PROCEEDINGS OF THE INTERNET ANALYSIS SEMINAR ON THE DIRICHLET SPACE

Contents

Overview of the Workshop

This workshop was part of the Internet Analysis Seminar that is the education component of the National Science Foundation – DMS # 0955432 held by Brett D. Wick. The Internet Analysis Seminar consists of three phases that run over the course of a standard academic year. Each year, a topic in complex analysis, function theory, harmonic analysis, or operator theory is chosen and an internet seminar will be developed with corresponding lectures. The course will introduce advanced graduate students and post-doctoral researchers to various topics in those areas and, in particular, their interaction.

This was a workshop that focused on the Dirichlet function space. Each of the participants was assigned one of the following papers to read.

[1] J. Agler, *Interpolation*, preprint (1988).
[2] J. Arazy and S. D. Fisher, *The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces*, Illinois J. Math. **29** (1985), no. 3, 449–462.
[3] N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures for analytic Besov spaces*, Rev. Mat. Iberoamericana **18** (2002), no. 2, 443–510.
[4] N. Arcozzi, R. Rochberg, E. Sawyer, and B. D. Wick, *Bilinear forms on the Dirichlet space*, Anal. PDE **3** (2010), no. 1, 21–47.
[5] L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 269–303.
[6] S.-Y. A. Chang and D. E. Marshall, *On a sharp inequality concerning the Dirichlet integral*, Amer. J. Math. **107** (1985), no. 5, 1015–1033.
[7] D. E. Marshall and C. Sundberg, *Interpolating sequences for the multipliers of the Dirichlet space*, preprint (1994), available at http://www.math.washington.edu/~marshall/preprints/interp.pdf.
[8] A. Nagel, W. Rudin, and J. H. Shapiro, *Tangential boundary behavior of functions in Dirichlet-type spaces*, Ann. of Math. (2) **116** (1982), no. 2, 331–360.
[9] S. Richter and C. Sundberg, *A formula for the local Dirichlet integral*, Michigan Math. J. **38** (1991), no. 3, 355–379.
[10] T. T. Trent, *A corona theorem for multipliers on Dirichlet space*, Integral Equations Operator Theory **49** (2004), no. 1, 123–139.

They were then responsible to prepare two one hour lectures based on the paper and an extended abstract based on the paper. This proceeding is the collection of the extended abstract prepared by each participant. The following people participated in the workshop:

Austin Anderson	University of Hawaii
Raphaël Clouâtre	Indiana University
Alberto Condori	Florida Gulf Coast University
Tim Feguson	University of Michigan
Constanze Liaw	Texas A&M University
Shuaibing Luo	University of Tennessee, Knoxville
Eyvindur Palsson	Cornell University
Mrinal Ragupathi	Vanderbilt University
James Scurry	Georgia Institute of Technology
Daniel Seco	Universitat Autónoma de Barcelona
Brett D. Wick	Georgia Institute of Technology

INTERPOLATION

JIM AGLER

presented by Alberto A. Condori

ABSTRACT. Let $\mathcal{H}$ be a Hilbert space of analytic functions in $\mathbb{D}$. Under mild conditions on the reproducing kernels of $\mathcal{H}$, we give a necessary and sufficient condition for the solution of the Nevanlinna-Pick problem for space of multipliers $\mathcal{M}_{\mathcal{H}}$ of $\mathcal{H}$.

NOTATION

(1) $\mathbb{D}$ denotes the unit disc in the complex plane $\mathbb{C}$.
(2) $\operatorname{span} S$ denotes the linear span of a subset S in a vector space.
(3) $\operatorname{clos} S$ denotes the closure of a set S in a Hilbert space.
(4) z denotes the identity map of $\mathbb{C}$ (or a subset of $\mathbb{C}$) onto itself.
(5) $\mathcal{M}_{\mathcal{H}}$ denotes the set of *multipliers* φ of a Hilbert space of analytic functions $\mathcal{H}$, i.e. $\varphi f \in \mathcal{H}$ when $f \in \mathcal{H}$.
(6) For $\varphi \in \mathcal{M}_{\mathcal{H}}$, $M_\varphi : \mathcal{H} \to \mathcal{H}$ is defined by $M_\varphi f = \varphi f$ for $f \in \mathcal{H}$.
(7) H^∞ is the space of bounded analytic functions on $\mathbb{D}$ equipped with the norm $\|\psi\|_{H^\infty} = \sup_{\zeta \in \mathbb{D}} |\psi(\zeta)|$.
(8) H^2 is the Hardy space of analytic functions f on $\mathbb{D}$ with power series $f = \sum_{n=0}^{\infty} a_n z^n$ for which $\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2$ is finite.
(9) $\mathcal{D}$ denotes the Dirichlet space of analytic functions f on $\mathbb{D}$ with power series $f = \sum_{n=0}^{\infty} a_n z^n$ for which $\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2$ is finite.

1. INTRODUCTION

The *Nevanlinna-Pick problem* for H^∞ can be stated as follows: given n distinct points $\zeta_1, \ldots, \zeta_n \in \mathbb{D}$, and $w_1, \ldots, w_n \in \mathbb{C}$, determine whether there is a $\varphi \in H^\infty$ with $\|\varphi\|_{H^\infty} \leq 1$ such that

$$\varphi(\zeta_j) = w_j \quad \text{for } 1 \leq j \leq n.$$

Pick's theorem provides a necessary and sufficient condition for the existence of such a function φ. This condition specifies that a certain matrix be positive semi-definite.

Based on the observation that H^∞ is the space of multipliers $\mathcal{M}_{H^2}$ of the Hardy space H^2, J. Agler reformulated the Nevanlinna-Pick problem in the context of the space of multipliers $\mathcal{M}_{\mathcal{D}}$ of the Dirichlet space $\mathcal{D}$ and found a necessary and sufficient condition for its solution. His condition was similar to that given in Pick's theorem and appears in his unpublished though influential paper [Ag].

In this note, we generalize Agler's result by reformulating the Nevanlinna-Pick problem in the context of the space of multipliers $\mathcal{M}_{\mathcal{H}}$ of Hilbert spaces $\mathcal{H}$ of analytic functions on $\mathbb{D}$ and provide an analogous necessary and sufficient condition for its solution.

2. HILBERT SPACES OF ANALYTIC FUNCTIONS

Let $\Omega \subseteq \mathbb{C}$. We say $\mathcal{H}$ is a *Hilbert space of analytic functions* on Ω if

(1) $\mathcal{H}$ is a Hilbert space consisting of analytic functions on Ω,
(2) point evaluation functionals are both non-zero and continuous on $\mathcal{H}$,
(3) $\mathcal{H}$ contains all analytic polynomials as a dense subset, and
(4) if $f \in \mathcal{H}$, then $zf \in \mathcal{H}$.

It is easy to verify that H^2 and $\mathcal{D}$ are Hilbert spaces of analytic functions on $\mathbb{D}$.

For a Hilbert space of analytic functions $\mathcal{H}$, the following assertions hold.

(1) For each $\lambda \in \Omega$, there is a $k_\lambda \in \mathcal{H}$, called the *reproducing kernel at* λ, such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$.
(2) For $\varphi \in \mathcal{M}_\mathcal{H}$, M_φ is a bounded operator on $\mathcal{H}$ (by the Closed Graph Theorem) and its adjoint satisfies $M_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$ for $\lambda \in \Omega$.
(3) If $\varphi \in \mathcal{M}_\mathcal{H}$, then φ belongs to $\mathcal{H}$ and is bounded on Ω by $\|M_\varphi^*\|$.
(4) The set $\mathcal{M}_\mathcal{H}$ is a Banach algebra when equipped with the norm $\|\cdot\|_{\mathcal{M}_\mathcal{H}}$, where $\|\varphi\|_{\mathcal{M}_\mathcal{H}} \stackrel{\text{def}}{=} \|M_\varphi\|$ for $\varphi \in \mathcal{M}_\mathcal{H}$.
(5) If T is a bounded operator on $\mathcal{H}$ that commutes with M_z, then there is a $\varphi \in \mathcal{M}_\mathcal{H}$ such that $T = M_\varphi$.

3. The Nevanlinna-Pick problem

Henceforth, $\mathcal{H}$ denotes a Hilbert space of analytic functions on $\mathbb{D}$.

The *Nevanlinna-Pick problem* is now reformulated as follows: given n distinct points $\zeta_1, \dots, \zeta_n \in \mathbb{D}$, and $w_1, \dots, w_n \in \mathbb{C}$, determine whether there is a $\varphi \in \mathcal{M}_\mathcal{H}$ such that $\|\varphi\|_{\mathcal{M}_\mathcal{H}} \leq 1$ (i.e. φ is *contractive*) and

$$\varphi(\zeta_j) = w_j \quad \text{for } 1 \leq j \leq n. \tag{3.1}$$

To simplify notation, we define the mth Pick matrix $\mathcal{P}_m$ induced by $\zeta_1, \dots, \zeta_m \in \mathbb{D}$ and $w_1, \dots, w_m \in \mathbb{C}$ to be

$$P_m[\zeta_1, \dots, \zeta_m; w_1, \dots, w_m] = \left((1 - \overline{w_i} w_j) k_{\zeta_i}(\zeta_j)\right)_{1 \leq j, i \leq m}. \tag{3.2}$$

When the data $\zeta_1, \dots, \zeta_m$ and $w_1, \dots, w_m$ are understood in the context, we write P_m instead of $P_m[\zeta_1, \dots, \zeta_m; w_1, \dots, w_m]$.

The Nevanlinna-Pick problem for H^∞ was placed into an operator-theoretic framework by D. Sarason [Sa]. His important, albeit simple, observations in the case $\mathcal{H} = H^2$ and $\mathcal{M}_\mathcal{H} = H^\infty$ are summarized and generalized in the lemmata below.

Lemma 3.1. *Let $\zeta_1, \dots, \zeta_m \in \mathbb{D}$ be distinct, and $w_1, \dots, w_m \in \mathbb{C}$. Define the operator A_m^* on $\mathcal{H}_m \stackrel{\text{def}}{=} \operatorname{span}\{k_{\zeta_j} : 1 \leq j \leq m\}$ by $A_m^* k_{\zeta_j} = \overline{w_j} k_{\zeta_j}$, $1 \leq j \leq m$. Then the Pick matrix $\mathcal{P}_m$ is positive semi-definite if and only if A_m^* is a contraction on $\mathcal{H}_m$.*

Lemma 3.2. *Let $\zeta_1, \dots, \zeta_m \in \mathbb{D}$ be distinct, and $w_1, \dots, w_m \in \mathbb{C}$. If there is a contractive $\varphi \in \mathcal{M}_\mathcal{H}$ that satisfies* (3.1)*, then the corresponding Pick matrix $\mathcal{P}_n$ is positive semi-definite.*

Intuition to prove a converse to Lemma 3.2 comes from the proof of Lemma 3.2 and the statement of Lemma 3.1; if the Pick matrix $\mathcal{P}_n$ is positive semi-definite and A_n^* can be extended from $\mathcal{H}_n$ to $\mathcal{H}$ in a norm preserving manner, then one should be able to find a $\varphi \in \mathcal{M}_\mathcal{H}$ such that $M_\varphi^* | \mathcal{H}_n = A_n^*$.

Theorem 3.3. *Suppose*

(1) *every finite set of reproducing kernels of $\mathcal{H}$ is linearly independent,*

(2) $k_\lambda(\zeta)$ *is never zero for* $\zeta, \lambda \in \mathbb{D}$, *and*
(3) $1 - 1/k_\lambda(\zeta)$ *is a positive semi-definite kernel.*

If $\zeta_1, \dots, \zeta_n \in \mathbb{D}$ *are distinct and* $w_1, \dots, w_n \in \mathbb{C}$ *are such that the corresponding Pick matrix* $\mathcal{P}_n$ *is positive semi-definite, then there is a contractive* $\varphi \in \mathcal{M}_{\mathcal{H}}$ *such that* (3.1) *is satisfied.*

Details of the proof of Theorem 3.3 are found in section 4. We now use Theorem 3.3 to prove two interpolation results.

3.1. **Interpolation in H^∞.** A simple computation reveals that the reproducing kernel at $\lambda \in \mathbb{D}$ for H^2 is

$$k_\lambda(\zeta) = \frac{1}{1 - \overline{\lambda}\zeta}.$$

It can be easily verified that this reproducing kernel satisfies the hypotheses of Theorem 3.3. Combining Theorem 3.3 and Lemma 3.2, we obtain

Theorem 3.4 (Pick)**.** *Given* $\zeta_1, \dots, \zeta_n \in \mathbb{D}$ *distinct and* $w_1, \dots, w_n \in \mathbb{C}$*, there is an analytic function* $\varphi : \mathbb{D} \to \mathbb{D}$ *satisfying* (3.1) *if and only if the induced Pick matrix*

$$\mathcal{P}_n = \left(\frac{1 - \overline{w_i} w_j}{1 - \overline{\zeta_i}\zeta_j} \right)_{1 \le j, i \le n}$$

is positive semi-definite.

3.2. **Interpolation in $\mathcal{M}_{\mathcal{D}}$.** Recall that the inner product of $f, g \in \mathcal{D}$ is $\langle f, g \rangle_{\mathcal{D}} = \sum_{n=0}^{\infty} (n+1) a_n \overline{b_n}$, where for $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$. Then the reproducing kernel at λ is

$$k_\lambda = \sum_{n=0}^{\infty} \frac{1}{n+1} \bar{\lambda}^n z^n.$$

Lemma 3.5. *There is a sequence* $(a_n)_{n \in \mathbb{N}}$ *of positive numbers such that*

$$1 - \frac{1}{k_\lambda(\zeta)} = \sum_{n=1}^{\infty} a_n \bar{\lambda}^n \zeta^n.$$

In light of Lemma 3.5, Theorem 3.3 and Lemma 3.2, we obtain

Theorem 3.6 (Agler)**.** *Given* $\zeta_1, \dots, \zeta_n \in \mathbb{D}$ *distinct and* $w_1, \dots, w_n \in \mathbb{C}$*, there a contractive* $\varphi \in \mathcal{M}_{\mathcal{D}}$ *satisfying* (3.1) *if and only if the induced Pick matrix*

$$\mathcal{P}_n = \left((1 - \overline{w_i} w_j) \frac{1}{\overline{\zeta_i}\zeta_j} \log \frac{1}{1 - \overline{\zeta_i}\zeta_j} \right)_{1 \le j, i \le n}$$

is positive semi-definite.

4. Proof of Theorem 3.3

In this section, we assume that the hypotheses of Theorem 3.3 are satisfied.

The analyticity of $\mathcal{H}$ implies that $\{\zeta_1, \dots, \zeta_n\}$ can be extended to a sequence $Z = \{\zeta_j : j \in \mathbb{N}\}$ such that $\mathcal{H} = \operatorname{clos\,span}\{k_{\zeta_j} : j \in \mathbb{N}\}$ (e.g. any sequence whose terms are distinct and tend to zero).

Thus, if we can extend the set $\{w_1, \dots, w_n\}$ to a sequence $\{w_j : j \in \mathbb{N}\}$ such that, for each $m \in \mathbb{N}$, the corresponding Pick matrix $\mathcal{P}_m$ is positive semi-definite, then the operator M^* defined on $\mathcal{H}$ by

$$M^* k_{\zeta_j} = \overline{w_j}\, k_{\zeta_j}$$

is a contraction and commutes with M_z^* on $\mathcal{H}$. In this case, there is a $\varphi \in \mathcal{M}_{\mathcal{H}}$ such that $M^* = M_\varphi^*$. Hence $\|\varphi\|_{\mathcal{M}_{\mathcal{H}}} = \|M_\varphi\| = \|M^*\| \le 1$ and (3.1) is satisfied.

We proceed by induction. Let $m \in \mathbb{N}$. Suppose $\{w_1, \dots, w_m\} \subseteq \mathbb{C}$ are such that the corresponding Pick matrix $\mathcal{P}_m$ is positive semi-definite. We show that there is a $w_{m+1} \in \mathbb{C}$ such that the corresponding Pick matrix $\mathcal{P}_{m+1}$ is positive. Equivalently, we find $w_{m+1} \in \mathbb{C}$ such that the induced operator A_{m+1}^* is a contraction (see Lemma 3.1).

For arbitrary $w \in \mathbb{C}$, consider $A^{(w)}$ defined on $\mathcal{H}_{m+1}$ by $A^{(w)} k_{\zeta_j} = \overline{w_j} k_{\zeta_j}$ for $1 \le j \le m$ and $A^{(w)} k_{\zeta_{m+1}} = \overline{w} k_{\zeta_{m+1}}$. We claim that there is a $w \in \mathbb{C}$ such that $A^{(w)}$ is a contraction.

The first assumption in Theorem 3.3 implies the existence of a $b_j \in \mathcal{H}_{m+1}$ such that $\langle k_{\zeta_i}, b_j\rangle = 1$ when $i = j$ and $= 0$ otherwise, $1 \le j \le m+1$.

Let P_m denote the orthogonal projection from $\mathcal{H}_{m+1}$ onto $\mathcal{H}_m$ and Q_m denote the orthogonal projection from $\mathcal{H}_{m+1}$ onto $\mathcal{H}_{m+1} \ominus \mathbb{C} \cdot k_{\zeta_{m+1}}$. Notice that

- $\operatorname{span}\{P_m b_j : 1 \le j \le m\} = \operatorname{Range} P_m = \mathcal{H}_m$ because $P_m b_{m+1} = 0$.
- $A^{(w)} P_m$ agrees with A_m^* on $\mathcal{H}_m$ (and so it is independent of w).
- $Q_m A^{(w)}$ is independent of w because $Q_m k_{\zeta_{m+1}} = 0$.
- $A^{(w)}$ admits the four-block representation

$$A^{(w)} = \begin{pmatrix} Q_m A^{(w)} P_m & Q_m A^{(w)} (I - P_m) \\ (I - Q_m) A^{(w)} P_m & (I - Q_m) A^{(w)} (I - P_m) \end{pmatrix}. \tag{4.1}$$

  where the first column agrees with A_m^* and the first row agrees with $Q_m A^{(w)}$ and thus are independent of w.
- $(I - Q_m) g = \langle g, k_{\zeta_{m+1}}\rangle \|k_{\zeta_{m+1}}\|^{-2} k_{\zeta_{m+1}}$ for $g \in \mathcal{H}_{m+1}$.
- The operator $(I - Q_m) A^{(w)} (I - P_m)$ has rank 1; in fact, if $d_{m+1} = \sum_{j=1}^{m+1} a_j k_{\zeta_j}$, then

$$(I - Q_m) A^{(w)} (I - P_m) d_{m+1} = \left(\overline{w} a_{m+1} + \sum_{j=1}^{m} \overline{w_j} a_j \frac{\langle k_{\zeta_j}, k_{\zeta_{m+1}}\rangle}{\|k_{\zeta_{m+1}}\|^2} \right) k_{\zeta_{m+1}}.$$

  Thus, the (2,2) block in the operator matrix representation (4.1) of $A^{(w)}$ runs over all rank one operators from $\mathcal{H}_{m+1} \ominus \mathcal{H}_m$ onto $\mathbb{C} \cdot k_{\zeta_{m+1}}$.

By Parrott's theorem [Pa], $\inf \left\{ \left\|A^{(w)}\right\| : w \in \mathbb{C} \right\} = \max \left\{ \left\|A_m^*\right\|, \left\|Q_m A^{(w)}\right\| \right\}$ and there exists a $w \in \mathbb{C}$ such that the infimum is attained. Since A_m^* is a contraction, it suffices to show that the operator $Q_m A^{(w)}$ (independent of w) is a contraction.

