# THE CALDERÓN-ZYGMUND DECOMPOSITION ON PRODUCT DOMAINS

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We recall that in a previous paper [1], the authors introduced an "atomic" decomposition for the space $H^1$ of the product of upper half space. The result in [1] of concern here is the following:

THEOREM. *If $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, then we may write $f = \Sigma\lambda_k a_k$ where $\lambda_k$ are constants with $\Sigma|\lambda_k| \leq c\|f\|_{H^1}$ and $a_k$ are atoms.*

In section 1 of the present note, we will recall the definitions of $H^1$ and atoms on the product space $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ as used in [1], and then proceed to prove the converse statement of above theorem. We then apply the theorem to prove the main result of this note, namely, the Calderón-Zygmund decomposition in the setting of products of upper half planes. The decomposition, together with some easy applications of it to interpolation problems, are carried out in section 2. Finally in section 3, we will briefly indicate how the result in section 1 can be modified to give $H^p$-atomic decomposition for all $0 < p \leq 1$.

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**1. Atomic decomposition of $H^1$.** In what follows, we shall work exclusively with the domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ and its distinguished boundary, $\mathbf{R}^2$. We will use the same definitions and notations as in [1]. A point of $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ will be denoted $(t, y)$ where $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2)$, $y_i \geq 0$, $i = 1, 2$. We shall often use the following notations: $\psi(t)$ will be a $C^1$ function on $\mathbf{R}^1$ supported on $[-1, 1]$ with $\psi$ even and $\int_{-1}^1 \psi(t)dt = 0$; if $y > 0$, $\psi_y(t) = (1/y)\psi(t/y)$ and if $y = (y_1, y_2)$ and $t = (t_1, t_2) \in \mathbf{R}^2$, then $\psi_y(t) = \psi_{y_1}(t_1)\cdot\psi_{y_2}(t_2)$. If $f$ is a function defined on $\mathbf{R}^2$ then $f(t, y)$ will, by

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definition, mean $f * \psi_y(t)$. Further, if $x = (x_1, x_2) \in \mathbf{R}^2$, $\Gamma(x)$ will denote the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$, where for $i = 1, 2$

$$\Gamma(x_i) = \{(t_i, y_i) \in \mathbf{R}^2_+ : |x_i - t_i| < y_i\}$$

Given a function $f$ on $\mathbf{R}^2$ we define its double $S$-function by

$$S^2(f) = \iint_{\Gamma(x)} |f(t, y)|^2 \frac{dt\,dy}{y_1^2 y_2^2}.$$

Then it is a fact that for $1 < p < \infty$

$$\|S(f)\|_p \leq c_p \|f\|_p$$

We may also define functions in $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, $0 < p < \infty$, as those functions $f$ with $S(f) \in L^p(\mathbf{R}^2)$ and define $\|f\|_{H^p} = \|S(f)\|_p$. As it turns out, this definition of $H^p$ spaces is equivalent to the one defined via boundary values of functions on $\mathbf{R}^2$ of bi-holomorphic functions on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ (c.f. Gundy-Stein [3], and Merryfield, (to appear)). We also adopt the following definition of "atom."

*Definition.* An "atom" is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set $\Omega$ of finite measure such that

(1) $\|a\|_2 \leq 1/|\Omega|^{1/2}$

(2) $a$ can be further decomposed into "elementary particles" $a_R$ as follows:

(i) $a = \Sigma_R\, a_R$ where $a_R$ is supported in the triple of a distinct dyadic rectangle $R \subset \Omega$ (say $R = I \times J$)

(ii) $\int_I a_R(x_1, \tilde{x}_2)dx_1 = \int_J a_R(\tilde{x}_1, x_2)dx_2 = 0$ for each $\tilde{x}_1 \in I$, $\tilde{x}_2 \in J$

(iii) $a_R$ is $C^1$ with $\|a_R\|_\infty \leq d_R$,

$$\left\|\frac{\partial a_R}{\partial x_1}\right\|_\infty \leq \frac{d_R}{|I|}, \qquad \left\|\frac{\partial a_R}{\partial x_2}\right\|_\infty \leq \frac{d_R}{|J|}$$

with $\Sigma d_R^2 |R| \leq A/|\Omega|$ ($A$ is an absolute constant).

With this definition we are ready to give the result of this section.

