A continuous version of duality of H^1 with BMO on the bidisc

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Introduction

The purpose of this paper is to give the atomic decomposition for the space H^1 of the polydisc and to obtain a boundary characterization of its dual space. To begin with, let us mention some of the history of these problems. The one variable version of the H^1 theory which we extend here to the case of product domains has, of course, a very rich and extensive history. But for our purposes here we wish to single out three main one variable results of a modern flavor. These are the duality of H^1 and BMO (Charles Fefferman [1]), the atomic decomposition of H^1 (Coifman [2] and Latter [3]) and the inequality of John-Nirenberg [4] for BMO functions. Now, in order to extend these results to the polydisc, one would probably think that it would be a routine matter of iterating one dimensional methods mentioned above. Perhaps the best illustration of why this is *not* the case is given by a counterexample of L. Carleson [5]. In trying to find a simple characterization of functions φ which are in the dual of H^1 of the polydisc, one would be tempted to look at the one-variable definition of BMO involving the expression

$$\frac{1}{|I|}\int_I |\varphi(x) - \varphi_I|\,dx$$

(here I is an interval, φ_I is the mean value of φ over I) and simply introduce an analogous expression involving some sort of mean-oscillation over *rectangles* rather than intervals. The effect of Carleson's work is to show that this approach fails. The resulting function $\varphi(x, y)$ may not act continuously on H^1 on the bidisc.

Despite this, all hope is not lost, but the situation becomes considerably more complicated. The results of the authors in [6] and [7] characterize those functions φ which are in the dual of H^1 of the bidisc in terms of the double Poisson integral of the function φ. In this theory the role in one dimension

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played by intervals was played not by rectangles but arbitrary open sets. The question of how to characterize functions in the dual of H^1 directly without considering Poisson integrals remained open. In this paper, we settle this question.

Let us discuss briefly the results of the paper in relation to their analogous one dimensional results.

1. The atomic decomposition for H^1 functions

In one variable, if $f(x)$ is a real valued function in $H^1(\mathbf{R}^1)$ then $f(x)$ can be written

$$f(x) = \sum \lambda_k a_k(x)$$

where $\sum |\lambda_k| \leqq C\|f\|_{H^1}$ and $a_k(x)$ are particularly simple functions called "atoms". An atom is a function $a(x)$ supported in an interval I such that $\int_I a(x)dx = 0$ and $|a(x)| \leqq 1/|I|$. (See Coifman [2], Latter [3].) This decomposition is intimately connected to the duality of H^1 with BMO and has a number of interesting applications. An analogous decomposition for functions f defined on $\mathbf{R}^2$ which are boundary values of functions in $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ is obtained in Section 1 of this paper. Such an f is written as $\sum \lambda_k a_k(x, y)$ where again $\sum |\lambda_k| \leqq C\|f\|_{H^1}$ and a_k is an atom. But in the product case an atom a is supported in an open set $\Omega \subset \mathbf{R}^2$ such that $\|a\|_{L^2} \leqq 1/|\Omega|^{1/2}$,

$$\int_I a(x_1, x_2)dx_1 = 0 \ , \qquad \int_J a(x_1, x_2)dx_2 = 0$$

where I is any component interval of the open set of $\mathbf{R}^1$ obtained by slicing Ω at height x_2 and J is defined similarly by slicing Ω vertically instead of horizontally. In addition the atom must have the stronger property of being itself decomposable into even simpler "elementary particles" which are functions $e(x_1, x_2)$, rather smooth, supported in *rectangles* $R = I \times J \subset \Omega$ so that

$$\int_I e(x_1, x_2)dx_1 = \int_J e(x_1, x_2)dx_2 = 0 \ .$$

2. Duality of H^1 and BMO

The problem of finding a description of those functions $\varphi(x)$ on $\mathbf{R}^1$ which act as bounded functionals on H^1 was solved by C. Fefferman in his fundamental work [1]. The answer as the reader knows, is that $\varphi \in \mathrm{BMO}(\mathbf{R}^1)$, i.e.,

$$\frac{1}{|I|}\int_I |\varphi(x) - \varphi_I|dx \leqq M \qquad \text{for all intervals} \quad I \subset \mathbf{R}^1 \ .$$