Recalling $Q_m k_{m+1} = 0$, we see that $Q_m A^{(w)}$ is a contraction on $\operatorname{Range} Q_m = \operatorname{span}\{Q_m k_{\zeta_j} : 1 \le j \le m\}$ (or equivalently, on $\mathcal{H}_{m+1}$) if and only if $\|Q_m A^{(w)} Q_m x\| \le \|Q_m x\|$ for all $x \in \mathcal{H}_m$, or equivalently, $Q_m^2 - (Q_m A^{(w)} Q_m)^* Q_m A^{(w)} Q_m \ge 0$ on $\mathcal{H}_m$. In fact, for $1 \le i, j \le m$, simple computations give

$$\langle Q_m A^{(w)} Q_m k_{\zeta_j}, Q_m A^{(w)} Q_m k_{\zeta_i}\rangle = \overline{w_j} w_i \langle k_{\zeta_j}, k_{\zeta_i}\rangle + \overline{w_j} w_i \frac{\langle k_{\zeta_j}, k_{\zeta_{m+1}}\rangle}{\|k_{\zeta_{m+1}}\|^2} \overline{\langle k_{\zeta_i}, k_{\zeta_{m+1}}\rangle}$$

$$\text{and } \langle Q_m k_{\zeta_j}, Q_m k_{\zeta_i}\rangle = \langle Q_m k_{\zeta_j}, k_{\zeta_i}\rangle = \langle k_{\zeta_j}, k_{\zeta_i}\rangle - \frac{\langle k_{\zeta_j}, k_{\zeta_{m+1}}\rangle}{\|k_{\zeta_{m+1}}\|^2} \langle k_{\zeta_{m+1}}, k_{\zeta_i}\rangle.$$

Thus, $Q_m^2 - (Q_m A^{(w)} Q_m)^* Q_m A^{(w)} Q_m \geq 0$ on $\mathcal{H}_m$ if and only if the $m \times m$ matrix with the entries below is positive semi-definite:

$$\begin{aligned}\langle Q_m k_{\zeta_j}, Q_m k_{\zeta_i}\rangle &- \langle Q_m A^{(w)} Q_m k_{\zeta_j}, Q_m A^{(w)} Q_m k_{\zeta_i}\rangle \\ &= (1 - \overline{w_j} w_i)\langle k_{\zeta_j}, k_{\zeta_i}\rangle \left[1 - \frac{\langle k_{\zeta_j}, k_{\zeta_{m+1}}\rangle\langle k_{\zeta_{m+1}}, k_{\zeta_i}\rangle}{\|k_{\zeta_{m+1}}\|^2 \langle k_{\zeta_j}, k_{\zeta_i}\rangle}\right]. \end{aligned} \tag{4.2}$$

Under the assumption that A_m^* is a contraction, $\mathcal{P}_m$ and the $m \times m$ matrix with the entries in (4.2) are positive semi-definite provided that the matrix

$$\mathcal{N}_m = \left(1 - \frac{\langle k_{\zeta_j}, k_{\zeta_{m+1}}\rangle\langle k_{\zeta_{m+1}}, k_{\zeta_i}\rangle}{\langle k_{\zeta_{m+1}}, k_{\zeta_{m+1}}\rangle\langle k_{\zeta_j}, k_{\zeta_i}\rangle}\right)_{1\leq j,i\leq m} = \left(1 - \frac{k_{\zeta_j}(\zeta_{m+1})\overline{k_{\zeta_i}(\zeta_{m+1})}}{k_{\zeta_{m+1}}(\zeta_{m+1}) k_{\zeta_j}(\zeta_i)}\right)_{1\leq j,i\leq m}$$

is positive semi-definite, by the Schur product theorem for matrices.

We now finish the proof using an argument due to Quiggin [Qu].

The third assumption on the reproducing kernels implies that the matrix $\frac{1}{K}$ whose entries are $1/k_{\zeta_j}(\zeta_i)$, $1 \leq j, i \leq m+1$, is the sum of a rank one positive and a negative semi-definite matrix. By Weyl's theorem for matrices, $\frac{1}{K}$ has at most one positive eigenvalue. Thus, $\frac{1}{K}$ must have exactly one positive eigenvalue because it has at least one positive eigenvalue.

On the other hand, $\frac{1}{K}$ admits the following block representation:

$$\frac{1}{K} = \begin{pmatrix} A & v \\ v^* & \frac{1}{k_{\zeta_{m+1}}(\zeta_{m+1})} \end{pmatrix}.$$

Let I_m denote the $m \times m$ identity matrix and $B = \begin{pmatrix} I_m & -k_{\zeta_{m+1}}(\zeta_{m+1}) v \\ 0 & 1 \end{pmatrix}$.

It is easy to see that $B\dfrac{1}{K}B^* = \begin{pmatrix} A - k_{\zeta_{m+1}}(\zeta_{m+1}) v v^* & 0 \\ 0 & \frac{1}{k_{\zeta_{m+1}}(\zeta_{m+1})} \end{pmatrix}$.

By Sylvester's law of inertia, the number of positive, zero and negative eigenvalues of $\frac{1}{K}$ and $B\frac{1}{K}B^*$ agree. Thus, $\frac{1}{K}$ has exactly one positive eigenvalue if and only if the block $A - k_{\zeta_{m+1}}(\zeta_{m+1}) v v^*$ is negative. On the other hand, the entries of $-(A - k_{\zeta_{m+1}}(\zeta_{m+1}) v v^*)$ have the form

$$\frac{k_{\zeta_{m+1}}(\zeta_{m+1})}{k_{\zeta_j}(\zeta_{m+1})\overline{k_{\zeta_i}(\zeta_{m+1})}} - \frac{1}{k_{\zeta_j}(\zeta_i)}.$$

Thus, $-(A - k_{\zeta_{m+1}}(\zeta_{m+1}) v v^*)$ is the Schur product of $\mathcal{N}_m$ and the rank one positive matrix whose entries are given by

$$\frac{k_{\zeta_{m+1}(\zeta_{m+1})}}{k_{\zeta_j}(\zeta_{m+1})\overline{k_{\zeta_i}(\zeta_{m+1})}}.$$

Again, by the Schur product theorem for matrices, $-(A - k_{\zeta_m}(\zeta_m) v v^*)$ is positive if and only if $\mathcal{N}_m$ is positive. Hence, we conclude that $\mathcal{N}_m$ must be positive because $\frac{1}{K}$ has exactly one positive eigenvalue. This completes the proof that $Q_m A^{(w)}$ is a contraction from which we conclude existence of a w such that $A^{(w)}$ is a contraction.

5. Appendix: Completing matrix contractions

Given $A_{i,j} \in \mathcal{B}(\mathcal{H}_i, \mathcal{K}_j)$ for $i, j = 1, 2$, we define the operator matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \tag{5.1}$$

from $\mathcal{H}_1 \oplus \mathcal{H}_2$ to $\mathcal{K}_1 \oplus \mathcal{K}_2$ by

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{1,1}x_1 + A_{1,2}x_2 \\ A_{2,1}x_1 + A_{2,2}x_2 \end{pmatrix} \quad \text{for } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2.$$

(As usual, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$.) We now discuss the following problem: given operators $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$, when is it possible to find $A_{2,2}$ so that the operator matrix A in (5.1) is a contraction?

It is easy to see that if A is a contraction, then

$$\begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_{1,1} & A_{1,2} \end{pmatrix} \tag{5.2}$$

are also contractions. It turns out that the converse is also true.

Theorem 5.1 (Parrott)**.** *Let $A_{1,1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_1)$, $A_{1,2} \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_2)$ and $A_{2,1} \in \mathcal{B}(\mathcal{H}_2, \mathcal{K}_1)$. If the operator matrices in* (5.2) *are contractions, then there is an operator $A_{2,2}$ such that the resulting operator matrix A in* (5.1) *is a contraction.*

We refer to [Pa] for a proof of this result and Chapter 2 in [Pe] for related results.

Corollary 5.2. *Let $A_{1,1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_1)$, $A_{1,2} \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_2)$ and $A_{2,1} \in \mathcal{B}(\mathcal{H}_2, \mathcal{K}_1)$. Then*

$$\inf\left\{ \left\| \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & B \end{pmatrix} \right\| : B \in \mathcal{B}(\mathcal{H}_2, \mathcal{K}_2) \right\} = \max\left\{ \left\| \begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix} \right\|, \left\| \begin{pmatrix} A_{1,1} & A_{1,2} \end{pmatrix} \right\| \right\}$$

and the infimum is attained.

Proof. Let M denote the maximum and I the infimum above. Evidently,

$$\left\| \begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix} \right\| \le \left\| \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & B \end{pmatrix} \right\|$$

$$\text{and} \quad \| (A_{1,1} \; A_{1,2}) \| = \left\| \begin{pmatrix} A_{1,1}^* \\ A_{1,2}^* \end{pmatrix} \right\| \le \left\| \begin{pmatrix} A_{1,1}^* & A_{2,1}^* \\ A_{1,2}^* & B^* \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & B \end{pmatrix} \right\|;$$

that is, $I \ge M$. After rescaling, we may assume that $M = 1$. Then the operators in (5.2) are contractions and so there is an $A_{2,2}$ such that the resulting operator matrix A in (5.1) is contraction too, by Parrott's theorem. Hence, $I \le M$. □

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THE UNIQUENESS OF THE DIRICHLET SPACE AMONG MÖBIUS INVARIANT HILBERT SPACES

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presented by Mrinal Ragupathi

Abstract. It is well known that the Dirichlet seminorm is invariant for the action of the group of Möbius transformations. There are other seminormed spaces with this property, a famous example being the Bloch space. In this paper, Arazy and Fisher show that the Dirichlet space is the only Hilbert space on which the Möbius group can act by composition in a uniformly bounded manner. This paper is one of the earliest results in the theory of Möbius invariant spaces, which has applications to Hankel operators, composition operators, and the duality theory of function spaces.

1. Möbius invariant spaces

1.1. **The Möbius group.** Let $\mathbb{D}$ denote the unit disk in the complex plane. Let M denote the group of Möbius transformations on the unit disk. A Möbius transformation is a function of the form

$$\phi(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

for $z \in \mathbb{D}$, where $a \in \mathbb{D}$ and $\theta \in \mathbb{T}$.

The Möbius transformations are the holomorphic automorphisms of the disk. The set of all Möbius transformations forms a group under function composition. There is a well-known identification between the group of Möbius transformations and the quotient of the group $SL(2,\mathbb{R})$ by its center. The group $SL(2,\mathbb{R})$ is the set of 2×2 matrices with real entries and determinant one. The group operation is matrix multiplication.

The group M is non-abelian, and in fact it is non-amenable. For the purposes of the presentation we will need to identify two abelian subgroups of M. The first of these is the group of rotations $R = \{\rho_\theta : \theta \in (-\pi,\pi)\}$ where $\rho_\theta(z) = e^{i\theta}z$. This group is identified with circle group $\mathbb{T}$.

The second group is $G = \{\phi_r : -1 < r < 1\}$, where $\phi_r(z) = (z-r)/(1-rz)^{-1}$. An easy computation shows that

$$\phi_r \circ \phi_s = \phi_{(r+s)/(1+rs)}.$$

This shows that the above group is abelian. These two groups generate the group M since any Möbius transformation can be written in the form

$$e^{i\theta}\frac{z-a}{1-\overline{a}z} = e^{i\theta}\frac{z-|a|e^{i\phi}}{1-e^{-i\phi}|a|z} = e^{i(\theta+\phi)}\frac{e^{-i\phi}z-|a|}{1-|a|e^{-i\phi}z} = \rho_{\theta+\phi}\circ\psi_{|a|}\circ\rho_{-\phi}(z)$$

1.2. **Group actions.** A group G is said to act on a Banach space X if there is a homomorphism ρ from G into the group of invertible linear transformations of X, i.e., given $g \in G$ there exists an invertible linear transformation $\rho(g) \in B(X)$ such that $\rho(gg') = \rho(g)\rho(g')$ for all $g, g' \in G$. The action is called *uniformly bounded* if there is a constant C such that

$\sup_{g\in G}\|\rho(g)\| \le C$. If the constant $C = 1$, then $\rho(g)$ is contractive for all $g \in G$. It follows that $\rho(g)$ and $\rho(g)^{-1}$ are both contractions, and hence $\rho(g)$ is an isometry for all $g \in G$. In this case we say that G acts isometrically on X.

The Möbius group M acts on the disk. If $\phi \in M$, then there is a natural map on the set of holomorphic functions $H(\mathbb{D})$ given by $f \mapsto f \circ \phi^{-1}$. The space $H(\mathbb{D})$ is not a Banach space, but there are several interesting Banach spaces of analytic functions on which the induced action of M is well-defined.

For us it will often be convenient to work not with the space X, but the quotient of the space X by the constants. Note that the constants are invariant for the action of M and so there is an induced action on the quotient space given by $\rho(\phi)([f]) = [f \circ \phi^{-1}]$. When X is a Hilbert space, the quotient can be identified with the orthogonal complement of the constant function $\mathbf{1}$, and the above action is $\rho(\phi)(f) = f \circ \phi^{-1} - f(\phi^{-1}(0))\mathbf{1}$.

1.3. **Examples.**

(1) The Bloch space $\mathcal{B}$ is the set of analytic functions such that $p_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|$ is finite. The function $p_{\mathcal{B}}$ is a seminorm and the kernel of this seminorm is the constant functions. If we set $\|f\|_{\mathcal{B}} = |f(0)| + p_{\mathcal{B}}(f)$, then $\mathcal{B}$ is a Banach space with respect to this norm.

Let $\phi_a(z) = \dfrac{a - z}{1 - \overline{a}z}$ and f be an analytic function on the unit disk. A simple calculation shows that $|(f \circ \phi_a)'(0)| = (1 - |a|^2)|f'(a)|$ and it follows that the Bloch seminorm can be written in the form $p_{\mathcal{B}}(f) = \sup_{\phi\in M}|(f \circ \phi)'(0)|$. It follows that $p_{\mathcal{B}}(f \circ \phi) = p_{\mathcal{B}}(f)$ for all $\phi \in M$. This shows that the Bloch seminorm is invariant under the Möbius group.

(2) The next example is $H^\infty(\mathbb{D})$, the Banach algebra of bounded analytic functions on $\mathbb{D}$, with the norm given by $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. If ϕ is an automorphism of the unit disk, it is surjective, and we see that $\|f \circ \phi\|_\infty = \|f\|_\infty$.

(3) Consider the set of H^2 functions f with boundary limit in BMO. This space can be given a norm that is equivalent to the usual BMO norm by using $p(f) = \sup_{\phi\in M}\|f \circ \phi - (f \circ \phi)(0)\|_2$. This is again a space of analytic functions and the seminorm is invariant under the action of M.

(4) The final example is, not surprisingly, the Dirichlet space $\mathcal{D}$. The space $\mathcal{D}$ is the set of analytic functions such that

$$p_{\mathcal{D}}(f) = \int_{\mathbb{D}} |f'(z)|^2\, dA(z) < +\infty.$$

This function $p_{\mathcal{D}}$ is a seminorm on $\mathcal{D}$. If we change variables to $z = \phi(w)$, then the Jacobian of this map is $|\phi'(w)|^2$ and the change of variables formula shows that $p_{\mathcal{D}}(f \circ \phi) = p_{\mathcal{D}}(f)$.

If we recall that the seminorm $p_{\mathcal{D}}$ gives the area of the image $f(\mathbb{D})$, counting multiplicity, then the above invariance of the seminorm also follows from the fact that $f \circ \phi$ has the same image f and the multiplicity is unchanged.

1.4. **Basic properties and Duality.** The starting point of the theory of Möbius invariant spaces is a result of Rubel and Timoney [6]. Let (X, p) be a seminormed space of analytic functions such that p is invariant for the Möbius group M, i.e., $p(f \circ \phi) = p(f)$ for all $\phi \in M$. If there exists an integer $n \ge 1$, such that the functional $f \mapsto f^{(n)}(0)$ is continuous with respect to p, then $X \subseteq \mathcal{B}$, as a set, and there is a constant C such that $p_{\mathcal{B}}(f) \le Cp(f)$ for all $f \in X$. In this sense the Bloch space is maximal among the M-invariant function spaces. From now on we assume that the space (X, p) is contained in the Bloch space as above. It follows that the kernel of the seminorm is either trivial or the space of constant functions.

This leads to the following definition [3]:

Definition 1. A complete seminormed space of analytic functions (X, p) is called a *Möbius (or M-) invariant space* if X is boundedly contained in the Bloch space, and there is a uniformly bounded action of M on X.

Arazy-Fisher-Peetre [3] went on to develop a natural duality theory for M-invariant spaces and use this theory to show that there is also a minimal element of the class of M-invariant spaces. This is the space of functions such that $f'' \in L^1(\mathbb{D}, dA)$. The work in [3] also shows that M-invariant spaces contain the set of analytic polynomials as a dense subspace.

Given an M-invariant space (X, p), its Banach space dual can also be viewed as an M-invariant space, via a canonical pairing that uses the inner product on the Dirichlet space.

The interested reader should consult [3] and the survey paper of Fisher [5].

2. The main result

The paper of Arazy-Fisher [2] addresses the following question: *Are there Hilbert spaces besides the Dirichlet space that are M-invariant?*

The striking answer to this question is: no. This result is the main theorem of Arazy-Fisher [2].

Theorem 1 (Arazy-Fisher (1985))**.** *Suppose that (H, p) is a semi-Hilbert space and that H is boundedly contained in the Bloch space B. Let $p_{\mathcal{D}}$ denote the Dirichlet seminorm.*

(1) *If $p(f \circ \phi) = p(f)$ for all $\phi \in M$, then p is a multiple of $p_{\mathcal{D}}$ and $H = \mathcal{D}$ with equality of norms.*
(2) *If there is a constant C such that $p(f \circ \phi) \le Cp(f)$ for all $\phi \in M$, then $H = \mathcal{D}$ as a set and there is a constant K such that $K^{-1}p(f) \le p_{\mathcal{D}}(f) \le Kp(f)$ for all $f \in H$.*

2.1. **The unitary case.** The first statement is a fairly elementary result. Here is a brief outline of the proof of this result. We begin with the observation that the the semi-inner product $(,)$ induced by p has the property that $(f \circ \phi, g \circ \phi) = (f, g)$ for all $\phi \in M$. Rotation invariance shows the the functions z^n are orthogonal with respect p. Hence, the norm on H is determined by the norm of the functions z^n. Applying the relation above with $f = g = 1 + z$ and the Möbius transformation ϕ_r gives us a power series in r. By comparing coefficients we obtain $(\mathbf{1}, \mathbf{1}) = 0$ and the recursion $(z^n, z^n) = n(z, z)$ for $n \ge 1$. It follows that $p(z^n) = nc$ for some constant c and hence $p(f)^2 = c^2 \sum_{n=1}^{\infty} |a_n|^2 n = c^2 p_{\mathcal{D}}(f)^2$.

2.2. **The uniformly bounded case.** If the action of the group is assumed only to be uniformly bounded, then the problem is more interesting. We digress for moment from function theory to mention some of the history of uniformly bounded representations.

A representation of a group on Hilbert space H is called uniformly bounded if and only if there is a constant C such that $\sup_{g \in G} \|\pi(g)\| \le C$. How does one obtain such a representation? Given a unitary representation $\pi : G \to B(H)$ and a bounded invertible linear transformation $S \in B(H)$ we defined $\rho(g) = S^{-1}\pi(g)S$. We have $\|\rho(g)\| \le \|S^{-1}\|\|S\|$ from which we get a uniformly bounded representation with $C = \|S^{-1}\|\|S\|$.

The Dixmier problem asks for the converse: is every uniformly representation unitarizable? Equivalently, does there exist a unitary representation π and a similarity S such that ρ is of the above form? The answer is in fact false. The first known counter-example was due to Ehrenpreis and Mautner [4] in 1955 and group under consideration is in fact $SL(2,\mathbb{R})$.

In light of this problem, it would seem natural to look for examples of Hilbert space on which M acts by composition and for which the representation is uniformly bounded. The Arazy-Fisher result shows that the only candidate is the Dirichlet space. This implies the unitarizability of a uniformly bounded representation of M on a space of analytic functions, where the action is composition.

We now return to the proof. There are some simple observations one can make. Assume that the map $\phi \mapsto f \circ \phi$ is continuous from M into H.

The continuity of the map $\theta \mapsto f \circ \rho_\theta$ shows that H contains some non-trivial power of z. The invariance under the group M implies that H contains all polynomials. It is also the case that the functions analytic in a neighborhood of the closed disk are dense in H. This follows from the fact that the function $f_r(z) = f(rz)$ is in H for all $r < 1$ and that $f_r \to f$ in H as r increases to 1. The classical proof, involving the Poisson kernel, establishes this fact. It is also not hard to establish the bound $p(z^n) \lesssim n$. In order to prove the theorem, this bound must be improved to $p(z^n) \approx \sqrt{n}$.