THEOREM 1. *$f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ if and only if $f$ can be written as $f = \Sigma\lambda_k a_k$ where $a_k$ are atoms and $\lambda_k \geq 0$ satisfy $\Sigma|\lambda_k| \leq A\|f\|_{H^1}$.*

As mentioned in the introduction, that each $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ could be decomposed as in Theorem 1 was proved in [1]. Before indicating the proof of the converse direction, we wish first to give a very pedantic treatment of a one-dimension analogue. Suppose, instead of dealing with the $S$-function definition of $H^1$, we can also regard $H^1(\mathbf{R}_2^+)$ to be functions whose Hilbert transforms are in $L^1(\mathbf{R})$. Let $a(x)$ be a function supported in an interval $I$ centered at 0, with $\int_I a(x)dx = 0$ and $\|a\|_2^2 \le 1/|I|$. Then if $x$ is not in the double of $I$,

$$\begin{aligned} |Ha(x)| &= \left| \int_I a(y)\left(\frac{1}{x-y} - \frac{1}{x}\right)dy \right| \le \frac{|I|}{|x|^2}\int_I |a(y)|\,dy \\ &= \frac{|I|}{|x|}\cdot\frac{1}{|x|}\int_I |a(y)|\,dy \\ &\le M(\chi_I)(x)\cdot M(a)(x) \end{aligned}$$

where $M$ is the Hardy-Littlewood maximal operator. Notice that, this shows that $H(a)$ is in $L^1$ since

$$\begin{aligned} \|H(a)\|_{L^1(\text{away from } I)} &\le \|M(\chi_I)\|_2 \|M(a)\|_2 \le c\|\chi_I\|_2\|a\|_2 \\ &\le c|I|^{1/2}\cdot\frac{1}{|I|^{1/2}} = c. \end{aligned}$$

We now wish to extend the estimate

$$|H(a)(x)| \le M(\chi_I)(x)\cdot M(a)(x)$$

to the product case. Here the atom $a$ is a sum of elementary particles $a_R$ satisfying conditions as in the definition of atoms. Suppose $a$ is supported in the open set $\Omega \subset \mathbf{R}^2$. It is clear that the first factor on the right hand side of the inequality above is to be replaced by some estimate involving $M(\chi_\Omega)(x)$, where $M = M_S$ is the strong maximal operator. And the second term should involve terms which are positive majorant of $a$, and which could be upper estimated by the bounds of the elementary particles $a_R$.

We will introduce some more notations before we start our estimate. Suppose $a$ is an atom supported in the open set $\Omega$, $a = \Sigma_{R\subset\Omega}\, a_R$. Let $d_R$

be the bound for the elementary particle $a_R$, i.e. $d_R$ is the constant as appeared in the definition of atom. For each rectangle $S = I \times J$ let

$$S_+ = \left\{(t, y) \in \mathbf{R}_+^2 \times \mathbf{R}_+^2 ; t \in S, \frac{|I|}{2} < y_1 \leq |I|, \frac{|J|}{2} < y_2 \leq |J|\right\}.$$

For two different dyadic rectangles $R = I_R \times J_R$ and $S = I \times J$, let

$$m(R, S) = \frac{\min(|I_R|, |I|) \min(|J_R|, |J|)}{\max(|I_R|, |I|) \max(|J_R|, |J|)}.$$

Finally, for each point $x \in \mathbf{R}^2$, let $S_x$ denote the collection of dyadic rectangles containing the point $x$. Then

$$S^2(a)(x) = \iint_{\Gamma(x)} |a(t, y)|^2 \frac{dt\,dy}{y_1^2 y_2^2}$$

$$\leq \sum_{S \in S_x} \iint_{S_+} |a(t, y)|^2 \frac{dt\,dy}{y_1^2 y_2^2}$$

Let

$$\tilde{\Omega} = \bigcup_{\substack{R \subset \Omega \\ R \text{ dyadic}}} \tilde{R}$$

where $\tilde{R}$ is the triple in both directions of the rectangle $R$. Fix a point $x$ with $M_S(\chi_{\tilde{\Omega}})(x) < 1/4$. Fix $S \in S_x$, $S = I \times J$, and suppose $R \subset \Omega$ is a dyadic rectangle with $\tilde{R} \cap \tilde{S} \neq \phi$, $R = I_R \times J_R$. Then there are four types of such rectangles $R$.