What is an analogue of this result for the product case? A function φ on $\mathbf{R}^2$ will be in the dual of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ if and only if for every open set

$\Omega \subset \mathbf{R}^2$

$$\frac{1}{|\Omega|}\int_\Omega |\varphi - \varphi_\Omega|^2 dxdy \leqq M$$

where φ_Ω is a function which can be described *completely in terms of the geometry of* Ω. (In the one variable case $\Omega = I$ is an interval and φ_Ω is any function which is constant on Ω.) More specifically φ_Ω is a sum of smooth functions $g(x, y)$ living in rectangles $R = I \times J$ such that most of R lies outside Ω and

$$\int_I g(x_1, x_2)dx_1 = \int_J g(x_1, x_2)dx_2 = 0 .$$

There is another way of characterizing the dual of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, which is motivated by ideas from double Haar expansions. In the one variable case, if I is a dyadic interval and $\varphi(x)$ a function on $\mathbf{R}^1$ then φ can be expanded in a Haar series, $\varphi = \sum \varphi_I$. If I_0 is a fixed dyadic interval then $\sum_{I\subset I_0} \varphi_I(x) = (\varphi(x) - c_{I_0})\chi_{I_0}$ where c_{I_0} is the mean value of φ over I_0 so that the BMO condition for I_0 reads $\|\sum_{I\subset I_0} \varphi_I\|_2^2 \leqq M|I_0|$. In Section 1, we define a *non-dyadic* version of the double Haar expansion $\varphi = \sum \varphi_R$ where R is a dyadic rectangle (φ_R will behave much like a Haar term—it will live on the triple of R, have mean value 0 over each horizontal and vertical segment of R, and the different φ_R will be "almost orthogonal"). We then prove that φ is in the dual of H^1 if and only if

$$\|\textstyle\sum_{R\subset\Omega} \varphi_R\|_2^2 \leqq M|\Omega| \qquad \text{for every open set} \quad \Omega \subset \mathbf{R}^2 .$$

3. The John-Nirenberg inequality

In the classical case for a function φ in BMO$(\mathbf{R}^1)$, we have

$$\frac{1}{|I|}\int_I |\varphi(x) - \varphi_I|^p dx \leqq M_p$$

not only for $p = 1$ (the definition) but for all $p < \infty$. In fact $\frac{1}{|I|}\int_I e^{c|\varphi(x)-\varphi_I|}dx \leqq M$ where c is a small constant depending only on the BMO norm of φ.

We show in Section 3 that for φ in the dual of H^1 we have $\|\sum_{R\subset\Omega} \varphi_R\|_p^p \leqq C_p M|\Omega|$ for all open $\Omega \subset \mathbf{R}^2$ and in fact we show the sharp result, namely, that $\sum_{R\subset\Omega} \varphi_R$ belongs to the class $e^{\sqrt{L}}$ of functions Q, for which $e^{c\sqrt{Q}}$ is integrable locally.

At this point, we wish to refer the reader to the work of A. Bernard [8] where the atomic decomposition and duality of H^1 and BMO spaces is obtained in the context of double martingales. Finally, the authors would

like to thank L. Carleson and E. M. Stein for some inspiring conversations about BMO and C. Coifman and Y. Meyer for some incisive comments which motivated our work.