The result of Arazy and Fisher now makes use of a clever averaging argument. A locally compact group Γ is called *amenable* if there is a finitely additive probability measure m on Γ that is translation invariant, i.e., $m(gE) = m(E)$ for all open sets E. If Γ is discrete, then this condition states that m is invariant on all sets. This property is equivalent to saying that there is a state $s : L^\infty(\Gamma) \to \mathbb{C}$ such that $s(f \circ g^{-1}) = s(f)$ for all $g \in \Gamma$. Compact groups are amenable, as are abelian groups. The property of being amenable passes to subgroups, quotients, extensions, and direct limits. The canonical example of a non-amenable group is $\mathbb{F}_2$, the free group on 2 generators. Since M contains a copy of $\mathbb{F}_2$, M is non-amenable.

The invariant mean m allows us to average over the group, thus when working with amenable groups we can often average a seminorm and make it Γ-invariant. The group M is not amenable and therefore we can not average over M and reduce the uniformly bounded case to the unitary case. However, M does contain the two abelian groups mentioned earlier: the rotation group R and the group G indexed by the interval $(-1, 1)$.

The first group is the circle and the invariant mean is just normalized arc-length measure. Thus we obtain a rotation invariant seminorm by

$$p_R(f) = \left(\int_{-\pi}^{\pi} p(f \circ \rho_\theta)^2 \, d\theta \right)^{1/2}.$$

Let m be an invariant mean on G, and average over G by

$$p_G(f) = m(p(f \circ \phi_r)^2)^{1/2}.$$

This seminorm is invariant for the group G. Note that neither of the two seminorms is invariant for the group M.

Since the action of M is uniformly bounded, the seminorms p, p_R and p_G are all equivalent to the original seminorm. Hence the representation of M in each of the resulting Hilbert spaces is also uniformly bounded.

Let $(,)$, $(,)_R$ and $(,)_G$ denote the sesquilinear forms that are induced by the three seminorms. Rotation invariant seminorms have the property that powers of z are pairwise orthogonal. Therefore, the rotation invariance of p_R allows us to reduce the problem to proving the estimate $p_R(z^n) \approx \sqrt{n}$. If we invoke the equivalence between the seminorm p_R and the seminorm p_G, then we see that it is enough to establish that $p_G(z^n) \approx \sqrt{n}$.

A fairly involved estimate is required and this estimate takes up most of the paper. Since p_G is invariant for the group G, we have $(z^n, z^m)_G = (\phi_r^n, \phi_r^m)_G$. This gives a relation between the inner products $(z^n, z^m)_G$. First set $\alpha_{m,n} = (z^m, z^n)_G$ and set $\beta_n = \alpha_{n,n}$. The relation between these numbers is

$$0 = n(\alpha_{n+1,m} - \alpha_{n-1,m}) + m(\alpha_{n,m+1} - \alpha_{n,m-1}).$$

Let $s_N = \sum_{k=1}^{N} \beta_k + \beta_0/2$. It is shown that $\beta_{N+1}/(N+1)$ can be written in terms of $s_N/(N(N+1))$ and $\alpha_{N,N+2}/N$. The problem of showing that $\beta_n \approx n$ is essentially replaced with the problem of showing that $s_n \approx n^2$. As is often the case, the upper bound on s_n is easier to obtain, while the lower bound requires more work. We omit the details from this abstract.

3. Some counter-examples

In the final section the authors present examples to show that some of the assumptions can not be weakened.

(1) Consider the space of analytic functions on the open disk such that

$$\|f\|_w := \left(\sum_{n=0}^{\infty} |a_n|^2 w_n \right)^{1/2}$$

is convergent, where w_n is a fixed sequence of non-negative weights. This space of analytic function is invariant under the rotation group. However, if we pick a sequence w_n such that $w_1 = 0$ and $w_2 = 1$, then this space is not equivalent to the Dirichlet space.

(2) Let us consider the one-parameter group C_r given by $C_r(f) = f \circ \phi_r$. Let T denote the infinitesimal generator of this semigroup, i.e., the (unbounded, densely defined operator) T given by $Tf = \left.\frac{d}{dr}(f \circ \phi_r)\right|_{r=0}$. Given a non-negative measurable function u on the imaginary axis, this operator is used to define and inner product on the domain of $u(T)$ by $(f, g)_u = (u(T)f, g)_{\mathcal{D}}$. This inner product is equivalent to the inner product on the Dirichlet space if and only if u is positive a.e. However, the inner product is invariant with respect to the group G_r.

(3) Consider the space of analytic functions f such that f' is in the Hardy space. Here the (square of the) norm of f is given by $\sum_{n=1}^{\infty} n^2 |a_n|^2$. The norm is rotation invariant and $\|f \circ \phi_r\| \le \frac{2}{(1-r^2)} \|f\|$. Hence, G does act continuously on the space H. The space H is not isomorphic to the Dirichlet space. Here the action is not *uniformly bounded*.

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CARLESON MEASURES FOR ANALYTIC BESOV SPACES

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presented by Tim Ferguson

ABSTRACT. The paper "Carleson measures for Analytic Besov Spaces" by Arcozzi, Rochberg, and Sawyer deals with measures on weighted analytic Besov spaces such that the L^q norm of an analytic function with respect to the measure is bounded by a constant times the Besov p space norm of the function, where $1 < p \leq q < \infty$. These measures are called Carleson measures. The paper considers an analogous question for measures on trees, and uses the results obtained in that case to answer the original question. Various applications are given, for example to multipliers and interpolating sequences.

The article "Carleson measures for Analytic Besov Spaces" by Arcozzi, Rochberg, and Sawyer [1] deals with certain characterizations of Carleson measures for weighted Besov spaces of holomorphic functions. The main technique is to look at an analogous problem for discrete measures on trees. The authors obtain several results for trees, which are then translated back to the case of Besov spaces. The paper also deals with questions about multipliers and interpolating sequences in both Besov spaces and on trees.

To explain this paper, we will first establish some basic notation. For p in the range $1 \leq p \leq \infty$, we define p' to be the conjugate exponent of p; in other words, $1/p + 1/p' = 1$. For $a \in \mathbb{D}$, we define the Carleson square with vertex a to be

$$S(a) = \{z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, |\arg(a) - \arg(z)| \leq (1 - |a|)\pi\}.$$

The heightened Carleson box for a is defined to be

$$\widetilde{S}(a) = \{z \in \mathbb{D} : 1 - |z| \leq 2(1 - |a|), |\arg(a) - \arg(z)| \leq (1 - |a|)\pi\}.$$

Let $\rho \geq 0$ be a positive Borel measurable function on $\mathbb{D}$, the unit disc. We call ρ a *weight*. For a function f analytic in $\mathbb{D}$, and for $1 < p < \infty$, define

$$\|f\|^*_{B_p(\rho)} = \left[\int_{\mathbb{D}} |f'(z)|^p \rho(z)(1 - |z|^2)^{p-2} dm(z)\right]^{1/p},$$

where m is normalized Lebesgue area measure on the disc. Next define

$$\|f\|_{B_p(\rho)} = |f(0)| + \|f\|^*_{B_p(\rho)}.$$

The weighted Besov space $B_p(\rho)$ consists of all function f analytic in $\mathbb{D}$ such that $\|f\|_{B_p(\rho)}$ is finite. We will also define the hyperbolic area measure by

$$m_h(dz) = (1 - |z|^2)^{-2} m(dz).$$

The paper deals only with *admissible* weights. These are weights ρ satisfying both of the following:

(i) The weight ρ is regular, meaning that there exists $\epsilon, C > 0$ with $\epsilon < 1$ such that $\rho(z_1) \le C\rho(z_2)$ whenever
$$\left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right| \le \epsilon.$$

(ii) The weight $\rho_p(z) = (1-|z|^2)^{p-2}\rho(z)$ satisfies the *Bekollè-Bonami B_p condition:* There is a constant C depending only on ρ and p such that
$$\left(\int_{S(a)} \rho_p(z)\, dm(z)\right)\left(\int_{S(a)} \rho_p(z)^{1-p'} dm(z)\right)^{1/(p'-1)} \le Cm(S(a))^p.$$

The B.-B. condition of (ii) is a "reverse Hölder Inequality." For a large class of weights, including admissible weights, $B_p(\rho)$ is a Banach space under the norm $\|\cdot\|_{B_p(\rho)}$. The B.-B. condition allows one to identify the dual space of $B_p(\rho)$ with $B_{p'}(\rho^{1-p'})$, where $g \in B_{p'}(\rho^{1-p'})$ corresponds to the functional $f \mapsto \int_{\mathbb{D}} f(z)\overline{g(z)}\, dm(z)$.

Let μ be a positive Borel measure on $\mathbb{D}$. Suppose that $1 < p \le q < \infty$. We say that μ is a Carleson measure for $(B_p(\rho), q)$ if there is a constant $C(\mu)$ such that
$$\|f\|_{L^q(\mu)} \le C(\mu)\|f\|_{B_p(\rho)} \tag{1}$$
for all $f \in B_p(\rho)$. For this to occur, the authors prove that a necessary and sufficient condition is that there exist a $C_1(\mu)$ such that, for all $a \in \mathbb{D}$,
$$\left\{\int_{\widetilde{S}(a)} \rho(z)^{-p'/p}(\mu(S(z)\cap S(a)))^{p'} dm_h(z)\right\}^{q'/p'} \le C_1(\mu)\mu(S(a)). \tag{2}$$

They also discuss the simpler condition
$$\mu(S(a))^{1/q} \le C_2(\mu)\left\{\int_{[0,a]} \rho(w)^{1-p'}(1-|w|^2)^{-1}|dw|\right\}^{1/p'}. \tag{3}$$
This condition is necessary and sufficient for a measure to be Carleson if $1 < p < q < \infty$. If $p = q$, the condition is still necessary, but is not sufficient, as a counter-example in the paper shows.

These conditions are related to certain conditions for a type of Carleson measure on tress. A *tree* is a connected graph without loops. We generally fix a vertex $o \in T$ and call it the root of the tree T. If x and y are vertices of the tree, we say that $x \le y$ if $x \in [o, y]$, where $[o, y]$ is the geodesic joining o to y. For $x \in T$, the *shadow* of x, also called the *Carleson square with vertex x*, is the set
$$S(x) = \{y \in T : y \ge x\}.$$
For a function f defined on (the vertices of) T, we define the *primitive* or *integral* of f by
$$\mathcal{I}f(x) = \sum_o^x f(y) = \sum_{y\in[o,x]} f(y).$$
A weight ρ on T is a positive function on T, and a measure μ is a nonnegative function on T.

Given a measure μ, a weight ρ, and p and q such that $1 < p \le q < \infty$, we say that μ is an $(\mathcal{I}, \rho, p, q)$ Carleson measure if there exists a constant $C(\mu)$ such that for all functions f, we have

$$\left(\sum_{x\in T} |\mathcal{I}f(x)|^q \mu(x)\right)^{1/q} \le C(\mu)\left(\sum_{x\in T} |\phi(x)|^p \rho(x)\right)^{1/p}. \tag{4}$$

Corresponding to condition (2) we have the condition that there exists a constant $C(\mu)$ such that, for all $r \in T$,

$$\left(\sum_{x\in S(r)} \left(\sum_{y\in S(x)} \mu(y)\right)^{p'} \rho(x)^{1-p'}\right)^{q'/p'} \le C(\mu) \sum_{x\in S(r)} \mu(x), \tag{5}$$

which is a necessary and sufficient condition for μ to be Carleson.

Corresponding to the condition (3) is the condition that there exists a constant $C(\mu)$ such that for all $r \in T$,

$$\left(\sum_{x\in S(r)}\right)^{1/q} \le C(\mu)\left(\sum_{0}^{r} \rho(x)^{1-p'}\right)^{-1/p'}. \tag{6}$$

This condition is necessary and sufficient if $p < q$. If $p = q$, it is still necessary, but not sufficient.

The condition (5) can be understood as follows. The measure μ being a $(\mathcal{I}, \rho, p, q)$ Carleson measure is equivalent to to operator $\mathcal{I} : L^p(\rho) \to L^q(\mu)$ being bounded. Now, we can identify the dual space of $L^p(\rho)$ with $L^{p'}(\rho^{1-p'})$, where $g \in L^{p'}(\rho^{1-p'})$ corresponds to the functional $f \mapsto \sum_{x\in T} f(x)\overline{g(x)}$. Also, we can identify the dual space of $L^q(\mu)$ with $L^{q'}(\mu)$, where $g \in L^{q'}(\mu)$ corresponds to the functional $f \mapsto \sum_{x\in T} f(x)\overline{g(x)}\mu(x)$.

Thus, the operator $\mathcal{I}$ being bounded is equivalent with the adjoint operator $\mathcal{I}^* : L^{q'}(\mu) \to L^{p'}(\rho^{1-p'})$ being bounded. Now this operator is given by

$$\mathcal{I}^* f(x) = \sum_{y\in S(x)} f(y)\mu(y).$$

Testing the boundedness of $\mathcal{I}^*$ on characteristic functions of the form $\chi_{S(r)}$ yields condition (5), so this condition is necessary. As stated above, the authors also prove it is sufficient.

The condition (6) is a discretization and generalization of the necessary and sufficient condition for a measure to be Carleson for the Hardy space. This condition is sometimes called a *single box condition*. It is interesting that this type of condition is sufficient for Hardy and Bergman spaces, and for weighted Besov spaces (with admissible weights) when $p < q$, but not for weighted Besov spaces when $p = q$.

To relate the results for trees to analytic functions in the disc, consider the Whitney squares

$$\Delta_{n,m} = \left\{ z \in \mathbb{D} : 2^{-(n+1)} \le 1 - |z| \le 2^{-n}|, \left| \frac{\arg(z)}{2\pi} - \frac{m - 1/2}{2^n} \right| \le 2^{-(n+1)} \right\}$$

for $n \ge 0$ and $1 \le m \le 2^n$, which form a dyadic Whitney decomposition of $\mathbb{D}$. Each of these Whitney squares has (roughly) the same area measure and diameter under the hyperbolic metric. We associate with this dyadic decomposition the tree T_2 with vertices given by

$$\{\alpha : \alpha = (n, m), n \ge 0, 1 \le m \le 2^n\}$$

(where n and m are integers.) We say there is an edge between two vertices (n, m) and (n', m') if $\Delta_{(n,m)}$ and $\Delta_{(n',m')}$ share an arc of a circle. We define the root of T_2 to be $(0, 1) \in \mathbb{N} \times \mathbb{N}$, which corresponds to the dyadic square $\{z \in D : |z| \leq 1/2\}$. Then T_2 is a dyadic tree, each of whose vertices has degree 3, except for the root, which has degree 2.

If μ is a measure on $\mathbb{D}$, and $\alpha \in T_2$, we can define $\mu(\alpha) = \mu(\Delta_\alpha)$, and $\rho(\alpha) = \rho(\xi_\alpha)$, where ξ_α is the center of Δ_α. The authors show the following:

Proposition 5. *Let ρ be an admissible weight. For $1 < p \leq q < \infty$, the measure μ is a Carleson measure for $(B_p(\rho), q)$ if and only if it is a Carleson measure for $(\mathcal{I}, \rho, p, q)$.*

This proposition allows them to translate results about trees to results dealing with Besov spaces.

The authors remark that this proposition may seem surprising, since some structure is lost in the passage from Besov spaces to trees. For example, $\mathcal{I}f$ does not have to satisfy a mean value property. They say that the fact that this proposition does hold is linked to the B.-B. B_p condition. For example, if $\rho(z) = 1 - |z|^2$, and $p = 2$, then the weighted Besov spaces is just the Hardy space and the B_2 condition does not hold. Thus, this weight is not admissible. Both Proposition 5 and the equivalence of (1) and (2) fail for the Hardy space. In section 8, the authors show that results for Carleson measures on trees can be used to give information about Carleson measures for the Hardy space if a sort of mean value property is required to hold for functions on trees.

The authors apply their results on Carleson measures to related problems. For example, they deal with multipliers on trees and in the weighted Besov spaces. In each case, they give a necessary and sufficient condition for a given function to be a multiplier. They also deal with interpolating sequences on Besov spaces and on trees. For trees, they give a necessary and sufficient condition for a sequence to be interpolating. They also give partial results for this problem in Besov spaces.

There had been a good deal of work done on the problem of characterizing Carleson measures for analytic Besov spaces before this paper was written. For example, let $\rho_\alpha(z) = (1 - |z|^2)^\alpha$, and let $D_a = B_2(\rho_\alpha)$. In 1980, Stegenga [4] obtained a characterization of the Carleson measures for D_α when $\alpha > 0$. Later, I. Verbitsky [5], J. Wang [6], and Z. Wu [7] each generalized his result to B_p for $1 < p < \infty$.

For $\alpha \geq 1$, Stegenga found that a necessary and sufficient condition for a measure to be Carleson is that $\mu(S(z))p_\alpha^{-1}(z) \leq C$. This is equivilent to

$$\mu(S(z)) \int_{[0,z]} \rho_\alpha^{-1}(w)(1 - |w|^2)|dw| \leq C, \tag{7}$$

which is simply condition (3) for $p = q = 2$. For $\alpha < 1$, this condition is not sufficient any longer. He was able to obtain a sufficient condition in terms of a capacity estimate. The authors show that ρ_α is admissible if and only if $-1 < \alpha < 1$, which explains the why (7) is necessary but not sufficient in this range.

Kerman and Sawyer [3] extended some of Stegenga's results in 1984. They worked in the case $p = q = 2$, but their results hold for *Dirichlet* weights. A Dirichlet weight ρ is of the form $\rho(z) = \phi(1 - |z|)$ where $0 \leq \phi \leq 1$ and ϕ is nondecreasing and concave. They gave a necessary and sufficient condition not involving capacities for a measure to be Carleson for a Dirichlet weight. It turns out that some weights are both Dirichlet and admissible, but neither implies the other. Thus, the results of Kerman and Sawyer and this paper provide two different ways of describing Carleson measures for admissible, Dirichlet weights.

In 1995, Evans, Harris, and Pick [2] showed that a measure μ is $(\mathcal{I}, \rho, p, q)$ Carleson on a tree if and only if μ satisfies a certain capacity condition. From this, it follows that their capacity condition must be equivalent to (2), although the authors do not know a way to show this directly.

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BILINEAR FORMS ON THE DIRICHLET SPACE

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presented by Austin Anderson

ABSTRACT. The bilinear form $T_b(f,g) = \langle fg, b\rangle$ acting on the Dirichlet space $\mathcal{D}$ is bounded if and only if $|b'|^2\,dx\,dy$ is a Carleson measure for $\mathcal{D}$. The proof uses Stegenga's capacity-theoretic characterization of such measures, which is aided by a discrete capacity model on D and techniques related to interpolating sequences for Besov spaces.

Let $\mathcal{D}$ be the Dirichlet space of analytic functions on the unit disk D. For an analytic function $b(z)$, $z \in D$, define the bilinear form T_b on $\mathcal{D} \times \mathcal{D}$ by

$$T_b(f,g) = \langle fg, b\rangle_{\mathcal{D}}, \quad f, g \in \mathcal{D}.$$

Define the measure μ_b on D by

$$d\mu_b(z) = |b'(z)|^2 dA(z),$$

where dA denotes Lebesgue area measure. The main result of this paper is that T_b is bounded on $\mathcal{D} \times \mathcal{D}$ if and only if μ_b is a Carleson measure for $\mathcal{D}$. An early result of this type is due to Nehari [6]. Let H^2 denote the Hardy space on D. For $f, g \in H^2$, the operator $T_b^{H^2}(f,g) = \langle fg, b\rangle_{H^2}$ is bounded if and only if $|b'(z)|^2(1-|z|^2)\,dA(z)$ indicates a Carleson measure for H^2, i.e., $b \in BMOA$. Coifman, Rochberg, and Weiss showed that analogous results hold for $H^2(\partial\mathbb{B}^n)$, the Hardy space on the sphere in $\mathbb{C}^n$ [2]. Ferguson, Lacey, and Terwilleger did the same for H^2 on the polydisk $H^2(\mathbb{D}^n)$ ([3] and [4]). The trend in these results continues for Schrödinger operators in work by Maz'ya and Verbitsky [5], but apparently the techniques in each proof vary greatly, and no generalization is known. The result on H^2 follows from the factorization of any H^1 function into a product of two H^2 functions, and the main result here can be seen as a statement about weak factorization in $\mathcal{D}$.

