{D}} f$ as a function of the argument $x \in \mathbb{R}^N$ with values in $l^2 = l^2(\mathbb{Z})$.

Clearly, $||\mathbf{S}_{\mathcal{D}} f(\cdot, x)||_{l^2} = S_{\mathcal{D}} f(x)$, so $f \in H^1_{\mathcal{D}}$ if and only if $\mathbf{S}_{\mathcal{D}} f \in L^1(l^2)$. Moreover, $||\mathbf{S}_{\mathcal{D}} f||_{L^1(l^2)} = ||S_{\mathcal{D}} f||_1 = ||f||_{H^1_{\mathcal{D}}}$.

Now let $\mathcal{D}(\omega)$ be the random dyadic lattice, and let $(\Omega, \mathbb{P})$ be the corresponding probability space. Consider the space $\mathcal{L} = L^1(l^2 \otimes L^2(\Omega, \mathbb{P}))$, which is an L^1 space with values in the Hilbert space $l^2 \otimes L^2(\Omega, \mathbb{P})$.

Define the vector-valued square function $\mathbf{S}$ with values in $l^2 \otimes L^2(\Omega, \mathbb{P})$ by

$$\mathbf{S} f(k, \omega, x) := \mathbf{S}_{\mathcal{D}(\omega)} f(k, x), \ k \in \mathbb{Z}, \ \omega \in \Omega, \ x \in \mathbb{R}^N.$$

Here and below we will use notation $\mathbf{S} f(x) := \mathbf{S} f(\cdot, \cdot, x) \in l^2 \otimes L^2(\Omega, \mathbb{P})$.

Clearly, $||\mathbf{S} f(x)||_{l^2 \otimes L^2(\Omega, \mathbb{P})} = [\mathbb{E}(|S_{\mathcal{D}(\omega)} f(x)|^2)]^{1/2}$, so $||\mathbf{S} f||_{\mathcal{L}} = \int_{\mathbb{R}^N} [\mathbb{E}(|S_{\mathcal{D}(\omega)} f(x)|^2)]^{1/2} dx$.

4.1.2. *Vector-valued Calderón-Zygmund operators.* Let us recall the vector-valued Carderón-Zygmund operator theory.

Definition 4.2. Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, an operator-valued *Calderón-Zygmund kernel* on $\mathbb{R}^N$ is the functional $K(\cdot,\cdot):\mathbb{R}^N\times\mathbb{R}^N\setminus\{(x,x):x\in\mathbb{R}^N\}\to B(\mathcal{X},\mathcal{Y})$ satisfying

(i) $||K(x,y)||_{\mathcal{X}\to\mathcal{Y}}\le C|x-y|^{-N}$
(ii) There exists $\delta>0$ such that
$$||K(x,y)-K(x_0,y)||_{\mathcal{X}\to\mathcal{Y}}+||K(y,x)-K(y,x_0)||_{\mathcal{X}\to\mathcal{Y}}\le C\frac{|x-x_0|^\delta}{|y-x_0|^{N+\delta}}$$
whenever $|y-x_0|\ge 2|x-x_0|$.

Definition 4.3. An operator-valued *Carlderón-Zygmund operator* with Calderón-Zygmund kernel K is a bounded operator $T:L^2(\mathbb{R}^N,\mathcal{X})\to L^2(\mathbb{R}^N,\mathcal{Y})$ such that for all $f\in L^2(\mathbb{R}^N,\mathcal{X})$ and $g\in L^2(\mathbb{R}^N,\mathcal{Y})$, with $\mathrm{supp(f)}\cap\mathrm{supp(g)}=\phi$
$$\langle Tf,g\rangle_{L^2(\mathbb{R}^N,\mathcal{Y})\times L^2(\mathbb{R}^N,\mathcal{Y}^*)}=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\langle K(x,y)\circ f(y),g(x)\rangle_{\mathcal{Y}\times\mathcal{Y}^*}dydx.$$

Theorem 4.4. *If T is an operator-valued Carlderón-Zygmund operator defind as above, and assume that both spaces $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then T is bounded*

(i) $T:L^p(\mathbb{R}^N,\mathcal{X})\to L^p(\mathbb{R}^N,\mathcal{Y})$ *for all* $1<p<\infty$
(ii) $T:H^1(\mathbb{R}^N,\mathcal{X})\to L^1(\mathbb{R}^N,\mathcal{Y})$
(iii) $T:L^\infty(\mathbb{R}^N,\mathcal{X})\to\mathrm{BMO}(\mathrm{R^N},\mathcal{Y})$, *see Section 4.2.2 for the precise definition of* $\mathrm{BMO}(\mathrm{R^N},\mathcal{Y})$