0. Preliminaries

In what follows, we shall work exclusively with the domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ and its distinguished boundary, $\mathbf{R}^2$. A point of $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ will be denoted (t, y) where $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2)$, $y_i \geqq 0$, $i = 1, 2$. We shall often use the following notation: $\psi(t)$ will be a C^1 function on $\mathbf{R}^1$ supported on $[-1, +1]$ with ψ even and $\int_{-1}^{+1} \psi(t)dt = 0$; if $y > 0$, $\psi_y(t) = (1/y)\psi(t/y)$ and if $y = (y_1, y_2)$ and $t = (t_1, t_2) \in \mathbf{R}^2$ then $\psi_y(t) = \psi_{y_1}(t_1) \cdot \psi_{y_2}(t_2)$. If f is a function defined on $\mathbf{R}^2$ then $f(t, y)$ will, by definition mean $f * \psi_y(t)$. Further, if $x = (x_1, x_2) \in \mathbf{R}^2$, $\Gamma(x)$ will denote the product cone, $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$, where for $i = 1, 2$

$$\Gamma(x_i) = \{(t_i, y_i) \in \mathbf{R}^2_+ \,|\, |x_i - t_i| < y_i\} .$$

Given a function f on $\mathbf{R}^2$ we define its double S-function by

$$S^2(f)(x) = \iint_{\Gamma(x)} |f(t, y)|^2 \frac{dtdy}{y_1^2 y_2^2} .$$

Then it is a fact that for $1 < p < \infty$,

$$||S(f)||_p \leqq C_p ||f||_p$$

(see [9] and [10]). If $(t, y) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+$, then $R_{t,y}$ will denote the rectangle centered at $t \in \mathbf{R}^2$ whose side lengths are y_1 and y_2. If $\Omega \subset \mathbf{R}^2$ is open, then $\{(t, y) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ \,|\, R_{t,y} \subset \Omega\}$ will be called $S(\Omega)$.

Finally, we should say a few words about the definition of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$. Although we consider this at first as the set of boundary value functions on $\mathbf{R}^2$ of biholomorphic functions on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$, the recent work of Gundy-Stein [9] shows that the various definitions via area integrals and maximal functions are equivalent. For example, we could define $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ as the class of all functions on $\mathbf{R}^2$ for which $A(f) \in L^1(\mathbf{R}^2)$ where

$$A^2(f)(x) = \iint_{\Gamma(x)} |\nabla_1 \nabla_2 u(t, y)|^2 dtdy$$

and u is the multiple Poisson integral of f. Finally if $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ then $S(f) \in L^1(\mathbf{R}^2)$. (See [9] and [10].)

I. The atomic decomposition for $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$

The most elementary construction of our program is the decomposition of a function f in $H^1((\mathbf{R}^2_+)^2)$ into atoms. In the classical case, say on $\mathbf{R}^1$ (C.

Fefferman-E. M. Stein [11], Coifman [2], Latter [3]), an atom is a function $a(x)$ supported on an interval I such that

$$\int_I a(x)dx = 0 \qquad \text{and} \qquad \|a\|_\infty \leqq \frac{1}{|I|} .$$

On the polydisk things are a bit more complicated. There we make the following definition:

Definition. An "atom" is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set, Ω, of finite measure such that:

(1)
$$\|a\|_{L^2} \leqq \frac{1}{|\Omega|^{1/2}} ,$$

(2)
$$\int_I a(x_1, x_2)dx_1 = 0$$

where I is any component interval of a set of the form $\{x_1 \in \mathbf{R}^1 \mid (x_1, x_2) \in \Omega\}$ (where x_2 is fixed), i.e., a has mean 0 over every component interval of every x_1-cross section of Ω.

(3)
$$\int_J a(x_1, x_2)dx_2 = 0$$

where J is any component interval of a set of the form $\{x_2 \mid (x_1, x_2) \in \Omega\}$.

(4) a can be further decomposed into "elementary particles" a_R as follows:

(i) $a = \sum_R a_R$, where a_R is supported in a rectangle $R \subset \Omega$ (say $R = I \times J$, and the R in the sum have the property that no one R is contained in the triple of any other).