To set up the statement of the main theorem, norm the space of Carleson measures for $\mathcal{D}$ by

$$\|\mu\|_{CM} = \sup_{\|f\|_{\mathcal{D}}=1} \int_D |f|^2\,d\mu.$$

Define the space of analytic functions X by

$$X = \{b : \|b\|_X = |b(0)| + \|\mu_b\|_{CM}^{1/2} < \infty\},$$

and define X_0 as the closure in X of the polynomials. The norm of T_b is

$$\|T_b\| = \sup\{|T_b(f,g)| : \|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1\}.$$

Main Result. Theorem 1.1

(1) *T_b is bounded if and only if $b \in X$, and $\|T_b\| \approx \|b\|_X$.*

(2) T_b *is compact if and only if* $b \in X_0$.

Part 2 of Theorem 1.1 follows readily from part 1, and the proof of one implication of part 1 is straightforward, namely $b \in X$ implies T_b is bounded. If μ_b is a Carleson measure for $\mathcal{D}$, then

$$
\begin{aligned}
|T_b(f,g)| &= |\langle fg, b\rangle| \\
&= \left| f(0)g(0)\overline{b(0)} + \int_D (f(z)g'(z) + f'(z)g(z))\overline{b'(z)}\, dA(z) \right| \\
&\le |(fg\bar{b})(0)| + (\|g\|_{\mathcal{D}}) \left(\int_D |f(z)|^2 |b'(z)|^2\, dA(z) \right)^{1/2} + (\|f\|_{\mathcal{D}}) \left(\int_D |g(z)|^2 |b'(z)|^2\, dA(z) \right)^{1/2} \\
&\le |(fg\bar{b})(0)| + (\|g\|_{\mathcal{D}})(\|f\|_{\mathcal{D}})(\|\mu_b\|_{CM}^{1/2}) + (\|f\|_{\mathcal{D}})(\|g\|_{\mathcal{D}})(\|\mu_b\|_{CM}^{1/2}) \\
&\le C(b(0) + \|\mu_b\|_{CM}^{1/2})\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}} = C\|b\|_X \|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}.
\end{aligned}
$$

Hence, T_b is bounded and $\|T_b\| \le C\|b\|_X$.

The difficult part of Theorem 1.1 is to prove μ_b is a Carleson measure if T_b is bounded. The proof uses the characterization of Carleson measures for $\mathcal{D}$ established by Stegenga in [7], which gives a condition involving the capacity of a set $E \subset \partial D$,

$$\mathrm{Cap}_D(E) = \inf\{\|\psi\|_{\mathcal{D}}^2 : \psi(0) = 0, \operatorname{Re}\psi(z) > 1 \text{ for } z \in E\}. \tag{1}$$

Stegenga proved that μ is a Carleson measure for $\mathcal{D}$ if and only if the following holds:

There exists C such that

$$\mu\left(\cup_{j=1}^N T(I_j)\right) \le C\ \mathrm{Cap}_D\left(\cup_{j=1}^N I_j\right) \tag{2}$$

for any finite set of disjoint intervals $\{I_j\}_1^N \subset \partial D$. Here $T(I)$ is the tent region corresponding to an arc $I \subset \partial D$, i.e., the convex hull of I and the point $z(I) = (1 - |I|)e^{i\theta_I}$, where $e^{i\theta_I}$ is the midpoint of I and $|I|$ is the length of I. For any open set $G \subset \partial D$, we denote

$$T(G) = \bigcup_{I \subset G} T(I).$$

Capacity estimates are key to this paper, and the next lemma will use the following defintion of a "disk blowup". In the definition $0 < \rho < 1$ and I^ρ is the arc with length $|I|^\rho$ having the same midpoint as I.

$$\text{For } G \text{ open in } \partial D, \text{ the disk blowup of } G \text{ is} \quad G_D^\rho = \bigcup_{I \subset G} T(I^\rho). \tag{3}$$

Lemma 2.2. *Let* G *be open in* ∂D. *If* $w \in T(G) = G_D^1$ *and* $z \notin G_D^\rho$, *then* $|z-w| \ge (1-|w|^2)^\rho$.

A main tool in this paper is the discretization of the disk by the Bergman tree $\mathcal{T}$. By making circular cuts of radii $1 - 2^{-k}$, $k \ge 1$, and dividing each annulus $\{z : 1 - 2^{-k} < |z| < 1 - 2^{-(k+1)}\}$ into 2^k equal pieces by radial cuts, we obtain a collection of boxes with approximately constant hyperbolic diameter. Index the center points of these boxes by a dyadic tree $\mathcal{T}$ with root o. The radial projection of each box onto ∂D creates a bijection between the boxes and dyadic intervals in ∂D. Except for the root o, each square/interval x has one immediate predecessor x^{-1} and two immediate successors x_+ and x_-. Define the successor set $S(x) = \{y \in \mathcal{T} : y \ge x\}$. A *stopping time* $W \in \mathcal{T}$ is a subset of the tree with no successors, i.e., for $x, y \in W$, $y \ge x$ implies $x = y$. Given two stopping times E and F, the notation $F \succ E$ means for every $x \in F$ there exists $y \in E$ such that $x > y$. For $x_1, x_2 \in \mathcal{T}$, define the interval $[x_1, x_2] = \{y : x_1 \le y \le x_2\}$, and $\mathcal{G}(E, F) = \{y : y \in [e, f]$ for some $e \in E, f \in F\}$. Since every open set $G \subset \partial D$ is a countable union of dyadic intervals and points in between them, for every G we can associate a unique stopping time $E \subset \mathcal{T}$ such that $G \setminus E$ is at most countable.

The concepts of integration, harmonic functions, and the mean value property exist on the tree $\mathcal{T}$. For a function k on $\mathcal{T}$, define the discrete integral

$$Ik(x) = \sum_{y \in [o,x]} k(y).$$

A *harmonic* function H on a subset $\mathcal{G} \subset \mathcal{T}$ satisfies

$$H(x) = \frac{1}{3}\left(H(x^{-1}) + H(x_-) + H(x_+)\right), \quad x \in \mathcal{G}.$$

The capacity of a stopping time $F \subset \mathcal{T}$ is defined to be

$$\operatorname{Cap}_{\mathcal{T}}(F) = \inf\{\|k\|^2_{\ell^2(\mathcal{T})} : Ik \ge 1 \text{ on } F\}.$$

A *condenser* is a pair $E, F \subset \mathcal{T}$ of two disjoint stopping times with $F \succ E$, and

$$\operatorname{Cap}_{\mathcal{T}}(E, F) = \inf\left\{\|k\|^2_{\ell^2(\mathcal{T})} : Ik \ge 1 \text{ on } F,\ \operatorname{supp}(k) \subset \bigcup_{e \in E} S(e)\right\} \ge \operatorname{Cap}_{\mathcal{T}}(\{o\}, F) = \operatorname{Cap}_{\mathcal{T}} F.$$

The estimates required for proving the main theorem are trivial when the capacities involved are bounded away from 0, so in subsequent statements we might assume the capacities are small. The important properties of tree capacities are summarized in the next proposition.

Proposition 2.3. *Let E, F be disjoint stopping times with $F \succ E$.*
(1) There is a function h on $\mathcal{T}$ such that $\operatorname{Cap}_{\mathcal{T}}(E, F) = \|h\|^2_{\ell^2(\mathcal{T})}$.
(2) The function $H = Ih$ is harmonic on $\mathcal{T} \setminus (E \cup F)$.
(3) If $S \subset \mathcal{T}$ is a stopping time, then $\sum_{x \in S} |h(x)| \le \operatorname{Cap}_{\mathcal{T}}(E, F)$.
(4) $h(x) \ge 0$ for all $x \in \mathcal{T}$, and $h(x) = 0$ unless $x \in \mathcal{G}(E, F)$.

The tree model provides a result whose analog in the disk has not been proven. If F^ρ is the "tree blowup" of a stopping time F, Lemma 2.8 in this paper states that

$$\operatorname{Cap}_{\mathcal{T}} F^\rho \le \rho^{-2} \operatorname{Cap}_{\mathcal{T}} F. \tag{4}$$

The significance of Lemma 2.8 is the existence of constants C_ρ (independent of F) such that $\operatorname{Cap}_{\mathcal{T}} F^\rho \le C_\rho \operatorname{Cap}_{\mathcal{T}} F$ with $\lim_{\rho \to 1^-} C_\rho = 1$. A tool in achieving Lemma 2.8 is the "capacitary blowup," which depends on the structure of F via the unique extremal function H in Proposition 2.3. For $0 < \rho < 1$ the capacitary blowup is

$$\widehat{F^\rho} = \{x \in \mathcal{G}(\{o\}, F) : H(x) \ge \rho \text{ and } H(t) < \rho \text{ for } t < x\}.$$

We wish to approximate the extremal $H = Ih$ from Proposition 2.3 by an extremal Φ defined on the disk. For $s > -1$ define

$$\Phi(z) = \sum_{\kappa \in \mathcal{T}} h(\kappa) \left(\frac{1 - |\kappa|^2}{1 - \bar{\kappa} z}\right)^{1+s}.$$

Adjusting by a constant c_s, the function Φ is a projection onto the holomorphic functions of an interpolating function for Besov spaces. By a result of Böe [1], Φ is in $\mathcal{D}$, and a corollary of Proposition 2.9 is that $\|\Phi\|^2_{\mathcal{D}} \le C \operatorname{Cap}_{\mathcal{T}}(\widehat{F^\rho_{\mathcal{T}}}, F)$.

Proposition 2.9. *Let $F = \{w_k\}_k$ be a stopping time in $\mathcal{T}$ and $E = \widehat{F^\rho_{\mathcal{T}}}$. Then the following are true:*
$|\Phi(z) - \Phi(w_k)| \le \mathrm{Cap}_{\mathcal{T}}(E, F), \quad z \in T(w_k),$
$\mathrm{Re}\,\Phi(w_k) \ge c > 0, \qquad k \ge 1,$
$|\Phi(w_k)| \le C, \qquad k \ge 1,$
$|\Phi(z)| \le C\,\mathrm{Cap}_{\mathcal{T}}(E, F), \quad z \notin F.$

The upshot of Proposition 2.9 is that $\Psi(z) = \frac{3}{c}(\Phi(z) - \Phi(0))$ contends for an extremal function in the definition of disk capacity (1), and we obtain the next comparison.

Corollary 2.12. *Let G be a finite union of arcs in ∂D. Then*

$$\mathrm{Cap}_D(G) \approx \mathrm{Cap}_{\mathcal{T}}(G), \tag{5}$$

where $\mathrm{Cap}_{\mathcal{T}}(G)$ is the tree capacity of the set of dyadic intervals E associated with G such that $G \setminus E$ is countable and minimal.

Let $\mathcal{T}_\theta$ be the rotation of $\mathcal{T}$ by angle θ, and for an open set $E \subset \partial D$ let $\mathrm{Cap}_\theta(E)$ and $T_\theta(E)$ denote the tree capacity and tent region associated with $\mathcal{T}_\theta$. Defining

$$M := \sup_{E \text{ open } \subset \partial D} \frac{\int_{\partial D} \mu_b(T_\theta(E))\, d\theta}{\int_{\partial D} \mathrm{Cap}_\theta(E)\, d\theta}, \tag{6}$$

Corollary 2.13 says

$$\|\mu_b\|^2_{CM} \approx M. \tag{7}$$

Proposition 2.14 shows that the disk blowup does not increase $\mu_b(T(G))$ too much for open sets $G \subset \partial D$ that are almost extremal in the definition of M (6). That is, given $\epsilon > 0$ we can ensure $\mu_b(G^\rho_D \setminus T(G)) \le \varepsilon \mu_b(T(G))$ when $\frac{\int_{\partial D} \mu_b(T_\theta(G))\, d\theta}{\int_{\partial D} \mathrm{Cap}_\theta(G)\, d\theta} \ge \delta M$ for a suitable choice of $\rho < 1$ and $\delta < 1$. Lemma 2.8 (4) is key to proving Proposition 2.14.

Theorem 1.1 is finally proved by showing that

$$\mu_b(T_\theta(G)) \le C\|T_b\|^2\, \mathrm{Cap}_D(G) \tag{8}$$

for any finite union G of arcs in ∂D that is almost extremal for (6). It is shown that we may restrict our attention to such sets. Then (7) and (5) give

$$\|\mu_b\|^2_{CM} \approx M \le C\frac{\int_{\partial D} \mu_b(T_\theta(G))\, d\theta}{\int_{\partial D} \mathrm{Cap}_\theta(G)\, d\theta} \le C\frac{\|T_b\|^2\, \mathrm{Cap}_D(G)}{\mathrm{Cap}_D(G)} = C\|T_b\|^2.$$

The test functions f and g used in the proof of (8) are approximations of $b'\chi_{T_\theta(G)}$ and $\chi_{T_\theta(G)}$. For $\frac{1}{2} < \beta < \gamma < \alpha < 1$, define

$$V_G = T_\theta(G), \quad V^\alpha_G = G^\alpha_D, \quad V^\gamma_G = \widehat{(V^\alpha_G)^{\gamma/\alpha}_{\mathcal{T}}}, \quad \text{and } V^\beta_G = (V^\gamma_G)^{\beta/\gamma}_D.$$

Starting with V_G, a region of tents of tree intervals corresponding to G, take the disk blowup V^α_G, then the capacitary blowup V^γ_G of the disk blowup, and finally another disk blowup V^β_G. Letting $E = V^\gamma_G$ and $F = V^\alpha_G$, Proposition 2.9 gives the appropriate function Φ. Define $g = \Phi^2$, and let f be a projection onto the holomorphic functions of $b'\chi_{V_G}$, i.e.,

$$f(z) = \int_{V_G} \frac{b'(\zeta)(1 - |\zeta|^2)^s}{(1 - \bar\zeta z)^{1+s}} \frac{dA}{(1+s)\bar\zeta}.$$

Then $f'(z) = \int_{V_G} b'(\zeta)k(z,\zeta)\,dA = b'(z) + \Lambda b'(z)$, where $\Lambda b'(z) = -\int_{D\setminus T_G} b'(z)k(z,\zeta)\,dA$ and $k(z,\zeta) = \frac{(1-|\zeta|^2)^s}{(1-\bar{\zeta}z)^{2+s}}$. Plugging in our test functions, we obtain $T_b(f,g) = \sum_{j=1}^{4} I_j$. The summands are $I_1 = (f\Phi^2\bar{b})(0)$, $I_2 = \int_D |b'(z)|^2\Phi^2(z)\,dA$, $I_3 = 2\int_D \Phi(z)\Phi'(z)f(z)\overline{b'(z)}\,dA$, and $I_4 = \int_D \Lambda b'(z)\overline{b'(z)}\Phi^2(z)\,dA$. The estimate on I_1 is easy. We split I_2 into the three regions V_G, $V_G^\beta \setminus V_G$, and $D \setminus V_G^\beta$, and use Propositions 2.9 and 2.14. Splitting I_3 similarly, its estimate also uses Lemma 2.2 and a bilinear version of Schur's integral test. Getting the appropriate bound on I_4 also uses Schur's test, Corollary 2.12, and Lemmas 2.7 (not stated here, but it relates to capacitary blowup and 2.8) and 2.2.

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CYCLIC VECTORS IN THE DIRICHLET SPACE

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presented by Constanze Liaw

Abstract. Cyclic vectors play an important role in mathematical physics, operator theory as well as in the theory of function spaces. Part of the basic structure of a function space is determined by whether or not all/some/any/which of its vectors are cyclic. Their applications range from shift-invariant subspaces to the famous problem of Anderson localization. In their paper "Cyclic Vectors in the Dirichlet Space", see [2], Brown and Shields investigated certain properties of cyclic vectors in a general Banach space of analytic functions in a bounded region of the complex plane, while their (and also our) main focus is on the classical Dirichlet space. In the paper, the cyclicity of a vector means with respect to the forward shift operator. This short survey is based on the following articles [1–4] (in the order of and the references within).

1. Introduction

The *Dirichlet space* consists of analytic functions f on $\mathcal{D}$ which have finite Dirichlet integral $D(f) := 1/\pi \int_{\mathbb{D}} |f'|^2 dA$. A vector $f \in \mathcal{D}$ is called *cyclic*, if its polynomial multiples are dense in $\mathcal{D}$, i.e. if $\mathcal{D} = \operatorname{clos\,span}\{p(z)f(z) : p \text{ is a polynomial}\}$. Clearly, this is equivalent to the cyclicity of f with respect to the forward shift operator $S : f \mapsto zf$ on $\mathcal{D}$. By $[f]$ we denote the cyclic subspace generated by f, i.e.

$$[f] = \operatorname{clos\,span}\{p(z)f(z) : p \text{ is a polynomial}\}. \tag{1.1}$$

With this notation f is cyclic if and only if $\mathcal{D} = [f]$.

The question and main goal of this theory is to characterize the vectors $f \in \mathcal{D}$ which are cyclic in $\mathcal{D}$.

At this, we should also keep in mind that these questions can be thought of as part of understanding the structure and the basic building blocks of the Dirichlet space (and the other function spaces mentioned below).

1.1. **Outline.**

We begin section 2 by presenting several simple cyclicity results on Banach spaces of analytic functions on a bounded region in the complex plane, but very quickly move to a special case. Further, we briefly mention some motivating results about cyclicity in the Hardy space and discuss some relations and implications for cyclicity on the Dirichlet space. Finally, we state a theorem which connects the problem of cyclicity with that of (logarithmic) capacity and raise an open question.

The results of Brown and Shields were the basis of further study. We give an account of the results in this field since 1984 in section 3. At this point, a Beurling-type characterization of the shift-invariant subspaces of $\mathcal{D}$ still remains one of the open questions of this theory. In fact, Theorem 3.2 (below) relates the question of cyclicity for the Dirichlet space with the problem of shift-invariant subspaces. We further mention three fields which are related: multipliers, sets of uniqueness and the cyclicity with respect to other operators.

In the appendix, we very briefly present the notion of (logarithmic) capacity which is often used in this context.

2. Pre-History and main results

2.1. **Cyclic vectors are separated from zero.** We will state part of the theory for a Banach space E of analytic functions on a bounded region in the complex plane G, i.e. E satisfies six conditions including continuity of point evaluations, containment and density of polynomials as well as invariance of the shift. Rather than going into detail, we merely mention that the holomorphic on $\mathbb{D}$ functions with

$$\|f\|_\alpha^2 = \sum_0^\infty (n+1)^\alpha |a_n|^2 < \infty, \quad \alpha \in \mathbb{R},$$

form a class of examples of such Banach spaces (denoted by $\mathcal{D}_\alpha$) and turn our attention to a few simple results.

(Notice that we obtain Brett's Besov spaces by setting $\sigma = (1-\alpha)/2$ and restricting $\alpha \in [0,1]$. For the choices $\alpha = -1, 0, 1$ we obtain the Bergman, Hardy and Dirichlet space, correspondingly. Finally notice that $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$ for $\alpha > \beta$.)

Proposition 2.1 (see [2], p. 273)**.** *Let E be a Banach space of analytic functions on a bounded region in the complex plane G.*

1) *The constant function* **1** *is cyclic for every space E.*
2) *If $f \in E$ is cyclic, then $f(z) \neq 0$ for all $z \in G$.*

For the scale of spaces $\mathcal{D}_\alpha$, we have many more results which vary greatly depending on the value of α. Roughly speaking, the larger α, the easier is the situation.