(i) $|I_R| \geq |I|, |J_R| \geq |J|$

This could not occur. For the condition (i) implies that $S \subset \tilde{R}$, hence

$$1 = \frac{|S \cap \tilde{R}|}{|S|} \leq \frac{|S \cap \tilde{\Omega}|}{|S|} \leq M_S(\chi_\Omega)(x) < \frac{1}{4},$$

which is a contradiction.

(ii) $|I_R| \leq |I|, |J_R| \geq |J|$

Then for $(t, y) \in S_+$, we have

$$|a_R(t, y)| = \left| \iint a_R(\alpha, \beta)\psi_{y_1}(t_1 - \alpha)\psi_{y_2}(t_2 - \beta)d\alpha d\beta \right|$$

$$= \left| \int_{\tilde{J}} \int_{\tilde{I}_R} (a_R(\alpha, \beta) - a_R(\alpha, \beta_{\tilde{J}}))(\psi_{y_1}(t_1 - \alpha) - \psi_{y_1}(t_1 - \alpha_{\tilde{I}_R}))\psi_{y_2}(t_2 - \beta)d\alpha d\beta \right|$$

$$\leq c\left( \frac{d_R}{|J_R|}|J| \cdot \frac{|I_R|}{|I|^2} \cdot \frac{1}{|J|} \right) \cdot |J||I_R|$$

$$\leq c\left( \frac{|I_R||J|}{|I||J_R|} \right) d_R \left| \frac{I_R}{I} \right|$$

where $\alpha_{\tilde{I}_R}$, $\beta_{\tilde{J}}$ are centers of the intervals $\tilde{I}_R$, $\tilde{J}$ respectively and $c$ is a constant depending only on $\|\psi\|_\infty$, $\|\psi'\|_\infty$. Notice that in this special case

$$\frac{|I_R||J|}{|I||J_R|} = m(S, R) \quad \text{and} \quad \left| \frac{I_R}{I} \right| = \frac{|R \cap S|}{|S|}$$

$$= \frac{1}{|S|} \int_S \chi_R(t)dt \leq M(\chi_R)(x).$$

Hence for any $r > 0$ we have

$$|a_R(t, y)| \leq cm(S, R)d_R \left| \frac{I_R}{I} \right|^{1/r} \left| \frac{I_R}{I} \right|^{1-1/r}$$

$$\leq cm(S, R)d_R M^{1/r}(\chi_R)(x) \cdot \left| \frac{I_R}{I} \right|$$

(iii) $|I_R| \geq |I|$, $|J_R| \leq |J|$

Similar estimates as in the case (ii) yield the estimate that for

$$(t, y) \in S_+, \quad r > 0, \quad |a_R(t, y)| \leq cm(S, R)d_R M^{1/r}(\chi_R)(x) \left| \frac{J_R}{J} \right|^{1-1/r}$$

(iv) $|I_R| \le |I|$, $|J_R| \le |J|$

For $(t, y) \in S_+$,

$$|a_R(t,y)| = \left|\iint a_R(\alpha,\beta)\psi_{y_1}(t_1-\alpha)\psi_{y_2}(t_2-\beta)\,d\alpha\, d\beta\right|$$

$$= \left|\iint a_R(\alpha,\beta)(\psi_{y_1}(t_1-\alpha)-\psi_{y_1}(t_1-\alpha_{I_R}))\right.$$

$$\left.\cdot(\psi_{y_2}(t_2-\beta)-\psi_{y_2}(t_2-\beta_{J_R})d\alpha\, d\beta\right|$$

$$\le c\left(\iint |a_R(\alpha,\beta)|\,d\alpha\, d\beta\right)\frac{|I_R|}{|I|^2}\,\frac{|J_R|}{|J|^2}$$

$$\lesssim c\left(\frac{1}{|I|x|J|}\iint |a_R(\alpha,\beta)|^r d\alpha\, d\beta\right)^{1/r}$$

$$\cdot\left(\frac{1}{|I|x|J|}\iint \chi_R^t\, d\alpha\, d\beta\right)^{1/t}\frac{|I_R|\,|J_R|}{|I|\,|J|}$$

$$\lesssim cM^{1/r}(a_R^r)(x)\cdot\left(\frac{|I_R|\,|J_R|}{|I|\,|J|}\right)^{1+1/t}$$

$$\lesssim c d_R M^{1/r}(\chi_R)(x)(m(S,R))^{2-1/r}$$

where $\alpha_{I_R}$, $\beta_{J_R}$ appear in the second identity are centers of the intervals $\tilde{I}_R$, $\tilde{J}_R$ respectively. And $r$ is any constant $\ge 1$, and $t$ is chosen so that $1/t + 1/r = 1$.