(ii)
$$\int_I a_R(x_1, x_2)dx_1 = 0 , \qquad \text{for each} \quad x_2 \in J ,$$
$$\int_J a_R(x_1, x_2)dx_2 = 0 , \qquad \text{for each} \quad x_1 \in I .$$

(iii) a_R is C^1 with $\|a_R\|_\infty \leqq |R|^{1/2}$,

$$\left\|\frac{\partial a_R}{\partial x_1}\right\|_\infty \leqq \frac{C_R}{|I|\,|R|^{1/2}} , \qquad \text{and} \qquad \left\|\frac{\partial a_R}{\partial x_2}\right\|_\infty \leqq \frac{C_R}{|J|\,|R|^{1/2}} ,$$

and

$$\left\|\frac{\partial^2 a_R}{\partial x_1 \partial x_2}\right\|_\infty \leqq \frac{C_R}{|R|^{3/2}} \qquad \text{with} \quad \sum_R C_R^2 \leqq \frac{A}{|\Omega|}$$

(A is an absolute constant).

With this definition we are ready to give the result of this section.

THEOREM. *Let $f \in H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$. Then f can be written as $f = \sum \lambda_k a_k$ where a_k are atoms and $\lambda_k \geqq 0$ satisfy*

(1.1) $$\sum \lambda_k \leqq A \| f \|_{H^1} .^{*)}$$

Proof. Let $\psi(x)$ be defined on $\mathbf{R}^1$, a C^1 function supported on $[-1, +1]$, even, with mean value 0. If $y = (y_1, y_2)$, $y_i > 0$ and $x = (x_1, x_2)$, we set $\psi_y(x) = (1/y_1y_2)\psi(x_1/y_1)\psi(x_2/y_2)$, and define $f(x, y)$ by $f(x, y) = f * \psi_y(x)$. We shall normalize ψ so that

(1.2) $$\int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = 1 .$$

Then we have the formula $\left(\text{here we assume without loss of generality that } \int f(x_1, x_2)dx_1 = \int f(x_1, x_2)dx_2 = 0\right)$

(1.3) $$f(x) = \iint_{(t,y)\in(\mathbf{R}^2_+ \times \mathbf{R}^2_+)} f(t, y)\psi_y(x - t)\frac{dtdy}{y_1y_2} .$$

In fact, to see this, we need only take the Fourier transform of the right-hand side which is

$$\begin{aligned}\int_{y_1, y_2>0} \widehat{f(\cdot, y) * \psi_y}(\xi)\frac{dy_1dy_2}{y_1y_2} &= \int_{y_1, y_2>0} \hat{f}(\xi)\hat{\psi}(y\xi)\hat{\psi}(y\xi)\frac{dy_1dy_2}{y_1y_2} \\ &= \hat{f}(\xi)\int_{y_1, y_2>0} |\hat{\psi}(y\xi)|^2\frac{dy_1dy_2}{y_1y_2} \\ &= \hat{f}(\xi)\int_{y_1>0} |\hat{\psi}(y_1\xi_1)|^2\frac{dy_1}{y_1}\cdot\int_{y_2>0} |\hat{\psi}(y_2\xi_2)|^2\frac{dy_2}{y_2} \\ &= \hat{f}(\xi) .\end{aligned}$$

Now consider the double S-function given by

$$S^2(f)(x_1, x_2) = \iint_{\Gamma(x_1)\times\Gamma(x_2)} |f(t_1, t_2, y_1, y_2)|^2 \frac{dt_1dt_2dy_1dy_2}{y_1^2y_2^2} .$$

Since $f \in H^1(\mathbf{R}^2_+)^2$, we have $S(f) \in L^1(\mathbf{R}^2)$. Also, set $\Omega_k = \{S(f) > 2^k\}$ and consider the collection R_k of all dyadic rectangles $R = I \times J$ with the property that

$$|R \cap \Omega_{k+1}| < \frac{1}{2}|R| \quad \text{and} \quad |R \cap \Omega_k| \geqq \frac{1}{2}|R| .$$