Proposition 2.2 (see e.g. [2], p. 274)**.** *Consider $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$.*

1) *For $\alpha > 1$, a function f is cyclic (in $\mathcal{D}_\alpha$) if and only if it has no zeros in the closed unit disc, or equivalently, $|f(z)| > c > 0$ on $\mathbb{D}$.*
2) *For $\alpha = 1$ and $\alpha \leq 0$, the condition $|f(z)| > c > 0$ on $\mathbb{D}$ is sufficient for a function f to be cyclic.*
3) *For $\alpha < 0$, some singular inner functions become cyclic.*
4) *For $\alpha = -1$, various sufficient conditions for cyclicity are known.*

In what follows, we restrict ourselves to the Dirichlet space $\mathcal{D} = \mathcal{D}_1$. Although we know that $|f(z)| > c > 0$ on $\mathbb{D}$ implies cyclicity, we have not identified many cyclic functions. For example, we do no know, whether the embarrassingly simple function $f(z) = 1 - z$ is cyclic or not. This question and related ones can be addressed by the means of the dual pairing of the Dirichlet with the Bergman space. In [2] the following results were obtained.

Proposition 2.3 (see [2], p. 289)**.**

1) *The function $z - a$ is cyclic in $\mathcal{D}$ if and only if $|a| \geq 1$.*
2) *If p is a polynomial with no zeros in $\mathbb{D}$, then p is cyclic in $\mathcal{D}$.*
3) *If f is analytic on $\mathbb{D} \cup \mathbb{T}$ and has no zeros in $\mathbb{D}$, then f is cyclic in $\mathcal{D}$.*

2.2. **Cyclic vectors are outer.** It is well known that Beurling's Theorem gives a complete characterization of all cyclic vectors on the Hardy space H^2.

Theorem 2.4 (Beurling, 1952)**.** *A closed subspace X of H^2 is invariant under the forward shift ($S : f \mapsto zf$) if and only if $X = \vartheta H^2$, where ϑ is an inner function.*

In other words, a function $f \in H^2$ is cyclic (on H^2) if and only if it is outer.

Remark. Let us view this condition in the light of the latter subsection. A function $f \in H^2$ is non-zero in $\mathbb{D}$ if and only if its Blaschke factor is trivial (equal to $\mathbf{1}$). For the cyclicity of an H^2-function we must also assume the absence of the singular inner factor.

The relation of the Hardy and the Dirichlet space implies that one direction of the latter Theorem still holds true, if we replace H^2 by $\mathcal{D}$.

Theorem 2.5 (see [2], p. 275)**.** *If $f \in \mathcal{D}$ is cyclic, then it is outer.*

Proof. Recall that $\mathcal{D} \subset H^2$. Indeed, the Hardy space norm is given by $\|f\|^2_{H^2} = \sum_{n=0}^{\infty} |a_n|^2$ (where the a_n are the Fourier coefficients with respect to the standard exponential basis), while norm on Dirichlet space is $\|f\|_{\mathcal{D}} = \sum_{n=0}^{\infty}(n+1)|a_n|^2$.

Suppose that $f \in \mathcal{D}$ is cyclic (on $\mathcal{D}$). Then there is a sequence of polynomials $\{p_n\}$ such that $p_n f \to 1$ in $\mathcal{D}$, which implies the convergence also in H^2-norm. Hence f is cyclic on H^2 and it remains to apply Beurling's Theorem 2.4. $\square$

Remarks. a) Within the latter proof we have also shown that a function which is cyclic for $\mathcal{D}$ is cyclic for H^2.
b) The same reasoning also shows that this result is true for $f \in \mathcal{D}_\alpha$ for $\alpha \in [0, 1]$.

It is worth mentioning that in $\mathcal{D}$ (unlike in H^2), the converse of the latter theorem is not true: There are outer non-cyclic functions in $\mathcal{D}$ (Richter–Sundberg, 1994).

2.3. **Cyclic vectors and capacity.** In 1939, Beurling proved that for $f \in \mathcal{D}$, the radial boundary limit $\lim_{r\to 1^-} f(re^{it})$ exists outside a set of logarithmic capacity zero. (For some information about logarithmic capacity, please refer to the appendix.)

Over 30 years later, in 1972, Carleson proved that this result is sharp and that for any closed set $F \subset \mathbb{T}$ of capacity zero, there is a function in Dirichlet space for which the radial boundary limits do not exist anywhere on F.

For $f \in \mathcal{D}$, let $Z(f)$ denote the subset of $\mathbb{T}$ where the radial boundary limits exist and vanish, i.e.

$$Z(f) = \{e^{it} : \lim_{r\to 1^-} f(re^{it}) = 0\}.$$

Theorem 2.6 (see [2], p. 293)**.** *If $f \in \mathcal{D}$ is cyclic, then $Z(f)$ has capacity zero.*

Brown and Shields raised the following question, which still remains unsolved. Is a vector $f \in \mathcal{D}$ cyclic if and only if it is outer and has capacity zero? In the next section we explain the current status on this problem.

3. Post-history and other related questions

Probably motivated by the results of Brown and Shields, mathematicians provided manyresults concerning the cyclicity of vectors in $\mathcal{D}$. Often simple conditions sufficient for cyclicity were found.

For example, if $|f(z)| \geq |g(z)|$, $z \in \mathcal{D}$, for two functions $f, g \in \mathcal{D}$, then their cyclic subspaces obey $[g] \subset [f]$. In particular, in this case, the cyclicity of g implies that of f (the latter statement also follows from part 2) of Proposition 2.2).

Another statement says that if both f and $1/f$ belong to the Dirichlet space, then f is cyclic.

3.1. **Examples of cyclic functions.** Related to the question raised at the end of the latter section, Brown and Cohn proved the following results.

Theorem 3.1 (see [1]). *Let $E \subset \mathbb{T}$ be a closed set with capacity zero. Then there exists a cyclic $f \in \mathcal{D}$ which is continuous on $\mathbb{D} \cup \mathbb{T}$ and with $Z(f) = E$. In particular, there exist cyclic vectors for which $Z(f)$ is uncountable.*

In the proof Brown and Cohn constructed explicit examples of such functions. It is based on a modification of an argument used by Carleson in 1952 who was concerned with uniqueness sets of Dirichlet functions. He proved that for any closed set E of capacity zero, there exists a non-zero function with $E \subset Z(f)$. The functions constructed by Carleson are not continuous on $\mathbb{D} \cup \mathbb{T}$.

Remark. Brown and Cohn mention that their proof can be adapted (using a suitable Bessel capacity) to find analog cyclic vectors for the spaces $\mathcal{D}_\alpha$ for $\alpha \in (0, 1)$.

3.2. **Cyclicity and shift invariant subspaces.** The following theorem implies that a complete identification of the cyclic vectors of $\mathcal{D}$ translates to significant progress toward a Beurling-type characterization of the shift-invariant subspaces of $\mathcal{D}$ (and vice versa).

Theorem 3.2 (Richter–Sundberg, 1992). *Closed shift-invariant subspaces of $\mathcal{D}$ are of the form $[f]_{\mathcal{D}} \cap \theta H^2$, where f is an outer function in $\mathcal{D}$ and θ is an inner function (not necessarily in $\mathcal{D}$).*

3.3. **New sufficient condition.** A weaker version of the Brown–Shields conjecture ('A vector $f \in \mathcal{D}$ cyclic if and only if it is outer and has capacity zero.') was proved.

Theorem 3.3 (Hedenmalm–Shields, 1990; improved by Richter–Sundberg, 1994). *A vector $f \in \mathcal{D}$ is cyclic (in $\mathcal{D}$), if it is outer and if the set $\{e^{it} : \liminf_{r \to 1^-} f(re^{it}) = 0\}$ is countable.*

In fact, in 2006 El-Fallah–Kellay–Ransford replaced the condition 'countable' by one which is closer to 'capacity zero'. The proof is based on the notion of the so-called Bergman–Smirnov exceptional set which was introduced by Hedenmalm and Shields.

3.4. **Other related questions.** Let us mention three fields related to the problem at hand (cyclicity).

Multipliers. For a Banach space E of analytic functions on a bounded region G in the complex plane, consider those complex valued functions φ in G which satisfy $\varphi E \subset E$. We call those functions multipliers of E. The set of multipliers is denoted by $M(E)$. The theory of multipliers for the spaces $\mathcal{D}_\alpha$ is well studied, though not completely understood. Especially the multipliers of D_α for $\alpha \in (0, 1]$ is complicated. There are several results concerning the cyclicity of the products of certain functions, and other results which are based on this theory.

Sets of uniqueness. The theory of sets of uniqueness for classes X of analytic functions asks the following question: Given $E \subset \mathbb{T}$ assume that the radial limit of $f \in X$ equals zero quasi-everywhere (up to a set of capacity zero) on E. Can we classify those E for which f is the zero function in X? In fact, Malliavin found such a classification for the Dirichlet space in terms of very complicated conditions.

It is worth mentioning that the proof of existence of outer non-cyclic vectors in $\mathcal{D}$ was accomplished using the result of Malliavin. Indeed, in their 1994 paper Richter and Sundberg find a set with positive capacity E for which is not a uniqueness set. In particular, there exists non-zero functions $f \in \mathcal{D}$ with $Z(f) = E$. Dividing by the inner factor now yields an outer function $\tilde{f}$ for which $Z(\tilde{f})$ has positive capacity.

Cyclicity with respect to other operators. As we had mentioned, cyclicity here refers to being cyclic with respect to the forward shift operator $S : f \mapsto zf$. But there is also rich literature concerning the cyclicity of vectors with respect to other operators. For example, the backward shift has enjoyed much attention, see e.g. [4] and the references within. The cyclic vectors of rank one perturbations and Anderson-type Hamiltonians are being studied. In this case, cyclicity is connected to the famous problem of Anderson localization.

Appendix: Logarithmic Capacity

Let E be a bounded Borel set. Consider the class Γ_E of positive measures μ with $\operatorname{supp}\mu \subset E$ and logarithmic potential $U_\mu(z) := \int \log^+(4/|z-w|)d\mu(w) \leq 1$. Then the (logarithmic) capacity of E is given by $\sup_{\mu\in\Gamma_E} \mu(E)$.

If E is compact, then the supremum is attained by a unique measure.

Remark. While literature provides many different notions of capacity (adapted to the specific problem), the sets of zero capacity are the same for the most common notions.

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ON A SHARP INEQUALITY CONCERNING THE DIRICHLET INTEGRAL

S.-Y. A. CHANG AND D. MARSHALL

presented by James Scurry

ABSTRACT. Attention is given to the main result from [2], a conformally invariant form of the Chang-Marshall inequality. Context is provided for this theorem and, in particular, the results of [2] are viewed within the framework of real analysis. Emphasis is placed on the influence of Trudinger and Moser, as well as A_p theory, on Chang and Marshall's work. Finally, the proof of Chang and Marshall's inequality presented in [2] is summarized.

Let $\mathbb{D}$ denote the unit disk and $\mathcal{D}$ be the collection of analytic functions f defined on $\mathbb{D}$ which satisfy

$$D(f) \;=\; \left(\int_{\mathbb{D}} |f'(z)|^2 dA(z)\right)^{\frac{1}{2}} < \infty,$$

where $dA(z)$ is normalized area measure. We will refer to $\mathcal{D}$ as the Dirichlet space and to $D(f)$ as the Dirichlet integral of f for $f \in \mathcal{D}$; further, denote by $\mathbb{B}_{\mathcal{D}}$ the unit ball of $\mathcal{D}$, i.e. the collection of all $f \in \mathcal{D}$ such that $D(f) \leq 1$. The focus of Chang and Marshall in [2] is on the exponential integrability of functions from $\mathcal{D}$. Stated in a conformally invariant form, their main result is the following:

Theorem 0.1. *There exists a constant C such that if f is in the unit ball of $\mathcal{D}$ and satisfies $f(0) = 0$, then*

$$\sup_{z\in\mathbb{D}} \int_0^{2\pi} e^{\alpha|f(e^{\imath\theta})-f(z)|^2} P_z(\theta)\frac{d\theta}{2\pi} \;\leq\; C \tag{0.2}$$

where P_z is the Poisson kernel for $z \in \mathbb{D}$ and $0 \leq \alpha \leq 1$. Moreover, if $\alpha > 1$, then the left-hand side of 0.2 can be made arbitrarily large using the functions

$$B_a(z) \;=\; \log\frac{1}{1-az}\left(\log\frac{1}{1-a^2}\right)^{\frac{-1}{2}}.$$

Chang and Marshall are heavily motivated by work in real analysis and their results in [2] exist as analogues of several theorems from this field. In particular, there are two trains of thought which provide context to Theorem 0.1: one is rooted in the theory of A_p weights while the other is based on [8] and [6]. Moreover, prior to [2] each of these perspectives suggested a different conclusion about the validity of 0.2 in the case $\alpha = 1$.

We consider Theorem 0.1 first from within the environment of A_p theory. Recall that a function w on $\mathbb{T}$ is a weight if $0 < w$ and w is locally integrable. Further, w is an A_p weight for $1 < p < \infty$ if

$$\sup_I \left(\int_I w^p(\theta)d\theta\right)\left(\int_I w^{\frac{-1}{p-1}}(\theta)d\theta\right)^{\frac{1}{p-1}},$$

where I is used to denote an arc of $\mathbb{T}$. For $p = 2$, the Helson-Szegö Theorem characterizes the weights w on $\mathbb{T}$ for which conjugation is bounded as an operator; specifically we have

Theorem 0.3. *Conjugation is bounded as an operator from $L^2(w)$ to $L^2(w)$ if and only if $\log w = u + \tilde{v}$ and $u, v \in L^\infty$ such that $\|v\|_\infty < \frac{\pi}{2}$, where $\tilde{v}$ is the conjugate of v.*

A theorem by Muckenhoupt, Hunt, and Wheeden gives a different characterization of the weights w for which conjugation acts as a bounded operator:

Theorem 0.4. *If w is a weight, then conjugation is bounded as an operator from $L^2(w)$ to $L^2(w)$ if and only if w is an A_2 weight.*

Using Jensen's inequality, Theorems 0.3 and 0.4 can be combined to show

Theorem 0.5. *The following are equivalent for a real-valued function $f \in L^1(\mathbb{T})$:*

a. $\sup_{z\in\mathbb{D}} \int_0^{2\pi} e^{|f(e^{\imath\theta})-f(z)|} P_z(\theta)\frac{d\theta}{2\pi} < \infty$

b. $f = u + \tilde{v}$ *for some* $u, v \in L^\infty$ *with* $\|v\|_\infty < \frac{\pi}{2}$

and where $\tilde{v}$ is the conjugate of v.

Extrapolating from Theorem 0.5 leads to the conclusion 0.2 holds for $0 < \alpha < 1$ and fails for $\alpha \geq 1$. In particular, the estimate

$$m_z(\{\theta : |f(e^{\imath\theta}) - f(z)| > \lambda\}) \lesssim e^{-\lambda},$$

where m_z is the measure $P_z(\theta)\frac{d\theta}{2\pi}$, suffices to give a from Theorem 0.5 and in fact

$$\sup_{z\in\mathbb{D}} \int_0^{2\pi} e^{\alpha|f(e^{\imath\theta})-f(z)|} P_z(\theta)\frac{d\theta}{2\pi} < \infty$$

provided $0 \leq \alpha \leq 1$. The parallel estimate for Chang and Marshall's inequality would be

$$m_z(\{\theta : |f(e^{\imath\theta}) - f(z)| > \lambda\}) \lesssim e^{-\lambda^2}$$

which is not strong enough to guarantee the integral from 0.2 is finite. Thus, consideration of Theorem 0.1 within the framework of weighted theory suggests 0.2 fails for $\alpha = 1$.

Now we examine an alternative motivation of Chang and Marshall, the work of Trudinger and Moser. The relevant portion of Trudinger's work in [8] deals with the 'limiting case' of Sobolev-type inequalities. Assume $2 \leq n$ and recall that if E is a bounded open subset of $\mathbb{R}^n$ with a suitable boundary, $\mathring{W}^1_q(E)$ consists of the closure of all compactly supported $C^1(E)$ functions with respect to the norm

$$\|u\|_{\mathring{W}^1_q(E)} = \left(\int_E |\nabla u|^q dx\right)^{\frac{1}{q}},$$

The space $\mathring{W}^1_q(E)$ can be understood in the cases $1 < q < n$ and $q > n$ through Sobolev-type inequalities. Namely, if $1 < q < n$ and $u \in \mathring{W}^1_q(E)$ we have $\|u\|_p \lesssim \|u\|_{\mathring{W}^1_q(E)}$ where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$; and provided $q > n$, then $u \in C^{0,\gamma}(E)$ with $\gamma = 1 - \frac{n}{q}$ and $\|u\|_{C^{0,\gamma}(E)} \lesssim \|u\|_{\mathring{W}^1_q(E)}$. In [8], Trudginer considered the case omitted by the aforementioned inequalities, i.e. $q = n$, and was able to establish the following

Theorem 0.6. *If $u \in \mathring{W}_q^1$ such that $\|u\|_{\mathring{W}_q^1(E)}$ then*

$$\int_E e^{u^p \alpha} dx \;\; \lesssim \;\; |E|$$

with $p = \frac{n}{n-1}$ and the implied constant independent of u.

Subsequent research by Moser yielded a simplified proof of Theorem 0.6 and found the best constant α. In particular, according to [6], we have 0.7 holds uniformly for u in the unit ball of $\mathring{W}_q^1(E)$ provided $\alpha \leq \omega_{n-1}^{\frac{1}{n}-1} n$, where ω_{n-1} is the surface area of the $(n-1)$-dimensional sphere; and if $\alpha > \omega_{n-1}^{\frac{1}{n}-1}$ then the left-hand side of 0.7 can grow arbitrarily large for u in the unit ball of $\mathring{W}_q^1(E)$. Further, [6] first articulated the problem examined by Chang and Marshall in [2]; and, viewing Theorem 0.1 as a counterpart to Moser's work suggests 0.2 holds for $\alpha = 1$.

Ultimately, Chang and Marshall validated the Trudinger/Moser train of thought (which suggested 0.2 held for $\alpha = 1$) by proving Theorem 0.1 (that this is the proper viewpoint of 0.1 is further evidenced by Marshall's later paper [5] which provides a shorter proof of 0.1 by reducing 0.1 to a special case of Moser's work in [6]). The proof of Theorem 0.1 presented in [2] is subtle, relying heavily on a a distributional inequality from A. Beurling's doctoral thesis, i.e.

Theorem 0.7 (Beurling)**.** *If $f \in \mathcal{D}$ such that $f(0) = 0$, then*

$$|E_M| \;\; \leq \;\; e^{-M^2+1}.$$

and three technical lemmas:

Lemma 0.8. *There exist constants c_0 and a_0 so that if $a_0 \leq a \leq 1$ and if $M \geq 1$ then*

$$|\{\theta \in \mathbb{T} : |B_a(e^{\imath\theta})| > M\}| \;\; \leq \;\; c_0 e^{-M\sqrt{N_a}}$$

with $N_a = \log \frac{1}{1-a^2}$.

Lemma 0.9. *If $B > A \geq 0$ then*

$$\int_A^B e^{(M-A)(M-B)} M dM \;\; \leq \;\; \frac{2(A+B)}{B-A}.$$

Lemma 0.10. *If $1 - r^2 = \frac{E_M}{e}$ then*

$$M \;\; \leq \;\; 2\left(\sum_{n=1}^{\infty} n|a_n|^2(1-r^{2n})\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} |a_n| r^n.$$

Using Lemmas 0.8-0.10 in concert with Theorem 0.7, the proof proceeds by cases, beginning with the simplest. That is, the authors first demonstrate the theorem is true for Beurling functions B_a via Lemma 0.8 and an estimate on the supremum norm of these functions. Then a function $f \in \mathbb{B}_{\mathcal{D}}$ satisfying $f(0) = 0$ is fixed and the remainder of the proof compares f to a particular Beurling function B_a, where B_a depends on f. The authors quantify the function f's proximity to the Beurling functions by considering a number δ defined by

$$\delta \;\; = \;\; 1 - \sup_{0 \leq r \leq 1} \left(\sum_{n=1}^{\infty} a_n r^n\right)^2 \left(\log \frac{1}{1-r^2}\right)^{-1}. \tag{0.11}$$

From continuity, there is a number a so that the supremum in (0.11) is attained, and it is to the function B_a that f is compared; further, an easy computation reveals $\|f - B_a\|_{\mathcal{D}} \leq (2\delta)^{\frac{1}{2}}$.