Combining our estimates from (i) to (iv), we obtain for $(t, y) \in S_+$

$$|a(t,y)|^2 = \Big|\sum_{R\subset\Omega} a_R(t,y)\Big|^2$$

$$\le c\Big(\Big|\sum_{R\in \text{Type (ii)}} a_R(t,y)\Big|^2 + \Big|\sum_{R\in \text{Type (iii)}} a_R(t,y)\Big|^2$$

$$+\Big|\sum_{R\in \text{Type (iv)}} a_R(t,y)\Big|^2\Big)$$

$$\lesssim \Big(\sum_{R\subset\Omega} m^{2-1/r}(S,R)\, d_R^2 M^{2/r}(\chi_R)(x)\Big)\Big(\sum_{R\subset\Omega}\Delta_{R,S}\Big)$$

where

$$\Delta_{R,S} = \begin{cases} \left(\dfrac{|I_R|}{|I|}\right)^{1-1/r}\left(\dfrac{|J|}{|J_R|}\right)^{1/r} & \text{when } R \text{ is of Type (ii) w.r.t.S.} \\ \left(\dfrac{|J_R|}{|J|}\right)^{1-1/r}\left(\dfrac{|I|}{|I_R|}\right)^{1/r} & \text{when } R \text{ is of type (iii) w.r.t.S.} \\ (m(S, R)))^{2-1/r} & \text{when } R \text{ is of Type (iv) w.r.t.S.} \end{cases}$$

We now observe that for any $r > 1$, similar arguments as in [1], section 2 indicates that $\Sigma_{R \subset \Omega} \Delta_{R,S}$ is universally bounded independently of $S$. Thus for each point $x$ with $M_S(\chi_{\tilde{\Omega}})(x) < 1/4$ we have for each $r > 1$.

$$(*) \qquad S^2(a)(x) \leq \sum_{S \in S_x} \iint_{S_+} |a(t, y)|^2 \frac{dt\,dy}{y_1^2 y_2^2}$$

$$\leq c \sum_{S \in S_x} \sum_{R \subset \Omega} m^{2-1/r}(S, R) d_R^2 M^{2/r}(\chi_R)(x)$$

$$= c \sum_{R \subset \Omega} \Big( \sum_{S \in S_x} m^{2-1/r}(S, R)\Big) d_R^2 M^{2/r}(\chi_R)(x)$$

$$\leq c \sum_{R \subset \Omega} M^{2-1/r}(\chi_\Omega)(x) d_R^2 M^{2/r}(\chi_R)(x)$$

$$= c M^{2-1/r}(\chi_\Omega)(x) \sum_{R \subset \Omega} d_R^2 M^{2/r}(\chi_R)(x)$$

Thus

$$\int_{\{M(\chi_{\tilde{\Omega}})(x) < 1/4\}} S(a)(x)\,dx$$

$$\leq \left(\int M^{2-1/r}(\chi_\Omega)(x)\,dx\right)^{1/2} \left(\sum_{R \subset \Omega} d_R^2 \int M^{2/r}(\chi_R)\,dx\right)^{1/2}.$$

If we choose $1 < r < 2$ so that $2 - 1/r$ and $2/r > 1$ then we have

$$(1) \qquad \int_{\{M(\chi_{\tilde{\Omega}})(x) < 1/r\}} S(a)(x)\,dx \leq c\,|\Omega|^{1/2} \Big(\sum_{R \subset \Omega} d_R^2 |R|\Big)^{1/2}$$

$$\leq c$$

The last step in (1) follows from our assumption that $a$ is an atom. We also have

$$(2)\quad \int_{\{M(\chi_{\tilde{\Omega}})(x) > 1/4\}} S(a)(x)dx \leq \left(\int S^2(a)(x)dx\right)^{1/2} \left|\left\{M(\chi_{\tilde{\Omega}})(x) > \frac{1}{4}\right\}\right|^{1/2}$$

$$\leq c\|a\|_2 |\Omega|^{1/2} \leq c.$$

Combining the estimates in (1) and (2), we obtain the desired conclusion that $\|S(a)\|_{L^1} \leq c$ for some constant $c$ which is the same for every atom $a$. We have thus finished the proof of the converse part of Theorem 1.