For each $R \in R_k$ let

(1.4) $$f_R(x) = \iint_{(t,y)\in R_+} f(t, y)\psi_y(x - t)\frac{dtdy}{y_1y_2}$$

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*) Since this was written the authors have established that this atomic decomposition *characterizes* H^1. That is, in addition to the theorem above, we have that if $\lambda_k \geqq 0$, $\sum \lambda_k < \infty$, and a_k are atoms, then

$$f = \sum \lambda_k a_k \in H^1 \quad \text{and} \quad \|f\|_{H^1} \leqq C \sum \lambda_k .$$

Details will appear elsewhere.

where

$$R_+ = \left\{(t, y) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ \,|\, t \in R, \frac{|I|}{2} < y_1 \leqq |I|, \frac{|J|}{2} < y_2 \leqq |J|\right\} .$$

Now $f(t, y) = (f * \psi_y)(t)$, ψ is defined as in Section 0. Then we intend to show that if $f_k = \sum_{R \in R_k} f_R$ and $\tilde{\Omega}_k = \{M_S(X_{\Omega_k}) > 1/100\}$ (M_S is the strong maximal operator) then f_k is supported in $\tilde{\Omega}_k$; also if $a_k = f_k/2^k|\tilde{\Omega}_k|$ then a_k is an atom. Then from (1.3), $f = \sum f_R = \sum f_k$, and $\sum 2^k|\tilde{\Omega}_k| \leqq A\|f\|_{H^1}$, which will establish our result.

To show that $f_k/2^k|\tilde{\Omega}_k|$ is an atom on $\tilde{\Omega}_k$ we first observe that the support of $\psi_y(x - t)$ as a function of x is contained in the ten-fold dilation of a rectangle $R \in R_k$ provided $t \in R = I \times J$ and $|I|/2 \leqq y_1 < |I|$, $|J|/2 \leqq y_2 < |J|$. It follows that the support of $\psi_y(x - t)$ is contained in $\tilde{\Omega}_k$ and hence the same is true for f_k.

Moreover, if $\tilde{I}$ is any component interval of $\{x_1 | (x_1, x_2) \in \tilde{\Omega}_k\}$ with x_2 fixed, we may observe that

$$\int_{\tilde{I}} \psi_y(x_1 - t_1, x_2 - t_2)dx_1 = 0$$

since either

(1) the support of $\psi_y(x - t)$ is contained in a rectangle $R = I \times J$ for which $I \subset \tilde{I}$ or else $I \cap \tilde{I} = \varnothing$; or

(2) $x_2 \notin J$.

In case (1) $\int_{\tilde{I}} \psi_y(x_1 - t_1, x_2 - t_2)dx_1 = 0$ because $\int_I \psi_{y_1}(x_1 - t_1)dx_1 = 0$, and in case (2) there is nothing to show. Then

$$\begin{aligned}\int_{\tilde{I}} f_k(x_1, x_2)dx_1 &= \int_{\tilde{I}}\iint f(t, y)\psi_y(x - t)\,\frac{dtdy}{y_1y_2}\,dx_1 \\ &= \iint\left(\int_{\tilde{I}} \psi_y(x - t)dx_1\right) f(t, y)\,\frac{dtdy}{y_1y_2} \\ &= 0 .\end{aligned}$$

Now let us show that $\|f_k\|_2 \leqq A2^k|\Omega_k|^{1/2}$. The trick here is to use duality. In fact, suppose $\|g\|_2 = 1$. Then

$$\begin{aligned}(1.5)\qquad \left|\int f_k g(x)dx\right| &= \left|\int \sum_{R \in R_k} \iint_{(t,y) \in R_+} f(t, y)\psi_y(x - t)\,\frac{dtdy}{y_1y_2}\,g(x)dx\right| \\ &= \sum_{R \in R_k} \iint_{(t,y) \in R_+} |f(t, y)g(t, y)|\,\frac{dtdy}{y_1y_2} \\ &\leqq \left(\iint_{A_k} |f(t, y)|^2\frac{dtdy}{y_1y_2}\right)^{1/2}\left(\iint_{A_k} |g(t, y)|^2\frac{dtdy}{y_1y_2}\right)^{1/2} .\end{aligned}$$

Here A_k denotes the union over all rectangles $R \in R_k$ of sets of the form R_+.