4.1.3. *Outline of the proof of necessity.* In our case, $\mathcal{X}=\mathbb{C}$, $\mathcal{Y}=l^2\otimes L^2(\Omega,\mathbb{P})$, and the verctorization $\mathbf{S}f$ has the kernel
$$K(x,y)=\mathbf{S}\delta_y(x)=\left\{\Delta_k^{\mathcal{D}(\omega)}\delta_y(x)\right\}_{k\in\mathbb{Z},\omega\in\Omega}\in l^2\otimes L^2(\Omega,\mathbb{P}).$$
Direct computation, see [1] for details, shows that $\mathbf{S}$ is exactly a Carlderón-Zygmund operator, so apply the Calderón-Zygmund operator theory we have
$$\int_{\mathbb{R}^N}[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2)]^{1/2}=||\mathbf{S}f||_{L^1(\mathbb{R}^N,l^2\otimes L^2(\Omega,\mathbb{P}))}\le C||f||_{H^1}.$$

4.2. **Multi-parameter case.** Proof for the multi-parameter case is just an "iteration" of the one-parameter case once we understand the definition of multi-parameter H^1-spaces.

4.2.1. *Multi-parameter H^1-spaces.* Recall that for one-parameter Hardy space $H^1(\mathbb{R}^N)$ the norm $||S^Lf||_1$ gives an equivalent H^1-norm, where S^L is the Lusin square function
$$S^Lf(x):=\int_{\Gamma_x}|\nabla f(y,t)|^2t^{1-N}dydt,$$
here $\Gamma_x:=\{(y,t):y\in\mathbb{R}^N,t>0,|y-x|<t\}$, and $f(y,t)$ is the harmonic extension of f from $\mathbb{R}^N$ to $\mathbb{R}^{N+1}_+=\mathbb{R}^N\times\mathbb{R}_+$.

One can consider the *vecterization* S^L of S^L as follows. Let $\Gamma=\Gamma_0$ and define
$$\mathbf{S}^Lf(x,y,t)=\nabla f(x+y,t)t^{(1-N)/2},\ x\in\mathbb{R}^N,\ (y,t)\in\Gamma.$$
By construction $\mathbf{S}^L(x,\cdot,\cdot)\in L^2(\Gamma)\otimes\mathbb{C}^{N+1}$ and $||\mathbf{S}^Lf(x,\cdot,\cdot)||_{L^2(\Gamma)\otimes\mathbb{C}^{N+1}}=S^Lf(x)$, therefore $||\mathbf{S}^Lf||_{L^1(L^2(\Gamma)\otimes\mathbb{C}^{N+1})}=||S^Lf||_1$.

For multi-parameter case, one can define S^L by

$$S^L f(x_1, x_2, ..., x_n) := \left[\int_{\Gamma_{x_1}\times\Gamma_{x_2}\times...\times\Gamma_{x_n}} |\nabla_1\nabla_2...\nabla_n f(y_1, t_1, y_2, t_2, ..., y_n, t_n)|^2 \times t_1^{1-N_1} t_2^{1-N_2} ... t_n^{1-N_n} dy_1 dt_1 dy_2 dt_2 ... dy_n dt_n\right]^{1/2}.$$

here $f(y_1, t_1, y_2, t_2, ..., y_n, t_n)$ is the harmonic in each variable (y_k, t_k), $y_k \in \mathbb{R}^{N_k}$, $t_k \in \mathbb{R}_+$ extension of f from $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times ... \times \mathbb{R}^{N_n}$ to $\mathbb{R}_+^{N_1+1} \times \mathbb{R}_+^{N_2+1} \times ... \times \mathbb{R}_+^{N_n+1}$ and ∇_k is the gradient in the variable (y_k, t_k).

Definition 4.5. Following the notation in Section 3.2, we say that $f \in H^1(X) = H^1(X_1 \otimes X_2 \otimes ... \otimes X_n)$ if $S^L f \in L^1(X)$ and $||S^L f||_1$ defines one of the possible equivalent norms in $H^1(X)$.

One can also define the *vectorization* $\mathbf{S}^L$ of S^L as $\mathbf{S}^L f(x) = \mathbf{S}_1^L \otimes \mathbf{S}_2^L \otimes ... \otimes \mathbf{S}_n^L f(x) \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n$, where $\mathbf{S}_k^L$ is one-parameter Lusin square function defined above taken in the variable x_k, $\mathcal{H}_k = L^2(\Gamma_k) \otimes \mathbb{C}^{N_k+1}$ and Γ_k is the cone in $\mathbb{R}_+^{N_k+1}$ with the vertex at 0.

Again, by construction $||\mathbf{S}^L f(x)||_{\mathcal{H}} = S^L f(x)$, so $||\mathbf{S}^L f||_{L^1(X,\mathcal{H})}$ gives the norm in $H^1(X)$.

4.2.2. *Outline of the proof of sufficiency.* Let us first recall the Hilbert-space-valued BMO space.