**2. The Calderón-Zygmund decomposition on product domain.** We shall now state the analogue for the product domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ of the Calderón-Zygmund decomposition.

CALDERÓN-ZYGMUND LEMMA. *Let $\alpha > 0$ be given and $f \in L^p(\mathbf{R}^2)$, $1 < p < 2$. Then we may write $f = g + b$ where $g \in L^2(\mathbf{R}^2)$ and $b \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ with $\|g\|_2^2 \leq \alpha^{2-p}\|f\|_p^p$; and $\|b\|_{H^1} \leq c\alpha^{1-p}\|f\|_p^p$, where $c$ is a universal constant.*

*Remarks.* (1) We will actually prove that there exist constants $\lambda_k$ and atoms $b_k$ with $\Sigma|\lambda_k| \leq \alpha^{1-p}\|f\|_p^p$ and $f = g + \Sigma\lambda_k b_k$. The conclusion that $b = \Sigma\lambda_k b_k$ is in $H^1$ then follows from Theorem 1.

(2) The proof represented below for the Lemma is a slight variation of the proof for the atomic $H^1$ decomposition in [1]. We include it here for the sake of completeness.

*Proof.* For the fix $\alpha > 0$, let $\Omega_k = \{x : S(f)(x) > \alpha \cdot 2^k\}$ and let $\mathcal{R}_0 = \{$all dyadic rectangle $R$ so that $|R \cap \Omega_0| < 1/2\,|R|.\}$ Let

$$f_R(x) = \iint_{R_+} f(t, y)\Omega_y(x - t)\frac{dt\,dy}{y_1 y_2}$$

where $R_+$ is the region defined $w, r, t.$ $R$ as in the proof of section 1. Let $g = \Sigma_{R \in \mathcal{R}_0} f_R$, then as in [1], we can estimate the $L^2$-norm of $g$ by duality. If $\|h\|_2 = 1$, then

$$\int_{\mathbf{R}^2} g(x)h(x)dx = \int_{\mathbf{R}^2}\left(\sum_{R \in \mathcal{R}_0} f_R(x)\right)h(x)dx$$

$$= \sum_{R\in\mathcal{R}_0} \int_{\mathbf{R}^2} \iint_{R_+} f(t, y)\psi_y(x-t)\frac{dt\,dy}{y_1y_2}h(x)dx$$

$$= \iint_{\substack{\cup R_+\\ R\in\mathcal{R}_0}} f(t, y)h(t, y)\frac{dt\,dy}{y_1y_2}$$

$$\leq \left(\iint_A |f(t, y)|^2\frac{dt\,dy}{y_1y_2}\right)^{1/2}\left(\iint_A |h(t, y)|^2\frac{dt\,dy}{y_1y_2}\right)^{1/2}$$

where we have set

$$A = \bigcup_{R\in\mathcal{R}_0} R_+$$

But

$$\iint_{(\mathbf{R}^2_+)^2} |h(t, y)|^2\frac{dt\,dy}{y_1y_2} = \int_{\mathbf{R}^2} S^2(h)(x)dx \leq c\|h\|_2^2 \leq c.$$

As for $\iint_A |f(t, y)|^2(dt\,dy/y_1y_2)$, we claim that

$$\int_{S(f)(x)\leq\alpha} S^2(f)dx \geq c\iint_A |f(t, y)|^2\frac{dt\,dy}{y_1y_2}$$

We have

$$\int_{S(f)\leq\alpha} S^2(f)dx = \int_{S(f)\leq\alpha}\left(\int_{\Gamma(x)} |f(t, y)|^2\frac{dt\,dy}{y_1^2y_2^2}\right)dx$$

$$\geq \iint |f(t, y)|^2 m\{x\,|\,(t, y)\in\Gamma(x), S(f)(x)\leq\alpha\}\frac{dt\,dy}{y_1^2y_2^2}.$$

By construction, however if $(t, y)\in A$, then $m\{x:(t, y)\in\Gamma(x), S(f)(x)\leq\alpha\} \geq \frac{1}{2}y_1y_2$, so this last integral $\geq c\iint_A |f(t, y)|^2(dt\,dy/y_1y_2)$.