Now

$$\iint_{(t,y)\in \mathbf{R}^2_+\times\mathbf{R}^2_+} |g(t,y)|^2 \frac{dtdy}{y_1y_2} \leqq \int_{\mathbf{R}^2} S^2(g)(x_1,x_2)dx_1dx_2$$
$$= \int_{\mathbf{R}^2} |g|^2 dx_1dx_2 = 1\,.$$

To control $\iint_{A_k} |f(t,y)|^2(dtdy/y_1y_2)$ we look at

$$\int_{\tilde{\Omega}_k-\Omega_{k+1}} S^2(f)(x_1,x_2)dx_1dx_2 \leqq (2^{k+1})^2|\tilde{\Omega}_k| \leqq A2^{2k}|\Omega_k|\,.$$

For each $(t,y)\in A_k$, the rectangle R_{ty} centered at t with side lengths y_1 and y_2 has at least a fixed fraction, c, of its area inside $\tilde{\Omega}_k$ and at the same time outside Ω_{k+1}. Therefore $\{x=(x_1,x_2)\,|\,(t,y)\in\Gamma(x_1)\times\Gamma(x_2)\}\cap(\tilde{\Omega}_k-\Omega_{k+1})$ has measure $\geqq cy_1y_2$ and

$$\int_{\tilde{\Omega}_k-\Omega_{k+1}} S^2(f)(x_1,x_2)dx_1dx_2$$
$$\geqq \iint |f(t,y)|^2 |\{(x\,|\,(t,y)\in\Gamma(x)),\, x\in\tilde{\Omega}_k-\Omega_{k+1}\}| \frac{dtdy}{y_1^2y_2^2}$$
$$\geqq \iint_{A_k} |f(t,y)|^2 \frac{dtdy}{y_1y_2}\,.$$

Finally, to establish that $f_k/2^k|\tilde{\Omega}_k|$ is an atom, we must verify the decomposition of this function into elementary particles.

We examine

$$f_R(x) = \iint_{(t,y)\in R_+} f(t,y)\psi_y(x-t)\frac{dtdy}{y_1y_2}\,.$$

This function is supported in the 3-fold dilation $\tilde{R}$ of R, has mean 0 over horizontal and vertical segments of R and since

$$\|\psi_y\|_\infty \leqq \frac{A}{|R|}\,,$$
$$\left\|\frac{\partial}{\partial x_1}[\psi_y(x-t)]\right\|_\infty \leqq \frac{A}{|I||R|}\,,$$
$$\left\|\frac{\partial}{\partial x_2}[\psi_y(x-t)]\right\|_\infty \leqq \frac{A}{|J||R|}$$

and

$$\left\|\frac{\partial^2\psi_y}{\partial x_1\partial x_2}\right\|_\infty \leqq \frac{A}{|R|^2}\,,$$

this will verify conditions 4 (ii) and (iii) if we get an estimate of the form

$$\sum_{R\in R_k}\left(\iint_{(t,y)\in R_+} |f(t,y)|^2\frac{dtdy}{y_1y_2}\right) \leqq A|\Omega_k|2^{2k}\,.$$

The left side of this inequality is clearly dominated by $\iint_{A_k} |f(t, y)|^2 (dtdy/y_1y_2)$ which, as we have already seen, satisfies the desired inequality. We should remark at this point that if the argument about is carried out in one dimension then the "atoms" are functions $a_k(x)$ on R^1 supported on open sets $\Omega_k = \bigcup_{j=1}^{\infty} I_{k_j}$ with I_{k_j} the component intervals of Ω_k and $\int_{I_{k_j}} a_k(x)dx = 0,\ k \geqq 1$. Then the atomic decomposition above says that if $f \in H^1(R^1)$ then $f = \sum \lambda_k a_k$ where a_k is an atom supported on Ω_k. If