Provided δ is at least some fixed positive number δ_0, then Theorem 0.1 can be shown using a simple application of Beurling's estimate. The balance of the paper is therefore devoted to showing the case $\delta \leq \delta_0$. This case bifurcates into two sub-cases depending on the size of a. In particular, if a is small enough, less than an appropriately chosen constant a_0, the proof is straightforward and only the case when a is larger than a_0 must be considered. To this end, the authors assume $a_0 \leq a$ and break up the integral $\displaystyle\int_1^\infty |E_M| e^{M^2} M dM$ into several pieces:

$$\int_1^\infty |E_M| e^{M^2} M dM = \int_1^{\sqrt{N}/2} + \int_{\sqrt{N}/2}^{(1-c\delta)\sqrt{N}} + \int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} + \int_{(1+c\delta)\sqrt{N}}^{3\sqrt{N}} + \int_{3\sqrt{N}}^{\infty}.$$

The authors handle the first integral by approximating f with a Beurling function and using Beurling's estimate in concert with Lemma 0.8. For the remaining integrals, a useful bound on M is obtained via Lemma 0.10:

$$\begin{aligned} M \leq & 2(2\delta)^{\frac{1}{2}} + 2\left(\log \frac{1-a^2r^2}{1-a^2} N^{-1}\right)^{\frac{1}{2}} \\ & \qquad + N^{\frac{1}{2}} \log \frac{1-a^2}{1-ra} N^{\frac{-1}{2}} + (2\delta)^{\frac{1}{2}} \left(\log \frac{(1-ra)^2}{(1-r^2)(1-a^2)}\right)^{\frac{1}{2}}. \end{aligned}$$

By the right-continuity of the distribution function (and the assumption that various constants have been chosen appropriately) a constant M_0 such that $|E_{M_0}| e^{-1} = (1-a^2)^{(1-c\delta)^2}$ can be found and elementary methods in concert with 0.7 gives the following

$$\begin{aligned} \int_{\sqrt{N}/2}^{M_0} |E_M| e^{M^2} M dM &\lesssim \int_{\sqrt{N}/2}^{\sqrt{N}} e^{(M-\sqrt{N}/3)(M-\sqrt{N})} dM \lesssim 1 \\ \int_{M_0}^{(1-c\delta)\sqrt{N}} |E_M| e^{M^2} M dM &\lesssim \int_{M_0}^{(1-c\delta)\sqrt{N}} e^{M^2} M dM \qquad \lesssim 1. \end{aligned}$$

The third integral is estimated again by considering cases. From 0.7,

$$\int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} |E_M| e^{M^2} M dM \lesssim 1$$

provided $N\delta$ is small enough, i.e. smaller than an appropriately chosen constant c; and if $N\delta > c$ Lemma 0.10 gives

$$\int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} |E_M| e^{M^2} M dM \lesssim \int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} e^{\left((\frac{-\delta}{1-\delta})(M - c\delta^{\frac{-1}{2}}\right)^2} M dM \lesssim 1.$$

For the fourth integral and fifth integrals, Beurling's Theorem combined with Lemma 0.10 and elementary methods shows

$$\begin{aligned} \int_{(1+c\delta)\sqrt{N}} |E_M| e^{M^2} M dM &\lesssim \int_{\sqrt{N}}^{3\sqrt{N}} e^{M^2 - 4M\sqrt{N} + 3N} M dM \lesssim 1 \\ \int_{3\sqrt{N}}^{\infty} |E_M| e^{M^2} M dM &\lesssim e^{-3(M-2\sqrt{N})^2} (M - 2\sqrt{N}) dM \lesssim 1. \end{aligned}$$

Combining each of the five estimates from above concludes the proof of Theorem 0.1 presented in [2].

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INTERPOLATING SEQUENCES FOR THE MULTIPLIERS OF THE DIRICHLET SPACE

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presented by Daniel Seco

Abstract. We will study the solution to the problem of interpolation on the Dirichlet space, following mainly the article by D. Marshall and C. Sundberg, but relating it with advances on the Hardy space (by Carleson, Shapiro and Shields) and with other results and proofs about the Dirichlet space (Boe and others). If there is time we will go through the research in progress done by our team about 2 problems on sampling and interpolation on the Dirichlet space.

1. On interpolating sequences on the Dirichlet space

1.1. **Introduction.** A classical problem of Complex Analysis is that of finding a function, living on a fixed space, assigning prescribed values to the elements of a sequence of points. Sequences for which this can be done with norm control are often called interpolating sequences (or sequences of interpolation).

More concretely, suppose that $\mathbb{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$ and l is a space of sequences. We say that $\{z_n\} \subset \mathbb{D}$ is an *interpolating sequence from* $\mathbb{H}$ *to* l if the restriction operator $R : \mathbb{H} \to l$ defined by $R(f) = \{f(z_n)\}$, is onto and bounded. That is, if, given $\{w_n\} \in l$ we can find $f \in \mathbb{H}$ such that $f(z_n) = w_n \forall n$.

The main result in this terms is due to L. Carleson ([8]), who proved two characterizations of interpolating sequences from $\mathbb{H}^\infty$ to l^∞:

Theorem 1 (Carleson, '58). *Let $\{z_n\} \subset \mathbb{D}$. The following are equivalent:*

- **(a):** $\{z_n\}$ *is an interpolating sequence from* $\mathbb{H}^\infty$ *to* l^∞
- **(b):** $\{z_n\}$ *is separated and* $\sum(1-|z_n|)\delta_{z_n}$ *is a Carleson measure for* $\mathbb{H}^\infty$
- **(c):** $\{z_n\}$ *is strongly separated (some property in terms of Blaschke products)*

This characterization happened to be also true for $\mathbb{H}^2$. To see this, we define a sequence of functions $\{u_n\}$ to be a *Riesz sequence for* $\mathbb{H}$ if $||\sum a_n u_n||^2_{\mathbb{H}} \approx \sum |a_n|^2 \forall \{a_n\}$. The following was another major step on the understanding of interpolation, allowing to solve the problem on the Hardy space. See [15].

Theorem 2 (Shapiro - Shields, '61). *Let $\{z_n\} \subset \mathbb{D}$. The following are equivalent:*

- **(a):** $\{z_n\}$ *is an interpolating sequence from* $\mathbb{H}^\infty$ *to* l^∞
- **(b):** $\{z_n\}$ *is an interpolating sequence from* $\mathbb{H}^2$ *to* l^2
- **(c):** $\{\widetilde{K_{z_n}}\}$ *is a Riesz sequence for* $\mathbb{H}^2$ *where* $\widetilde{K_{z_n}}(z) = \frac{\frac{1}{1-\overline{z_n}z}}{(\frac{1}{1-|z|^2})^{1/2}}$ *is the normalized reproducing kernel of the Hardy space at the point* z_n

Our purpose is to obtain similar results to these ones for the Dirichlet space. It actually took 33 years from Carleson's result until someone proved a useful characterization of interpolating sequences for the Dirichlet space $\boldsymbol{D}$. Why is this? One can think of $\boldsymbol{D}$ as a vector subspace of $\mathbb{H}^2$ and therefore, guaranteeing existence of a function with given values will be more difficult than in the Hardy space, but some analogy between the two spaces is useful: the space $\mathbb{H}^\infty$ is, in fact, the algebra of multipliers of $\mathbb{H}^2$, that is, the algebra of all functions ϕ such that $\forall f \in \mathbb{H}^2 \phi f \in \mathbb{H}^2$. It is hence, natural to look for a characterization in terms of the multiplier algebra of $\boldsymbol{D}$, which we denote $\boldsymbol{M}$.

Also finding the analogue of Carleson measures will prove useful. Consider a measure μ on the unit disc. We say that μ is *Carleson for* $\boldsymbol{D}$ if the embedding of $\boldsymbol{D}$ into $L^2(\mu)$ is continuous, or equivalently, if for all functions of the space we have:

$$\int_{\mathbb{D}} |f|^2 d\mu \leq C||f||^2_{\boldsymbol{D}}$$

It is not difficult to see that a function $\phi \in \boldsymbol{D}$ is a multiplier (is in $\boldsymbol{M}$) if and only if $|\nabla\phi|^2 dA(z)$ is a Carleson measure and ϕ is a bounded analytic function. Also, fortunately enough, Stegenga provided a characterization of Carleson measures for $\boldsymbol{D}$. See [16].

Theorem 3 (Stegenga, '80)**.** *μ is a Carleson measure for* $\mathbf{D}$ *if and only if*

$$\mu(\bigcup_i S(I_i)) \leq CCap(\bigcup_i I_i)$$

where Cap is the logarithmic capacity and $S(I)$ denotes the Carleson square of the interval I.

Observe that this is a much more complicated situation than in the Hardy space, where the characterization refers to only one interval. Other characterizations have been found more recently by Arcozzi, Rochberg and Sawyer ('02, [2] and '07, [3]).

It is, in general, common to find analogies between logarithmic versiones of the properties of the Hardy space and the properties of the Dirichlet space. This is the case of the reproducing kernel, which is given by the formula $K_{z_n}(z) = \log \frac{1}{1-\overline{z_n}z}$.

In the same line, the natural concept to use when looking for separation conditions is not pseudohyperbolic but rather hyperbolic distance. Notice that $||K_z||^2_{\boldsymbol{D}} = K_z(z) \approx \beta(z, 0)$.

Before we can state the theorem we need to define two more concepts. We say that a sequence of unitary functions $\{u_n\}$ in $\mathbb{H}$ (for us, normalized reproducing kernels on the points $\{z_n\}$) form an *unconditional basic sequence (UBS)* for $\mathbb{H}$ if for a fixed $C < \infty$ we have:

$$||\sum b_n u_n||_{\mathbb{H}} \leq C||\sum a_n u_n||_{\mathbb{H}} \forall |b_n| \leq |a_n|$$

If $\{\widetilde{K_{z_n}}\}$ is a UBS for $\mathbb{H}$, we say that there is *free interpolation* for $\{z_n\}$.

1.2. **Main Theorem.** We are ready to state now the main theorem. We've chosen the form of [6] although we will follow mainly the article [11].

Theorem 4 (Marshall - Sundberg, '91)**.** *The following are equivalent:*

- **(1):** $\{z_n\}$ *is an interpolating sequence from* $\mathbf{M}$ *to* l^∞
- **(2):** $\{z_n\}$ *is a sequence of free interpolation*
- **(3):** $\{\widetilde{K_{z_n}}\}$ *is a Riesz sequence*
- **(4):** *The normalized restriction from* $\mathbf{D}$ *to* l^2, $\widetilde{R} : \widetilde{R}(f) = \{\frac{f(z_n)}{||K_{z_n}||_{\mathbf{D}}}\}$ *is bounded and onto.*

(5): *a) Separation: There exists a* $C > 0$ *with:*

$$\frac{\beta(z_n, z_m)}{\beta(z_n, 0)} \geq C$$

b) Carleson: The measure

$$\sum \frac{\delta_{z_n}}{||K_{z_n}||^2_{\mathbf{D}}}$$

is a Carleson measure for $\mathbf{D}$.

1.3. **Comments.**

Remark 1. **(a):** *The boundedness of the restriction operator is not automatic in* $\mathbf{D}$ *as opposed to the Hardy space.*

(b): *The result has a long story. Four different proofs are known (Marshall and Sundberg, '91,* [11]*; Bishop, '94,* [5]*; Boe, '02,* [6]*; and Boe, '05,* [7]*).*

(c): *Also, K. Seip ('04,* [14]*), provided a proof for RKHS of* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5)*, thus it is understood that* (5) $\Rightarrow$ (1) *is the most intrincate part.*

Following Peter Jones's solution to the $\overline{\partial}$ - equation, which can be used to construct an explicit proof of the Carleson Interpolation Theorem (CIT), Xiao published a solution for the $\overline{\partial}$ - equation for multipliers of Sobolev spaces which include boundary values of the Dirichlet space functions. See [10], [18]. This has encouraged people to look at the still open problem of constructing the interpolating functions explicitly.

There are also a branch of different open problems in relation to other definitions of interpolation. We refer to the recent survey [4].

1.4. **Necessity of (5).** We will check that all of the four first statements are equivalent, and that they imply the fifth.

We start with the two parts that are true for general RKHS.

To see (1) implies (2), we choose a $\phi : \overline{\phi(z_n} a_n = b_n$. Using the fact that the reproducing kernel is an eigenvector of the adjoint of the multiplication operator $(M^*_\phi(\overline{z_n})K_{z_n})$ we have:

$$||\sum b_n K_{z_n}||_D = ||M^*_\phi(\sum a_n K_{z_n})||_D \leq ||M_\phi|| \cdot ||\sum a_n K_{z_n}||_D \leq$$
$$\leq C \cdot ||\phi(z_n)||_{l^\infty} \cdot ||\sum a_n K_{z_n}||_D$$

Then we have a result which is the basis of the proof of our theorem (see, for instance, [12]):

Theorem 5 (Koethe - Toeplitz)**.** *For any RKHS* $\mathbb{H}$*, we have* (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4)*.*

We will go through most of the implications on the proof.

The property that (1) be equivalent to all the others is true also on a very general environment of RKHS with the Pick property:

Definition 1. *We say that* H *has the* Pick property *if always that for a finite family of points* $\{z_i\}_{i=1}^n \subset \mathbb{D}$ *and any set of complex values* $\{w_i\}_{i=1}^n$ *one has that* $\{(1-\overline{w_i}w_j)K_{z_i}(z_j)\} \geq 0$*, then there exists a multiplier* $\phi : ||\phi||_M \leq 1$ *which interpolates the values* $\phi(z_i) = w_i, \quad i = 1, ..., n$.

If $\mathbb{H}$ has the Pick property, many of the features of the space correspond with similar behavior on the multiplier algebra. One of these properties will allow us to see that (4) $\Rightarrow$ (1).

The reason why we can use this is in [1]:

Theorem 6 (Agler, '86)**. D** *has the Pick property.*

Now we turn to (4) $\Rightarrow$ (5). The Carleson condition is a direct change of wording on the boundedness of the map. For the separation, we interpolate the $\delta_{n,i}$ on z_i. We need a hyperbolic Lipschitz-type property of Dirichlet functions due to Zhu ([19]):

Lemma 1. *For* $f \in \mathbf{D}$*:*

$$|f(z) - f(w)| \leq C||f||_D \beta(z,w)^{1/2}$$

This statement together with the OMT can be used to get:

$$||f||_D \leq C\frac{|f(z_n)|}{||K_{z_n}||_D} \leq C||f||_D (\frac{\beta(z_n,z_m)}{\beta(z_n,0)})^{1/2}$$

1.5. **Sufficiency of (5).** The first reduction of the problem is to study a real harmonic relaxation of the problem. The objective is to prove that the normalized reproducing kernels of the real harmonic version of the Dirichlet space form an UBS for $Re(D_h)$. Applying the KTT we will deduce that they are indeed a Riesz sequence.

Therefore, the second step is to prove the following:

Proposition 1. *There exists* $C < \infty$ *such that if* $u \in Re(D_h), \{t_n\} \subset \mathbb{R}, |t_n| \leq 1 \forall n$ *then there exists a function* $v \in D_h : v(z_n) = t_n U(z_n) \forall n : ||v||_D \leq C$

The proof of the claim consists on actually building functions ϕ_n which approximate the function δ_{z_n} with an error that will be overall controllable thanks to the Carleson condition.

To define $\phi_n(z)$, we design a region V_n associated with the point z_n in such a way that not many of the different regions are intersecting at a particular point (this is possible thanks to the separation condition). Moreover, the regions will be some logarithmic version of the top halves of Carleson squares as in CIT.

Each region V_n will classify the disc on 3 areas: an inner part, where $\phi_n(z) = 1$, a middle one where ϕ varies linearly from the interior to the exterior and an outer area where $\phi_n(z) = 0$. In all cases there will be an admitted error of the order of the hyperbolic distance to the origin.

Next, with an iterative process we will define ϕ as a weighted sum of the previous ones $\phi = \sum a_n \phi_n$, so that it interpolates given bounded values at the sequence points, but also permitting $|a_n| \leq 3$ (allowing that there exist weakly convergent subsequences of functions) and $|| \sum_{j=0}^{\infty} a_j \phi_j ||_\infty \leq 2$ so that $\phi \in \mathbb{H}^\infty$.

A delicate step is then to show that $|\nabla \phi|^2 dA$ is a Carleson measure for $\boldsymbol{D}$, assuring that ϕ is a multiplier.

Finally by an argument based on Poisson kernels, one can extend the harmonic function from its boundary values to an analytic function satisfying similar properties. ■

1.6. **Further comments.** If there is some time left, we can study an approach to sampling and interpolation from $\boldsymbol{D}$ to a different kind of spaces (instead of l^2, $l^2(w)$ for some weight w), which we find naturally arising from the boundedness of some operator used by Rochberg and Wu ([13]). Interpolation happens to be, unfortunately, an empty concept but sampling becomes more interesting.

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TANGENTIAL BOUNDARY BEHAVIOR OF FUNCTIONS IN DIRICHLET-TYPE SPACES

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presented by Eyvindur Ari Palsson

Abstract. We describe some results on the boundary behavior of harmonic functions in classes modeled on the classical Dirichlet space. The main result is the strong boundedness of maximal function operators in certain approach regions.

1. Introduction

Consider a bounded holomorphic function on the open unit disc $\mathbb{D}$ in $\mathbb{C}$. It is of interest to understand the behavior of such functions near the boundary $\mathbb{T}$ of the unit disc. In 1906 Fatou showed [1] that any such function has a non-tangential limit at a.e. $e^{i\theta} \in \mathbb{T}$. Littlewood proved in 1927 [3, 4] that Fatou's result was sharp. More precisely, then Littlewood showed that for any curve γ in $\mathbb{D}$, that approaches the point 1 tangentially, then there exists a bounded holomorphic function f in $\mathbb{D}$ such that the limit of f along the rotation γ_θ does not exist for a.e. $e^{i\theta} \in \mathbb{T}$. Nagel, Rudin and Shapiro consider a smaller class of functions, that is, harmonic functions that belong to the analogues of the classical Dirichlet space $\mathcal{D}(\mathbb{D})$, and obtain results for the existence of limits and estimates of the supremum within regions that allow tangential approach to the boundary $\mathbb{T}$.

2. Main Result

Most of the theory is developed in the upper half-space $\mathbb{R}^{n+1}_+$ which is given by

$$\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, y > 0\}.$$

Simple arguments then allow the authors to push their results to the unit disc.

A kernel for us will be a nonnegative L^1-function, which is radial and decreasing. We will use the letter K to denote a kernel. Usually we will normalize the kernel so that $\|K\|_1 = 1$.

The space of all K-potentials associated to the exponent p is denoted by L^p_K and is given by

$$L^p_K = \{K * F : F \in L^p\}$$

The norm

$$\|f\|_{K,p} = \inf\{\|F\|_p : f = K * F\}$$

then makes L^p_K into a Banach space. It is clear that $L^p_K \subseteq L^p$ but one can also see that to every $f \in L^p$ corresponds a kernel K and an $F \in L^p$ such that $f = K * F$.

Since we want to work with harmonic functions then we recall that the definition of the Poisson kernel for $\mathbb{R}^{n+1}_+$ is

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{(n+1)/2}}$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-(n+1)/2}$ is chosen such that $\|P_y\|_1 = 1$ for $0 < y < \infty$. Then

$$K_y(x) = (P_y * K)(x)$$

is the harmonic extension of K to $\mathbb{R}^{n+1}_+$.

We denote the space of all Poisson integrals $P[f]$ of functions $f \in L^p_K$ by h^p_K. Thus, saying that $u \in h^p_K$ means that

$$u(x,y) = P[f](x,y) = (P_y * f)(x) = (K_y * F)(x)$$

for some $f \in L^p_K$, $F \in L^p$, and all $(x,y) \in \mathbb{R}^{n+1}_+$. We norm the space with

$$\|u\|_{K,p} = \|f\|_{K,p}.$$

The spaces h^p_K are the Dirichlet-type spaces from the title.