Now in turn we must estimate $\int_{S(f)\leq\alpha} S^2(f)(x)dx$. Clearly

$$\int_{S(f)\leq\alpha} S^2(f)(x)dx \leq \alpha^{2-p}\int_{\mathbf{R}^2} S^p(f)(x)dx \leq c\alpha^{2-p}\|f\|_p^p$$

and this gives the desired estimate for $g$.

To define the $b_k$, let $\mathcal{R}_k = \{$all dyadic rectangle $R$ with $|R \cap \Omega_{k-1}| \geq \frac{1}{2}|R|$ but $|R \cap \Omega_k| < \frac{1}{2}|R|\}$. Let $\tilde{b}_k = \Sigma_{R \in \mathcal{R}_k} f_R$. Let $A_k = \bigcup_{R \in \mathcal{R}_k} R_+$. Then if $c_R = (\iint_{R+} |f(t, y)|^2 (dt dy/y_1 y_2)^{1/2}$ we see that $\Sigma_{R \in \mathcal{R}_k} c_R^2 \leq \iint_{A_k} |f(t, y)|^2 (dt dy/y_1 y_2)$. And since for any $(t, y) \in A_k$, $m\{x \in R^2, M_S(\chi_{\Omega_{k-1}})(x) > \frac{1}{2}, x \notin \Omega_k$ and $(t, y) \in \Gamma(x)\} \geq \frac{1}{2} y_1 y_2$. We have

$$\iint_{A_k} |f(t, y)|^2 \frac{dt dy}{y_1 y_2} \leq \iint_{\{M(\chi_{\Omega_{k-1}}) > 1/2\} \setminus \Omega_k} S^2(f)(x) dx \leq c|\Omega_{k-1}|(2^k \alpha)^2.$$

Hence $b_k = \tilde{b}_k / 2^k \alpha |\Omega_{k-1}|$ is an atom and so for $\lambda_k = 2^k \alpha |\Omega_{k-1}|$ we have $f = g + \Sigma \lambda_k b_k$ where

$$\Sigma \lambda_k \leq \int_{S(f) > \alpha} S(f)(x) dx \leq \alpha^{1-p} \int S^p(f) dx \leq c \alpha^{1-p} \| f \|_p^p.$$

And where $\| g \|_2^2 \leq \alpha^{2-p} \| f \|_p^p$.

As a trivial corollary to the Calderón-Zygmund decomposition, we obtain the following:

THEOREM 2. *Let $T$ be a linear operator which is bounded from $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ to $L^1(\mathbf{R}^2)$ and bounded on $L^2(\mathbf{R}^2)$. Then $T$ is bounded on $L^p(\mathbf{R}^2)$ for all $1 < p < 2$.*

*Proof.* Let $f \in L^p(\mathbf{R}^2)$ and $\alpha > 0$. According to the Calderón-Zygmund Lemma, we may write $f = g + b$ where $\| g \|_2^2 \leq \alpha^{2-p} \| f \|_p^p$ and $\| b \|_{H^1} \leq c \alpha^{1-p} \| f \|_p^p$. Then

$$\begin{aligned} m\{|Tf| > \alpha\} &\leq m\{|Tg| > \alpha/2\} + m\{|Tb| > \alpha/2\} \\ &\leq c\left(\frac{1}{\alpha^2} \| Tg \|_2^2 + \frac{1}{\alpha} \| Tb \|_1\right) \\ &\leq c\left(\frac{1}{\alpha^2} \| g \|_2^2 + \frac{1}{\alpha} \| b \|_{H^1}\right) \\ &\leq c \frac{1}{\alpha^p} \| f \|_p^p \end{aligned}$$

$T$ is therefore weak-type $(p, p)$ for $1 < p < 2$ and is bounded on $L^p$ in the same range of $p$ by the Marcinkiewicz Theorem.

Notice that the double Hilbert transform $Tf = H_{x_1}H_{x_2}f$ obviously satisfies the condition in Theorem 2. The theorem is thus a generalization of the classical M. Riesz Theorem to the setting of the product domain.

**3. Atomic decomposition for $H^p$, $0 < p < 1$.** We wish to make a few final remarks concerning $H^p$ for $p < 1$. In particular, indicate that our methods extend to give the atomic decomposition for $H^p$ for $0 < p < 1$. Notice that for the same reasons as in the classical upper half plane, we need some higher orders vanishing property of a $p$-atom for $H^p$-decomposition (c.f. Coifman-Weiss [2]).