$$a_{k_j}(x) = a_k(x) \cdot \chi_{I_{k_j}}(x) \Big/ \left(\int_{I_{k_j}} a_k^2(t)dt\right)^{1/2} |I_{k_j}|^{1/2} ,$$

then

$$f(x) = \sum_k \sum_j \lambda_k \left(\int_{I_{k_j}} a_k^2(t)dt\right)^{1/2} |I_{k_j}| a_{k_j}(x)$$

is a *classical* atomic decomposition of f into classical L^2 atoms supported on intervals. With only minor modifications, we obtain the extension to R^n, $n > 1$, by letting the Whitney decomposition of Ω replace the decomposition into component intervals. On the polydisc however, there is no neat way of decomposing an open set into its maximal subrectangles.

II. The space BMO and its duality with H^1

Recall that if φ is a locally integrable function on $\mathbf{R}$, then φ is of bounded mean oscillation (abbreviated as BMO($\mathbf{R}$)) if

$$\sup_I \frac{1}{|I|} \int_I |\varphi - \varphi_I|^2 dx = \|\varphi\|_*^2 < \infty , \tag{2.1}$$

where the supremum ranges over all finite intervals I in $\mathbf{R}$, and $\varphi_I = 1/|I| \int_I \varphi(x)dx$. C. Fefferman and E. Stein proved in [10] that BMO($\mathbf{R}$) is the dual space of $H^1(\mathbf{R}_+^2)$. In this section, we will propose two kinds of definitions about boundary behavior of BMO functions defined on $\mathbf{R}^2$. The main result is that both kinds of definitions characterize the dual space of $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$.

The first definition is motivated by the version of dyadic BMO defined on $\mathbf{R}^2$ (cf. the work of Bernard [8], Decomp [11]), where we use all the notations as introduced in Section 0.

Definition. $\mathrm{BMO}_{(a)}$ is the space of locally integrable functions φ defined on $\mathbf{R}^2$, satisfying:

$$\sup_\Omega \frac{1}{|\Omega|} \|\textstyle\sum_{R \subset \Omega} \varphi_R\|_2^2 = \|\varphi\|_*^2 < \infty ,$$

where the supremum ranges over all open sets Ω of finite measure in $\mathbf{R}^2$. And for each dyadic rectangle R, φ_R is defined with respect to φ as in (1.4).

The second definition is motivated by the work on atomic decomposition of H^1 in Section 1. It has the advantage of not depending on the particular function ψ, and also appears closer to the expression (2.1).

Definition. $\mathrm{BMO}_{(b)}$ is the space of locally integrable functions φ defined on $\mathbf{R}^2$ such that given any open set $\Omega \subset \mathbf{R}^2$, there exists a function $\tilde{\varphi}_\Omega$ so that

$$\frac{1}{|\Omega|}\int_\Omega |\varphi(t) - \tilde{\varphi}_\Omega(t)|^2 dt \leqq M \quad \text{for some} \quad M$$

independent of Ω where $\tilde{\varphi}_\Omega$ satisfies

(a) $\tilde{\varphi}_\Omega = \sum \tilde{\varphi}_i$; where each $\tilde{\varphi}_i$ is supported on the triple $\tilde{R}_i$ of distinct dyadic rectangles R_i with $|\tilde{R}_i \cap \Omega| < 1/2|\tilde{R}_i|$, and $\tilde{\varphi}_i$ has mean value zero over each horizontal and vertical segment of $\tilde{R}_i$.
Furthermore, if $R_i = I_i \times J_i$

(b) $$\|\varphi_i\|_\infty \leqq \frac{C_{R_i}}{|R_i|^{1/2}}, \qquad \left\|\frac{\partial\varphi_i}{\partial x_1}\right\|_\infty \leqq \frac{C_{R_i}}{|R_i|^{1/2}}\frac{1}{|I_i|},$$

$$\left\|\frac{\partial\varphi_i}{\partial x_2}\right\|_\infty \leqq \frac{C_{R_i}}{|R_i|^{1/2}}\frac{1}{|J_i|}, \quad \text{and} \quad \left\|\frac{\partial^2\varphi_i}{\partial x_1\partial x_2}\right\|_\infty \leqq \frac{C_{R_i}}{|R_i|^{3/2}}$$

for some C_{R_i}.