If K is a kernel, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \beta < \infty$, and $x_0 \in \mathbb{R}^n$, we define the approach region

$$\Omega^p_{K,\beta}(x_0) = \{(x,y) \in \mathbb{R}^{n+1}_+ : |x - x_0|^{n/p}\|K_y\|_q < \beta\}$$

and the associated maximal function

$$\left(\mathcal{M}_{K,p,\beta} f\right)(x_0) = \sup\{|u(x,y)| : (x,y) \in \Omega^p_{K,\beta}\}$$

where $u = P[f]$.

The main result of the paper is the boundedness of the maximal functions as stated in the following theorem.

Theorem 1. *If $p > 1$, there is a constant $A = A(n,K,p,\beta) < \infty$ such that*

$$\|\mathcal{M}_{K,p,\beta} f\|_p \le A\|f\|_{K,p}$$

for every $f \in L^p_K$. A corresponding weak-type inequality holds when $p = 1$.

3. Tangential Convergence

An important preliminary result to the main result is that the maximal function operators $\mathcal{M}_{K,p,\beta}$ are of weak type (p,p).

Theorem 2. *If $p \ge 1$, there is a constant $A = A(n,K,p,\beta) < \infty$ such that*

$$|\{\mathcal{M}_{K,p,\beta} f > \lambda\}| \le A\lambda^{-p}\|f\|^p_{K,p}$$

for all $f \in L^p_K$ and all $\lambda \in (0,\infty)$.

In order to state the tangential convergence result we need the following definition.

Definition 3. *A function u with domain $\mathbb{R}^{n+1}_n$ is said to have Ω^p_K-limit L at a point $x_0 \in \mathbb{R}^n$ if it is true for every $0 < \beta < \infty$ that $u(x,y) \to L$ as $(x,y) \to (x_0, 0)$ within $\Omega^p_{K,\beta}$.*

The weak type result, which is considerably weaker than the main result, is sufficient to prove the tangential convergence result. With a standard argument we get the following theorem.

Theorem 4. *If $1 \leq p < \infty$, $f \in L^p_K$, and $u = P[f]$, then, for almost every $x_0 \in \mathbb{R}^n$, the Ω^p_K-limit of u exists at x_0 and equals $f(x_0)$.*

If for simplicity we look at the set of all $x \in \mathbb{R}^n$ for which $(x, y) \in \Omega^p_{K,1}(0)$ then by definition we see it is an open ball, centered at 0, whose radius is $r(y) = \|K_y\|_q^{-p/n}$. The authors then prove

$$\frac{r(y)}{y} \to \infty \text{ as } y \to 0$$

which geometrically means that $\Omega^p_{K,1}(0)$ admits curves that approach the point $(0, 0)$ tangentially. Furthermore the authors show that the preceding theorem is optimal with regard to the size of the approach regions.

4. Factored kernels

We now consider "factored" kernels $K = H * G$, where both H and G are kernels. Since $\|K\|_q \leq \|G\|_q$ then the approach regions Ω_G are narrower than the Ω_K's (in other words, less tangential). In this case we are able to get a stronger result than just saying that the set, on which a function $u \in h^p_K$ fails to have Ω^p_G-limits, is of measure zero. To state this theorem precisely we need the concept of capacities.

Definition 5. *Suppose K is a kernel, $1 < p < \infty$, and $E \subseteq \mathbb{R}^n$. Let $T(K, p, E)$ be the set of all $F \in L^p$ such that $F \geq 0$ on $\mathbb{R}^n$ and $(K * F)(x) \geq 1$ for every $x \in E$. We then define the (K, p)-capacity of E to be*

$$C_{K,p}(E) = \inf\{\|F\|_p^p : F \in T(K, p, E)\}$$

with the understanding that the inf of the empty set is $+\infty$.

It is easy to see that if $C_{K,p}(E) = 0$ then the Lebesgue measure of E is also 0.

Theorem 6. *Suppose $K = H * G$, $1 < p < \infty$, $f \in L^p_K$, and $u = P[f]$. There is a set $E \subseteq \mathbb{R}^n$, with $C_{H,p}(E) = 0$, such that the Ω^p_G-limit of u exists and equals $f(x)$ at every $x \in \mathbb{R}^n \setminus E$.*

The key ingredient in the proof of this theorem is the main result of the paper on the strong boundedness of the maximal function operators. For comparison then the following theorem is classical (see [2], for example, for the case of the unit circle).

Theorem 7. *Suppose $K = H * G$, $1 < p < \infty$, $f \in L^p_K$, and $u = P[f]$. There is a set $E \subseteq \mathbb{R}^n$, with $C_{H,p}(E) = 0$, such that the non-tangential limit of u exists and equals $f(x)$ at every $x \in \mathbb{R}^n \setminus E$.*

5. Carleson measures

In this section we present a theorem that exhibits a geometric condition which implies that μ is a Carleson measure for h^p_K. We start with a couple of definitions.

Definition 8. *A positive Borel measure μ on $\mathbb{R}^{n+1}_n$ is said to be a Carleson measure for h^p_K if there is a constant $A < \infty$ such that*

$$\int_{\mathbb{R}^{n+1}_n} |u|^p d\mu \leq A\|u\|^p_{K,p} \text{ for every } u \in h^p_K.$$

Definition 9. *For $E \subseteq \mathbb{R}^n$ and K, p, β as before,*

$$Q^p_{K,\beta}(E) = \mathbb{R}^{n+1}_n \setminus \bigcup_{x \notin E} \Omega^p_{K,\beta}$$

Theorem 10. *Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_n$. If $p > 1$, and if for some $\beta > 0$ there is a constant $A < \infty$ such that*

$$\mu\left(Q^p_{K,\beta}(B)\right) \leq A|B|$$

for every open ball $B \subseteq \mathbb{R}^n$, then μ is a Carleson measure for h^p_K.

The authors present an example, due to David Stegenga, that shows that this sufficient condition, is not necessary. The proof of the theorem uses the main result of the paper on the strong boundedness of the maximal function operators.

6. Applications using Bessel kernels

For $0 < \alpha \leq n$, G_α is the function on $\mathbb{R}^n$ whose Fourier transform is

$$\widehat{G_\alpha}(\xi) = (1 + |\xi|^2)^{-\alpha/2}.$$

Each G_α is a positive, radial, decreasing L^1-function. These functions are thus kernels and are called the Bessel kernels. We denote the potential spaces generated by these kernels by $\mathfrak{L}^p_\alpha$, rather than $L^p_{G_\alpha}$. Applying theorem 4 to this concrete case and by carefully analyzing the harmonic extension of G_α we obtain the following theorem.

Theorem 11. *Suppose $1 \leq p < \infty$, $0 < \alpha \leq n$, $f \in \mathfrak{L}^p_\alpha$ and $u = P[f]$. Except possibly on a set of x_0's of measure 0, $u(x, y)$ converges to $f(x_0)$ when $(x, y) \to (x_0, 0)$ within the regions defined by*

$$\begin{array}{ll} y > c|x - x_0|^{n/(n-\alpha p)} & \textit{if } \alpha p < n \\ y > \exp\{-c|x - x_0|^{-n(q-1)}\} & \textit{if } \alpha p = n,\ p > 1 \\ y > \exp\{-c|x - x_0|^{-n}\} & \textit{if } \alpha = n,\ p = 1 \end{array}$$

where c is any positive number.

To obtain tangential convergence on the unit disc we use a function

$$\tilde{g_\alpha}(\theta) = (1 - e^{i\theta})^{\alpha - 1}$$

for $0 < \alpha < 1$. This function is not a kernel. It is however dominated by a kernel that has essentially the same asymptotic relations as G_α and thus we obtain a corresponding theorem for the unit disc.

An easy calculation shows that the Cauchy integral of $\tilde{g_\alpha} * F$ is the same as its Poisson integral, namely

$$(P_r * \tilde{g_\alpha} * F)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(t)dt}{(1 - e^{-it}z)^{1-\alpha}}.$$

We define the exponential contact region as

$$\mathcal{E}_{\gamma,c}(\theta) = \left\{ re^{i\phi} : 1 - r > \exp\left(-c\left|\sin\left(\frac{\phi - \theta}{2}\right)\right|^{-\gamma}\right)\right\}.$$

This region makes exponential contact with the unit circle $\mathbb{T}$ at $e^{i\theta}$. We also define regions

$$\mathcal{A}_{\gamma,c}(\theta) = \left\{ re^{i\phi} : 1 - r > c \left| \sin\left(\frac{\phi - \theta}{2}\right) \right|^{\gamma} \right\}$$

which have order of contact γ. We say that a function h, defined in $\mathbb{D}$, has $\mathcal{A}_\gamma$-limit L at $e^{i\theta}$ if $h(z) \to L$ as $z \to e^{i\theta}$ within $\mathcal{A}_{\gamma,c}(\theta)$ for every c. $\mathcal{E}_\gamma$-limits are defined analogously.

Now we can finally present the theorem for the tangential convergence on the unit disc.

Theorem 12. *Suppose* $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $F \in L^p(\mathbb{T})$, $0 < \alpha < 1$ *and*

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(t)dt}{(1 - e^{-it}z)^{1-\alpha}}$$

for all $z \in \mathbb{D}$.

a) *If* $\alpha p < 1$ *and* $\gamma = 1/(1 - \alpha p)$ *then the* $\mathcal{A}_\gamma$*-limit of* h *exists almost everywhere on* $\mathbb{T}$.

b) *If* $\alpha p = 1$ *then the* $\mathcal{E}_{q-1}$*-limit of* h *exists almost everywhere on* $\mathbb{T}$.

We note that when $p = 2$ *and* $\alpha = \frac{1}{2}$ *we get exactly information about the classical Dirichlet space* $\mathcal{D}(\mathbb{D})$. *We also remark that the function* h *in the theorem is continuous on* $\overline{\mathbb{D}}$ *when* $\alpha p > 1$.

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A FORMULA FOR THE LOCAL DIRICHLET INTEGRAL

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presented by Shuaibing Luo

Abstract. We study a different norm on the $D(\mu)$ spaces using the local Dirichlet integral, it turns out to be equal to the original norm. We then derive a formula for the local Dirichlet integral, obtaining Carleson's formula [3] for the Dirichlet integral as a Corollary and answering some questions of [2] about cyclic vectors.

1. The $D(\mu)$ spaces

Let $H^2(\mathbf{D})$ denote the Hardy space of the open unit disc $\mathbf{D}$. The Dirichlet space D is the space of analytic functions f in $\mathbf{D}$ with finite Dirichlet integral; That is,

$$D(f) = \iint_{\mathbf{D}} |f'(z)|^2 \, dA(z) < \infty,$$

where $dA(re^{it}) = (1/\pi)r \, dr \, dt$ denotes the normalized area measure on $\mathbf{D}$. In his investigating about minimal surfaces, Douglas [4] used the following formula for the Dirichlet integral of f:

$$D(f) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |\frac{f(e^{it}) - f(e^{is})}{e^{it} - e^{is}}|^2 \, dt \, ds.$$

Let $f \in L^1(= L^1(\mathbf{T}))$, for $\zeta \in \mathbf{T}$ we define the *local Dirichlet integral* of f at ζ by

$$D_\zeta(f) = \frac{1}{2\pi} \int_0^{2\pi} |\frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta}|^2 \, dt.$$

Note that it follows from Douglas' formula that one can obtain the Dirichlet integral of f by integrating the local Dirichlet integral with respect to normalized Lebesgue measure on $\mathbf{T}$.

In this paper, we are interested in the general Dirichlet type spaces - the $D(\mu)$ spaces. For a nonnegative finite Borel measure μ on $\mathbf{T}$, define the harmonic function φ_μ by

$$\varphi_\mu(z) = \int_{\mathbf{T}} \frac{1 - |z|^2}{\zeta - z}|^2 \, d\mu(\zeta).$$

If $\mu = 0$ then let $D(\mu) = H^2$; otherwise define $D(\mu)$ to be the space of all H^2 functions f such that

$$\iint_{\mathbf{D}} |f'(z)|^2 \varphi_\mu(z) \, dA(z) < \infty.$$

Here we use dA to denote normalized area measure on $\mathbf{D}$, $dA(z) = (1/\pi)r \, dr \, dt$ if $z = re^{it}$. A norm on $D(\mu)$ is given by

$$||f||_\mu^2 = ||f||_{H^2}^2 + \iint_{\mathbf{D}} |f'(z)|^2 \varphi_\mu(z) \, dA(z), \quad f \in D(\mu).$$

In particular, the classical Dirichlet space D equals the space $D(m)$, where m denotes normalized Lebesgue measure on $\mathbf{T}$.

We already know that one obtains Douglas' formula for the Dirichlet integral by integrating the local Dirichlet integrals $D_\zeta(f)$ with respect to Lebesgue measure on $\mathbf{T}$. In general, we have for the $D(\mu)$ - integrals:

Proposition 1.1. *Let μ be a nonnegative finite Borel measure on $\boldsymbol{T}$. If $f \in H^2$, then*

$$\int_{\boldsymbol{T}} D_\zeta(f)\ d\mu(\zeta) = \iint_{\boldsymbol{D}} |f'(z)|^2 \varphi_\mu(z)\ dA(z).$$

From this, we obtain the following Corollary:

Corollary 1.2. *let $f \in D(\mu)$. Then the oricyclic limit of f exists at μ - a.e. boundary points $\zeta \in \boldsymbol{T}$. Furthermore, the boundary value function f is in $L^2(\mu)$ and*

$$||zf||_\mu^2 - ||f||_\mu^2 = \int_{\boldsymbol{T}} |f(\zeta)|^2\ d\mu(\zeta).$$

Where oricyclic limits means: $f(z) \to f(\zeta)$ if $z \to \zeta$ in any oricyclic approach region

$$O_k(\zeta) = \{z \in \boldsymbol{D} : |z-\zeta|^2 < k(1-|z|^2)\}.$$

There is a connection of the spaces $D(\delta_\zeta)$ to the de Branges spaces $M(\overline{\zeta} - S^*) = \{(\overline{\zeta} - S^*)g,\ g \in H^2\}$, where S^* is the backward shift on H^2. Fix $\zeta \in \mathbf{T}$ and let S denote the unilateral shift, that is, the operator of multiplication by z on H^2. The space $M(\overline{\zeta} - S^*) \subseteq H^2$ is defined to be the range of the operator $\overline{\zeta} - S^*$. A Hilbert space norm $||\ ||_M$ can be defined on $M(\overline{\zeta} - S^*)$ so that $\overline{\zeta} - S^*$ acts as an isometry from H^2 onto $M(\overline{\zeta} - S^*)$. Thus, if $f = (\overline{\zeta} - S^*)g \in M(\overline{\zeta} - S^*)$ for some $g \in H^2$, then

$$||f||_M^2 = ||(\overline{\zeta} - S^*)g||_M^2 = ||g||_{H^2}^2.$$

Proposition 1.3. *The spaces $M(\overline{\zeta} - S^*)$ and $D(\delta_\zeta)$ coincide with equivalence of norms. More precisely, we have*

$$||f||_M^2 = |f(\zeta)|^2 + D_\zeta(f) = D_\zeta(zf) \quad \textit{for any } f \in H^2.$$

2. A Formula for the Local Dirichlet Integral

§2. In 1960 Carleson [3] proved a formula that expresses the Dirichlet integral of f as a sum of three nonnegative terms, involving respectively the Blaschke factor of f, the singular inner factor, and the outer factor. In this paper, we derive a formula for $D_\zeta(f)$ for an arbitrary H^2 function f:

Theorem 2.1. *Let $\zeta \in \boldsymbol{T}$, let $f \in H^2$, and let $f = BSf_0$, that is, let*

$$f(z) = \prod_{j=1}^{\infty} \frac{\overline{\alpha_j}}{|\alpha_j|} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z}\ exp\{-\int \frac{e^{it}+z}{e^{it}-z}\ d\sigma(t)\}\ exp\{\frac{1}{2\pi}\int \frac{e^{it}+z}{e^{it}-z} log|f(e^{it})|\ dt\}$$

be the factorization of f into a Blaschke product, a singular inner factor, and an outer factor. Write $u(e^{it}) = log|f_0(e^{it})|$. Then

$$D_\zeta(f) = \sum_{j=1}^{\infty} P_{\alpha_j}(\zeta)|f_0(\zeta)|^2 + \int_0^{2\pi} \frac{2}{|e^{it}-\zeta|^2}\ d\sigma(t)|f_0(\zeta)|^2$$
$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{2u(e^{it})} - e^{2u(\zeta)} - 2e^{2u(\zeta)}(u(e^{it}) - u(\zeta))}{|e^{it}-\zeta|^2}\ dt.$$

If either of the canonical factors of f is absent, then the corresponding summand in the expression for $D_\zeta(f)$ will be 0.

As a corollary we obtain Carleson's formula [3] and we note that the integrand of the last integral is always nonnegative.

Proof. We shall prove the formula for the local Dirichlet integral via a sequence of lemmas and propositions. The strategy is as follows: First we shall show that $D_\zeta(f) = ||(f(z) - f(\zeta))/(z-\zeta)||^2_{H^2}$ is the limit of $||(f(z) - f(\lambda))/(z-\lambda)||^2_{H^2}$ as $\lambda \to \zeta$ nontangentially. From this it will show that for any H^2 function of the form φf, where φ is inner, one has

$$D_\zeta(\varphi f) = D_\zeta(\varphi)|f(\zeta)|^2 + D_\zeta(f).$$

Thus, it can be proved by considering the inner and outer factor of f separately. □

In [5] it was shown that every function in the Dirichlet space D can be written as the quotient of two bounded functions in D. We now show that one can use the formula of $D_\zeta(f)$ at the beginning of this section to show that the local Dirichlet integral of the cut-off functions is bounded by the local Dirichlet integral of the original function. This implies that every function in $D(\mu)$ can be written as the quotient of two bounded functions in $D(\mu)$ for any nonnegative finite Borel measure μ.

Corollary 2.2. *Let $\alpha \in (0,\infty)$ and $f \in D(\delta_\zeta)$. Suppose $f = If_0$ is the inner-outer factorization of f. Let φ_0 be the outer function determined by $|\varphi_0| = min\{|f_0|, \alpha\}$, let $\varphi = I\varphi_0$, and let $\psi = \varphi_0/f_0$. Then $f = \varphi/\psi, \varphi, \psi \in H^\infty$,*

$$||\varphi||_\infty \le \alpha, \quad ||\psi||_\infty \le 1,$$

and $\varphi, \psi, 1/\psi \in D(\delta_\zeta)$ with

$$D_\zeta(\varphi) \le D_\zeta(f),$$
$$D_\zeta(\psi) \le D_\zeta(1/\psi) \le (1/\alpha^2) D_\zeta(f_0).$$

Consequently, every function in $D(\mu)$ can be written as the quotient of two bounded functions in $D(\mu)$.

3. Inner Functions and 2-Isometries

From our formula for the local Dirichlet integral in section 2, one can easily deduce that any inner function which defines a bounded multiplication operator on $D(\mu)$ must be a 2-isometric multiplication operator. By 2-isometry, we mean: An operator T on a separable complex Hilbert space H is called a 2-isometry if

$$||T^2x||^2 - ||Tx||^2 = ||Tx||^2 - ||x||^2 \quad \text{for all } x \in H.$$

This definition is due to Agler [1].

In this section, we shall prove a result that the converse to this statement is true as well: If a multiplication operator M_φ on $D(\mu)$ acts as a 2-isometry, then φ must be an inner function, This can be viewed as an extension of the well-known fact that the isometric multiplication operators on H^2 are given exactly by multiplications with inner functions.

Theorem 3.1. *Let μ be a nonnegative finite Borel measure on $\boldsymbol{T}$, and let φ be a complex function on $\boldsymbol{D}$ such that $f, \varphi f, \varphi^2 f \in D(\mu)$ for some nonzero function f. Then φ is the quotient of two inner functions if and only if*

$$||\varphi^2 f||^2_\mu - ||\varphi f||^2_\mu = ||\varphi f||^2_\mu - ||f||^2_\mu$$

holds for f and zf.

In particular, a multiplier φ on $D(\mu)$ acts as a 2-isometry if and only if φ is an inner function.