*Definition.* A $p$-atom is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set $\Omega$ of finite measure such that

(1) $\|a\|_2^2 \leq |\Omega|^{1-2/p}$

(2) $a$ can be further decomposed into "elementary particles" $\alpha_R$ as follows as in the case of an atom, with $a = \Sigma_{R\in\Omega}\, a_R$ and

i) $\int_I a_R(x_1, \tilde{x}_2)x_1^k dx_1 = \int_J a_R(\tilde{x}_1, x_2)x_2 dx_2 = 0$ for each $\tilde{x}_1 \in I$, $\tilde{x}_2 \in J$ and $= 0, 1, 2, \ldots, k(p)$, where $k(p)$ is an integer depending on $p$ $(k(p) \leq [2/p - 3/2])$.

ii) $a_R$ is a $c^m$ function with $\|a_R\|_\infty \leq d_R$

$$\left\| \frac{\partial^m a_R}{\partial x_1} \right\|_\infty \leq \frac{d_R}{|I|^m}, \qquad \left\| \frac{a^m a_R}{\partial x_2} \right\|_\infty$$

$$\leq \frac{d_R}{|J|^m} \qquad m \leq k(p) + 1$$

with $\Sigma d_R^2 |R| \leq A|\Omega|^{1-2/p}$.

With this definition of $p$-atoms, we can then state the parallel result of Theorem 1 for $H^p$ atomic decomposition.

THEOREM 3. *If $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, then we may write $f = \Sigma_k \lambda_k a_k$ where $\lambda_k$ are constants with $\Sigma\lambda_k^p \leq c_p \|f\|_{H^p}^p$ and $a_k$ are $p$-atoms.*

*Proof.* When $f$ is in $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, with the same notation as in previous sections, we let $\tilde{a}_k = \Sigma_{R\in\mathcal{R}_k} f_R$ and say that the elementary particle bound of $f_R$ is $d_R$. Then $\Sigma_{R\in\mathcal{R}_k} d_R^2|R| \leq 2^{2k}|\Omega_k|$. If we set $a_k = \tilde{a}_k/2^k|\Omega_k|^{1/p}$ and $\lambda_k = 2^k|\Omega_k|^{1/p}$ then $\Sigma\lambda_k^p \leq c_p\|f\|_{H^p}^p$, while $\|a_k\|_{L^2}^2 \leq |\Omega_k|^{1-2/p}$. To prove that $f_R$ satisfies the higher moments vanishing property, we need only require the same property on the kernel $\psi$

defining the area function $S$. (Since the $S$-function characterization of $H^p$ is independent of the choice of $\psi$, this could be done.) Thus $a_k$ is a $p$-atom.

We also should remark that a $p$-atom $a$ is in $H^p$ with $H^p$-norm essentially bounded by 1. To see this, we will only slightly vary the proof in section 1. If $a$ is a $p$-atom living in $\Omega$, $a = \Sigma a_R$, $a_R$ having bound $d_R$, we claim that the estimate (*) in section 1 could be modified to the following estimate (**).

(**) For each point $x$ satisfies $M(\chi_{\tilde{\Omega}})(x) < \frac{1}{4}$

$$S^2(a)(x) \leq cM^{k+2-1/r}(\chi_\Omega)(x)(\sum_R d_R^2 M^{2/r}(\chi_R)(x))$$

where $k = k(p)$ is the index appeared in the definition of $p$-atoms. To see how the index $k$ occurs in the estimate (**), we can trace back the proof of (*) to check that the index $k$ actually appears in each of the estimates for type (ii), (iii), (iv) rectangles there. We will skip some details and only indicate the proof for type (ii) rectangles here. Suppose $M(\chi_{\tilde{\Omega}})(x) < 1/4$, $S \in S_x$, $S = I \times J$ and $R = I_R \times J_R$ with $\tilde{R} \cap \tilde{S} \neq \phi$, and $|I_R| \leq |I|$ while $|J_R| \geq |J|$. Then for $(t, y) \in S_+$, *we have*

$$|a_R(t, y)| = \left| \iint a_R(\alpha, \beta)\psi_{y_1}(t_1 - \alpha)\psi_{y_2}(t_2 - \beta)d\alpha d\beta \right|$$