Definition 4.6. A function on $\mathbb{R}^N$ with values in a Hilbert space $\mathcal{H}$ belongs to the space BMO=BMO$(X, \mathcal{H})$ if

$$||f||_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q ||f(x) - f_Q||_{\mathcal{H}} dx < \infty,$$

here $f_Q := |Q|^{-1} \int_Q f(x)dx$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^N$.
If we fix a dyadic lattice $\mathcal{D}$ and take the supremum only over dyadic cubes $Q \in \mathcal{D}$, we get the dyadic BMO$_{\mathcal{D}}$ space associated to this lattice.

As in the scaler-valued case, H^1-BMO duality holds, which are $H^1(\mathbb{R}^N, \mathcal{H})^* = \mathrm{BMO}(\mathrm{\mathbb{R}^N}, \mathcal{H})$ and $H^1_{\mathcal{D}}(\mathbb{R}^N, \mathcal{H})^* = \mathrm{BMO}_{\mathcal{D}}(\mathrm{\mathbb{R}^N}, \mathcal{H})$.

Let $\mathbf{S}_k^L$, $\mathbf{S}_{\mathcal{D}_k}$ be the Lusin and dyadic square functions as above, taken in the variable x_k, and let $\mathcal{H}_k := L^2(\Gamma_k) \otimes \mathbb{C}^{N_k+1}$ and $\mathcal{H}_k' := l^2$ be the corresponding target spaces. Moreover, denote $\widetilde{\mathbf{S}_k}$ to be either one-parameter $\mathbf{S}_k^L$ or one-parameter $\mathbf{S}_{\mathcal{D}_k}$, $\widetilde{\mathcal{H}_k}$ to be either $\mathcal{H}_k$ or $\mathcal{H}_k'$, and let $\widetilde{\mathcal{H}^1} := \widetilde{\mathcal{H}_2} \otimes ... \otimes \widetilde{\mathcal{H}_n}$.

By the H^1-BMO duality, one can show, see [1] for details

Lemma 4.7.

$$\int_X ||\boldsymbol{S}_1^L \otimes \widetilde{\boldsymbol{S}_2} \otimes ... \otimes \widetilde{\boldsymbol{S}_n} f(x)||_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}^1}} dx \leq C \int_X ||\boldsymbol{S}_{\mathcal{D}_1} \otimes \widetilde{\boldsymbol{S}_2} \otimes ... \otimes \widetilde{\boldsymbol{S}_n} f(x)||_{\mathcal{H}_1' \otimes \widetilde{\mathcal{H}^1}} dx.$$

Applying this Lemma successively to each factor, we get

$$\int_X ||\mathbf{S}_1^L \otimes \mathbf{S}_2^L \otimes ... \otimes \mathbf{S}_n^L f(x)||_{\mathcal{H}} dx \leq C \int_X ||\mathbf{S}_{\mathcal{D}_1} \otimes \mathbf{S}_{\mathcal{D}_2} \otimes ... \otimes \mathbf{S}_{\mathcal{D}_n} f(x)||_{\mathcal{H}'} dx,$$

which by definition yields the sufficiency part.

4.2.3. *Outline of the proof of necessity.* Similarly as in the proof of sufficiency, but let now $\mathbf{S}_k^L$, $\mathbf{S}_k$ be the Lusin and random square functions, taken in the variable x_k, and let $\mathcal{H}_k := L^2(\Gamma_k) \otimes \mathbb{C}^{N_k+1}$ and $\mathcal{H}'_k := l^2 \otimes L^2(\Omega_k, \mathbb{P}_k)$ be the corresponding target spaces. Moreover, denote $\widetilde{\mathbf{S}_k}$ to be either one-parameter $\mathbf{S}_k^L$ or one-parameter $\mathbf{S}_k$, $\widetilde{\mathcal{H}_k}$ to be either $\mathcal{H}_k$ or $\mathcal{H}'_k$, and let $\widetilde{\mathcal{H}^1} := \widetilde{\mathcal{H}_2} \otimes ... \otimes \widetilde{\mathcal{H}_n}$.

By the Calderón-Zygmund operator theory, one can show, see again [1] for details

Lemma 4.8.

$$\int_X ||\boldsymbol{S}_1 \otimes \widetilde{\boldsymbol{S}_2} \otimes ... \otimes \widetilde{\boldsymbol{S}_n} f(x)||_{\mathcal{H}'_1 \otimes \widetilde{\mathcal{H}^1}} dx \leq C \int_X ||\boldsymbol{S}_1^L \otimes \widetilde{\boldsymbol{S}_2} \otimes ... \otimes \widetilde{\boldsymbol{S}_n} f(x)||_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}^1}} dx.$$

Applying this Lemma successively to each factor, we get

$$\int_X ||\mathbf{S}_1 \otimes \mathbf{S}_2 \otimes ... \otimes \mathbf{S}_n f(x)||_{\mathcal{H}'} dx \leq C \int_X ||\mathbf{S}_1^L \otimes \mathbf{S}_2^L \otimes ... \otimes \mathbf{S}_n^L f(x)||_{\mathcal{H}} dx,$$

which by definition is exactly what we need for the necessity part.

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