4. Approximation of Functions in $D(\delta_\zeta$

In this section, we apply the local Dirichlet integral to some questions about cyclic vectors in the Dirichlet space. In fact, all of our results are true in the generality of all $D(\mu)$ spaces. Recall that a function f in $D(\mu)$ is called a *cyclic vector* if the polynomial multiples of f are dense in $D(\mu)$, and we denote by $[f]$ the smallest invariant subspace of the operator of multiplication by z.

Lemma 4.1. *Let $f \in D(\delta_\zeta)$ and $0 < r < 1$. Then $D_\zeta(f_r) \leq 4D_\zeta(f)$, where $f_r(z) = f(rz)$.*

Lemma 4.2. *$f \in D(\mu)$ and $\varphi \in H^\infty$ such that $\varphi f \in D(\mu)$, then $\varphi_r f \to \varphi f$ (weakly) in $D(\mu)$.*

Corollary 4.3. *Let $f, g \in D(\mu)$, If $|f(z)| \geq |g(z)|$ for all $z \in \boldsymbol{D}$, then $[g] \subseteq [f]$. In particular, if g is cyclic then f is cyclic.*

Corollary 4.4. *Let $f, g \in D(\mu) \cap H^\infty$. Then fg is cyclic if and only if both f and g are cyclic.*

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A CORONA THEOREM FOR MULTIPLIERS ON DIRICHLET SPACE

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presented by Raphaël Clouâtre

Abstract. Using operator theoretic methods, we extend the corona theorem to the setting of infinitely many multipliers on the Dirichlet space. The key ingredient is an operator factorization which allows one to write down explicitely the solution to a $\overline{\partial}$-problem.

1. Introduction

The purpose of the paper (see [4]) is to extend a result of Tolokonnikov (see [3]) stating that the Corona Theorem holds for multipliers on the Dirichlet space. More precisely, the author establishes the result in the case of infinitely many functions. The corresponding results in the Hardy space setting are due to Carleson in the finite case, and to Rosenblum and Tolokonnikov in the infinite case (see [2] and [3]).

Trent's proof has three main ingredients. First, a version of the commutant lifting theorem due to Clancy and McCullough (see [1]), which allows one to reduce the $\mathscr{M}(\mathcal{D})$-corona problem to a $\mathcal{D}$-corona problem. This has the advantage of providing us with familiar Hilbert space techniques. Then, one needs to consider the extension of multipliers on $\mathcal{D}$ to multipliers on the harmonic dirichlet space $\mathcal{HD}$. These extension properties are a technical tool used in the proofs. Finally, an operator factorization allows one to write explicitely the solution of a $\overline{\partial}$-problem arising in the proof.

2. Notation and Preliminaries

Given a Hilbert space H, we denote by $\mathcal{B}(H)$ the algebra of bounded linear operators on H. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc in the complex plane, and let $\mathbb{T}$ be the unit circle. The Dirichlet space is denoted by $\mathcal{D}$ and consists of the functions $f : \mathbb{D} \to \mathbb{C}$ which can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$, with

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty.$$

For $z = x + iy \in \mathbb{C}$ and $t \in [0, 2\pi)$, set $d\sigma(t) = (1/2\pi)dt$ and $dA(z) = (1/\pi)dxdy$. It is easily verified that

$$\|f\|_{\mathcal{D}}^2 = \int_0^{2\pi} |f(e^{it})|^2 d\sigma(t) + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

An equivalent norm on $\mathcal{D}$ is given by

$$\int_0^{2\pi} |f(e^{it})|^2 d\sigma(t) + \int_0^{2\pi}\int_0^{2\pi} \left|\frac{f(e^{it}) - f(e^{i\theta})}{e^{it} - e^{i\theta}}\right|^2 d\sigma(t)d\sigma(\theta).$$

In an analogous fashion, the space $\mathcal{HD}$ consists of harmonic functions $f : \mathbb{D} \to \mathbb{C}$ with the property that the number

$$\|f\|^2_{\mathcal{HD}} = \int_0^{2\pi} |f(e^{it})|^2 d\sigma(t) + \int_0^{2\pi}\int_0^{2\pi} \left|\frac{f(e^{it}) - f(e^{i\theta})}{e^{it} - e^{i\theta}}\right|^2 d\sigma(t) d\sigma(\theta)$$

is finite.

The object we will be interested is the multiplier algebra on the Dirichlet space, namely

$$\mathscr{M}(\mathcal{D}) = \{\phi \in \mathcal{D} : \phi f \in \mathcal{D} \text{ for every } f \in \mathcal{D}\} = \{\phi \in \mathcal{D} : M_\phi \in \mathcal{B}(\mathcal{D})\}$$

where M_ϕ is the multiplication operator $M_\phi f = \phi f$. Given a sequence $\{f_j\}_{j=1}^\infty \subset \mathscr{M}(\mathcal{D})$, we define

$$F(z) = (f_1(z), f_2(z), \ldots).$$

Moreover, we define the row multiplication operator

$$M_F^R : \bigoplus_{j=1}^\infty \mathcal{D} \to \mathcal{D}$$

as

$$M_F^R((h_j)_{j=1}^\infty) = \sum_{j=1}^\infty f_j h_j.$$

Similarly, we define the column multiplication operator

$$M_F^C : \mathcal{D} \to \bigoplus_{j=1}^\infty \mathcal{D}$$

as

$$M_F^C h = (f_j h)_{j=1}^\infty.$$

The Dirichlet space $\mathcal{D}$ is a reproducing kernel Hilbert space with reproducing kernel

$$k_w(z) = \frac{1}{\overline{w}z} \log \frac{1}{1 - \overline{w}z}.$$

This kernel satisfies the Nevanlinna-Pick property and we have

$$1 - \frac{1}{k_w(z)} = \sum_{n=1}^\infty c_n \overline{w}^n z^n \tag{1}$$

where each c_n is positive.

3. Statement of main result

Let us first state the infinite version of the $H^\infty(\mathbb{D})$-corona theorem due to Rosenblum and Tolokonnikov (see [2] and [3]).

Theorem 3.1. *Let $\{f_j\}_{j=1}^\infty \subset H^\infty(\mathbb{D})$. Assume that there exists $0 < \varepsilon < e^{-1/2}$ such that*

$$0 < \varepsilon^2 \leq \sum_{j=1}^\infty |f_j(z)|^2 \leq 1$$

for every $z \in \mathbb{D}$*. Then, there exist* $\{g_j\}_{j=1}^{\infty} \subset H^{\infty}(\mathbb{D})$ *such that* $\sum_{j=1}^{\infty} f_j g_j = 1$ *and*

$$\sup_{z\in\mathbb{D}} \left\{ \sum_{j=1}^{\infty} |g_j(z)|^2 \right\} \leq \frac{9}{\varepsilon^2} \log \frac{1}{\varepsilon^2}.$$

The author mentions that the pointwise boundedness assumption

$$\sum_{j=1}^{\infty} |f_j(z)|^2 \leq 1$$

allows one to infer

$$\|T_F^R\| = \|T_F^C\| = \left(\sup_{z\in\mathbb{D}} \left\{ \sum_{j=1}^{\infty} |f_j(z)|^2 \right\} \right)^{1/2} < \infty$$

where T_F^R and T_F^C are defined in analogy with M_F^R and M_F^C on $\bigoplus_{j=1}^{\infty} H^2(\mathbb{D})$ and $H^2(\mathbb{D})$ respectively. In the case at hand, however, such a pointwise boundedness assumption does not suffice to ensure the boundedness of the operators M_F^R and M_F^C. Indeed, the author shows that for $f_j = c_j^{1/2} z^j$ where c_j is as in (1), we have

$$\sup_{z\in\mathbb{D}} \left\{ \sum_{j=1}^{\infty} |f_j(z)|^2 \right\} = 1$$

and $\|M_F^R\| = 1$, but M_F^C is unbounded. The replacement assumption shall be taken to be that M_F^C has norm at most 1. It stems from the following lemma.

Lemma 3.2. *Assume that* M_F^C *is bounded. Then,* M_F^R *is also bounded and* $\|M_F^R\| \leq \sqrt{18}\|M_F^C\|$.

We can now state the main result of the paper.

Theorem 3.3. *Let* $\{f_j\}_{j=1}^{\infty} \subset \mathscr{M}(\mathcal{D})$*. Assume that there exists* $\varepsilon > 0$ *such that*

$$\sum_{j=1}^{\infty} |f_j(z)|^2 \geq \varepsilon^2 > 0$$

for every $z \in \mathbb{D}$*. Assume also that* $\|M_F^C\| \leq 1$*. Then, there exist* $\{g_j\}_{j=1}^{\infty} \subset \mathscr{M}(\mathcal{D})$ *such that* $\sum_{j=1}^{\infty} f_j g_j = 1$ *and* $\|M_G^C\| \leq 1500\,\varepsilon^{-3}$*.*

To establish this result, the author breaks it into two parts, the combination of which clearly implies Theorem 3.3.

Theorem 3.4. *Let* $\{f_j\}_{j=1}^{\infty} \subset \mathscr{M}(\mathcal{D})$*. Assume that there exists* $\varepsilon > 0$ *such that*

$$\sum_{j=1}^{\infty} |f_j(z)|^2 \geq \varepsilon^2 > 0$$

for every $z \in \mathbb{D}$*. Assume also that* $\|M_F^C\| \leq 1$*. Then,*

$$\left(1500\,\varepsilon^{-3}\right)^{-2} \mathrm{Id} \leq M_F^R (M_F^R)^* \leq \mathrm{Id}\,.$$

The theorem above is the reduction we announced in the introduction: checking its conclusion only involves functions from the Hilbert space $\mathcal{D}$ itself, not from $\mathscr{M}(\mathcal{D})$, which makes it somewhat simpler to handle. To perform this reduction, we need the following.

Theorem 3.5. *Assume that*

$$\delta^2 \operatorname{Id} \leq M_F^R (M_F^R)^* \leq \operatorname{Id}.$$

Then, there exist $\{g_j\}_{j=1}^\infty \subset \mathscr{M}(\mathcal{D})$ *such that* $\sum_{j=1}^\infty f_j g_j = 1$ *and* $\|M_G^C\| \leq \delta^{-1}$.

Theorem 3.5 follows easily from a version of the commutant lifting theorem due to Clancy and McCullough (see Corollary 2.3 in [1]) which we now state. First note that the function $z \mapsto z$ is obviously a multiplier on $\mathcal{D}$, so that $M_z^* \in \mathcal{B}(\mathcal{D})$.

Theorem 3.6. *Let* $1 \leq m, n \leq \infty$ *be natural numbers. Let* $E \subset \bigoplus_{j=1}^m \mathcal{D}$ *be a closed invariant subspace for* $\bigoplus_{j=1}^m M_z^*$ *and* $F \subset \bigoplus_{j=1}^n \mathcal{D}$ *be a closed invariant subspace for* $\bigoplus_{j=1}^n M_z^*$. *Assume that* $X^* \in \mathcal{B}(E, F)$ *satisfies* $X^*((\bigoplus_{j=1}^m M_z^*)|E) = ((\bigoplus_{j=1}^n M_z^*)|F)X^*$. *Then, there exists an operator* $Y^* \in \mathcal{B}(\bigoplus_{j=1}^m \mathcal{D}, \bigoplus_{j=1}^n \mathcal{D})$ *such that* $Y^*|E = X^*$, $Y^*(\bigoplus_{j=1}^m M_z^*) = (\bigoplus_{j=1}^n M_z^*)Y^*$ *and* $\|Y^*\| = \|X^*\|$.

Let us make a remark. Let $T \in \mathcal{B}(\mathcal{D})$ be an operator that commutes with M_z. Then, for any polynomial p we have

$$Tp = Tp(M_z)1 = p(M_z)T1 = p\phi$$

where $\phi = T1$. Note now that $\{z^k : k \geq 0\}$ is an orthogonal basis of $\mathcal{D}$, so that the holomorphic polynomials are dense in $\mathcal{D}$. In particular, we conclude that $Tf = \phi f$ for every $f \in \mathcal{D}$ and thus $T = M_\phi$ is a multiplier on $\mathcal{D}$. This shows that an operator $T \in \mathcal{B}(\mathcal{D})$ is a multiplier if and only if T commutes with M_z. The conditions on Y^* in Theorem 3.6 thus imply that Y is a $(m \times n)$-matrix of multipliers.

4. The $\mathcal{D}$-corona theorem: The proof of Theorem 3.4

A standard operator theoretic argument (see page 3 of Lecture 11 in Professor Wick's notes) shows that Theorem 3.4 is actually equivalent to the following.

Theorem 4.1. *Let* $\{f_j\}_{j=1}^\infty \subset \mathscr{M}(\mathcal{D})$. *Assume that there exists* $\varepsilon > 0$ *such that*

$$\sum_{j=1}^\infty |f_j(z)|^2 \geq \varepsilon^2 > 0$$

for every $z \in \mathbb{D}$. *Assume also that* $\|M_F^C\| \leq 1$. *Then, for every* $h \in \mathcal{D}$ *there exists* $u_h \in \bigoplus_{j=1}^\infty \mathcal{D}$ *such that* $M_F^R u_h = h$ *and* $\|u_h\|_{\bigoplus_{j=1}^\infty \mathcal{D}} \leq 1500\, \varepsilon^{-3}\, \|h\|_{\mathcal{D}}$.

The purpose of the next sequence of lemmas is to show that in the case where F and h are smooth across $\mathbb{T}$, Theorem 4.1 holds. The smoothness assumption on h is inconsequential: it is sufficient to establish Theorem 4.1 for a dense set of functions, so we may assume that $h \in \mathcal{D}$ is holomorphic across $\mathbb{T}$. As for F, we will assume it to be smooth across $\mathbb{T}$ for now, and later use a compactness argument to show that this assumption can be removed. In other words, we assume henceforth that $\{f_j\}_{j=1}^\infty \subset \mathscr{M}(\mathcal{D}) \cap C^\infty(\overline{\mathbb{D}})$ with $\sum_{j=1}^\infty |f_j(z)|^2 \geq \varepsilon^2$ and $h \in \mathcal{D} \cap C^\infty(\overline{\mathbb{D}})$.

Since the function $FF^* = \sum_{j=1}^\infty |f_j|^2$ isn't holomorphic in general, it will prove useful to deal with the larger space $\mathcal{H}\mathcal{D}$. Consequently, we need a quantity of technical lemmas regarding the extension of multipliers on $\mathcal{D}$ to multipliers on $\mathcal{HD}$, the results of which we summarize here.

(i) Given $\{f_j\}_{j=1}^{\infty} \subset \mathscr{M}(\mathcal{D})$, we have $\|M_F^C\|_{\mathcal{B}(\mathcal{HD},\bigoplus_{j=1}^{\infty}\mathcal{HD})} \leq \sqrt{20}\|M_F^C\|_{B(\mathcal{D},\bigoplus_{j=1}^{\infty}\mathcal{D})}$.

(ii) Let $\{f_j\}_{j=1}^{\infty} \subset \mathscr{M}(\mathcal{D})$. Assume that there exists $\varepsilon > 0$ such that

$$\sum_{j=1}^{\infty} |f_j(z)|^2 \geq \varepsilon^2 > 0$$

for every $z \in \mathbb{D}$. Assume also that $\|M_F^C\| \leq 1$. Then, for every $h \in \mathcal{D}$ we have

$$\left\| \frac{F^*}{FF^*} h \right\|_{\mathcal{HD}}^2 \leq 10 \cdot 86^2 \cdot 20\, \varepsilon^{-4}\, \|h\|_{\mathcal{D}}^2.$$

The next step is the operator factorization discussed in the introduction.

Lemma 4.2. *Let $\{c_j\}_{j=1}^{\infty} \in \ell^2$ and define $C = (c_1, c_2, \dots) \in \mathcal{B}(\ell^2, \mathbb{C})$. Then, there exists a bounded linear operator $Q : \bigoplus_{j=1}^{\infty} \ell^2 \to \ell^2$ with entries being either 0 or $\pm c_i$ for some $i \in \mathbb{N}$, with the additional property that*

$$CC^* \operatorname{Id}_{\ell^2} - C^*C = QQ^*.$$

Upon identifying $\bigoplus_{j=1}^{\infty} \ell^2$ with ℓ^2, for each $z \in \overline{\mathbb{D}}$ we may apply Lemma 4.2 to $F(z)$ to obtain $Q(z) \in \mathcal{B}(\ell^2)$ with entries being either 0 or $\pm f_i(z)$ for some $i \in \mathbb{N}$, and such that

$$F(z)F^*(z) \operatorname{Id}_{\ell^2} - F(z)^* F(z) = Q(z)Q(z)^*,$$

whence $\operatorname{ran} Q(z) \subset \ker F(z)$ for every $z \in \overline{\mathbb{D}}$. It follows easily from the proof of Lemma 4.2 that $Q \in \mathcal{O}(\mathbb{D}, \mathcal{B}(\ell^2)) \cap C^{\infty}(\overline{\mathbb{D}}, \mathcal{B}(\ell^2))$. Additionally, we have the following estimate.

Lemma 4.3. *Assume that $\|M_F^C\| \leq 1$. Given $u = (u_j)_{j=1}^{\infty} \in \bigoplus_{j=1}^{\infty} \mathcal{HD}$, define $Qu : \mathbb{D} \to \ell^2$ as $(Qu)(z) = Q(z)u(z)$. Then, $Q \in \mathcal{B}(\bigoplus_{j=1}^{\infty} \mathcal{HD})$ and*

$$\|Q\|_{\mathcal{B}(\bigoplus_{j=1}^{\infty} \mathcal{HD})} \leq \sqrt{86}.$$

In order to prove Theorem 4.1, the author solves a $\overline{\partial}$-problem. The main tool here is the Cauchy transform. Given a function $k : \mathbb{D} \to \ell^2$, we define

$$\widehat{k}(z) = P.V. \int_{\mathbb{D}} \frac{k(w)}{z - w} dA(w) = \lim_{\varepsilon \to 0} \int_{\mathbb{D} \cap \{|z-w| \geq \varepsilon\}} \frac{k(w)}{z - w} dA(w).$$

In case $k \in C^{\infty}(\overline{\mathbb{D}}, \ell^2)$, we have that $\overline{\partial}\widehat{k} = k$ on $\mathbb{D}$.

Lemma 4.4. *Let $k : \overline{\mathbb{D}} \to \ell^2$ be a smooth function. Then,*

$$\|\widehat{k}\|_{\mathcal{HD}}^2 \leq \|k\|_{L^2(\mathbb{D}, dA)}^2 + \|\widehat{k}\|_{L^2(\mathbb{T}, d\sigma)}^2.$$

We can now proceed to the proof of Theorem 4.1 in the case where F is smooth across $\mathbb{T}$.

Proof. We have $F(z)F(z)^* \geq \varepsilon^2$ for every $z \in \mathbb{D}$, so that $(F(z)F(z)^*)^{-1} = \left(\sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{-1}$ makes sense. We set

$$u_h = (FF^*)^{-1} F^* h - Q \left((FF^*)^{-2} Q^* F'^* h \right)\widehat{}\,.$$

By construction we have that $M_F^R u_h = h$, so the only thing left to prove is that

$$\|u_h\|_{\mathcal{D}} \leq (1500\, \varepsilon^{-3}) \|h\|_{\mathcal{D}}.$$

Since Q and F are smooth on $\overline{\mathbb{D}}$, a straightforward calculation yields $\overline{\partial} u_h = 0$, and thus $\|u_h\|_{\mathcal{D}} = \|u_h\|_{\mathcal{HD}}$. The previously established estimates yield the result.

$\square$

All that is now left to do is to remove the assumption of F being smooth across $\mathbb{T}$. This is done via an approximation argument, based on the following lemma.

Lemma 4.5. *Let* $\{f_j\}_{j=1}^{\infty} \subset \mathscr{M}(\mathcal{D})$ *with* $\|M_F^C\| \leq 1$. *For* $0 \leq r \leq 1$ *and* $z \in \mathbb{D}$, *set* $F_r(z) = F(rz)$. *Then,*

(i) $M_{F_r}^C \in \mathcal{B}(\mathcal{D}, \bigoplus_{j=1}^{\infty} \mathcal{D})$ *and* $\|M_{F_r}^C\| \leq \|M_F^C\|$

(ii) $(M_{F_r}^R)^* \to (M_F^R)^*$ *in the strong operator topology as* $r \to 1$.

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