$$= \int_{\tilde{J}}\int_{\tilde{I}_R} \left( a_R(\alpha, \beta) - a_R(\alpha, \beta_J) - (\beta - \beta_J)\frac{\partial a_R}{\partial x_2}(\alpha, \beta_J) - \cdots - (\beta - \beta_J)^k \frac{\partial^k a_R}{\partial x_2}(\alpha, \beta_J)\right)$$

$$\cdot(\psi_{y_1}(t_1 - \alpha) - \psi_{y_1}(t_1 - \alpha_I) - (\alpha - \alpha_I)\psi'_{y_1}(t_1 - \alpha_I) - \cdots - (\alpha - \alpha_I)^k\psi^{(k)}_{y_1}(t_1 - \alpha_I))$$

$$\cdot\psi_{y_2}(t_2 - \beta)d\alpha d\beta$$

$$\leq c\left( \frac{d_R}{|J_R|^{k+1}} |J|^{k+1} \cdot \frac{|I_R|^{k+1}}{|I|^{k+2}} \cdot \frac{1}{|J|}\right)|J||I_R|$$

$$\leq c\left(\frac{|I_R||J|}{|I||J_R|}\right)^{k+1} d_R \left|\frac{I_R}{I}\right| = cm^{k+1}(S, R)d_R\left|\frac{I_R}{I}\right|.$$

$$\leq cm^{k+1}(S, R) d_R M^{1/r}(\chi_R)(x) \left| \frac{I_R}{I} \right|^{1-r} \quad \text{for any} \quad r > 1$$

where $\alpha_I$, $\beta_J$ are centers of $\tilde{I}_R$, $\tilde{J}$ respectively, and $c$ is a constant depending on $\| \psi^{(k+1)} \|_\infty$. From this point on, the same proof as in the estimate (*) works. Thus

$$\int_{\{M(\chi_{\tilde{\Omega}})(x) < 1/4\}} S^p(a)(x) dx$$

$$\leq \int M^{p/2(k+2-1/r)}(\chi_\Omega)(x) (\sum_R d_R^2 M^{2/r}(\chi_R)(x))^{p/2} dx$$

$$\leq \left( \int M^{p/2(k+2-1/r)s}(\chi_\Omega)(x) dx \right)^{1/s} \cdot \left( \int (\sum_R d_R^2 M^{2/r}(\chi_R)(x))^{p/2 \cdot t} dx \right)^{1/t}$$

where $s$, $t$ are any constants satisfying $1/s + 1/t = 1$. If we choose $t$ so that $(p/2)t = 1$, choose $r$ so that $1 < r < 2$; and finally choose $k$ so that $p/2(k + 2 - 1/r)s > 1$. Then

$$\int_{\{M(\chi_{\tilde{\Omega}})(x) < 1/4\}} S^p(a)(x) dx \leq \left( \int M^{p/2(k+2-1/r)s}(\chi_\Omega)(x) dx \right)^{1/s} (\sum_R d_R^2 |R|)^{1/t}$$

$$\leq c |\Omega|^{1/s} |\Omega|^{(1-2/p)1/t}$$

$$= c |\Omega|^{1-1/t} |\Omega|^{1/t-1} = c.$$

Notice that when $pt = 2$

$$\int_{\{M(\chi_{\tilde{\Omega}})(x) > 1/4\}} S^p(a)(x) dx \leq c \left( \int S^{pt}(a)(x) dx \right)^{1/t} |\Omega|^{1/s}$$

$$\leq c(\|a\|_2^{2/t}) |\Omega|^{1-1/t}$$

$$\leq c(\|a\|_2^2)^{p/2} |\Omega|^{1-p/2}$$

$$\leq c |\Omega|^{(1-2/p)p/2} |\Omega|^{1-p/2} \leq c.$$

We have thus verified that for an atom $a$, $\|a\|_{H^p} \simeq \|S(a)\|_p \leq c$. It remains to find the least lower bound for $k$ i.e. The smallest $k$ which allow the existence of some $1 < r < 2$, $1/s + 1/t = 1$, $pt = 2$ so that $p/2(k + 2 - 1/r)s > 1$. Simple calculation shows that $k = [2/p - 3/2]$ would be sufficient, where $[x]$ denotes the greatest integer $\leq x$. It remains open whether this choice of $k = k(p)$ is the best possible for $H^p$-atomic decomposition.

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