(c) $\sum_{|\tilde{R}_i \cap \Omega| \sim (1/2^k)|\tilde{R}_i|} C_{R_i}^2 \leqq c2^k \cdot k \cdot |\Omega|$ for each $k = 1, 2, \cdots$ and some absolute constant c.

The following theorem is our main result in this section.

THEOREM. *Assume $\varphi \in L^2(\mathbf{R}^2)$ satisfying*

$$\int \varphi(x_1, x_2)dx_2 = \int \varphi(x_1, x_2)dx_1 = 0$$

for all $(x_1, x_2) \in \mathbf{R}^2$. Then the following conditions on φ are equivalent:

(i) $\varphi \in \mathrm{BMO}_{(a)}$;

(ii) $\varphi \in \mathrm{BMO}_{(b)}$;

(iii) $\sup 1/|\Omega| \sum_{R\subset\Omega} S_R^2(\varphi) < \infty$, *where the supremum ranges over all finite open sets Ω in $\mathbf{R}^2$, and for each dyadic rectangle R,*

$$S_R^2(\varphi) = \iint_{R_+} |\varphi(t, y)|^2 \frac{dtdy}{y_1y_2};$$

(iv) *φ is in the dual of $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$.*

Remark. Condition (iii) could be reformulated as

$$\sup_\Omega \frac{1}{|\Omega|}\iint_{S(\Omega)} |\varphi(t, y)|^2 \frac{dtdy}{y_1y_2} < \infty,$$

where $S(\Omega)$ is the Carleson region associated with Ω (i.e., $S(\Omega) = \{(t, y): R_{t,y} \subset \Omega\}$). This condition is analogous to the Carleson measure condition appearing in [6], [7], [12].

The various implications in the above theorem will be proved in the order (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Proof of the theorem. To begin the proof of (i) $\Longrightarrow$ (iii), we need some elementary lemmas.

LEMMA 1. *Let ψ be an even, C^1 function supported on $[-1, 1]$ with mean value 0 as before. We set $\psi_y(x) = (1/y)\psi(x/y)$. Then for any s, t in $\mathbf{R}^1$, and $y \geqq z > 0$, we have*

$$\left|\int_{\mathbf{R}} \psi_y(x - s)\psi_z(x - t)dx\right| \leqq c\,\frac{z}{y^2} \tag{2.2}$$

where c is a constant depending only on $\|\psi\|_\infty$, $\|\psi'\|_\infty$.

Proof. After a simple dilation argument, to prove (2.2) one needs only to estimate an integral of the form $\int_{-1}^{1} \psi(x)\varphi(x)dx$ where $\varphi(x) = \psi((y/z)(x - r))$ for some $r \in \mathbf{R}$. Such an integral could be estimated in the obvious way by

$$\left|\int_{-1}^{1} (\psi(x) - m)\varphi(x)dx\right| \leqq 4\|\psi'\|_\infty \|\psi\|_\infty \frac{z^2}{y^2}\,,$$

where m can be chosen to be the value $\psi(r + z/y)$.

For each dyadic rectangle $R = I \times J$, let $\mathcal{G}_R = \{R_1\colon R_1$ is a dyadic rectangle with $\tilde{R}_1 \cap \tilde{R} \neq \emptyset\}$. For each $R_1 = I_1 \times J_1 \in \mathcal{G}_R$, let

$$r(R_1, R) = \left(\frac{\min(|I|, |I_1|)}{\max(|I|, |I_1|)}\,\frac{\min(|J|, |J_1|)}{\max(|J|, |J_1|)}\right)^{3/2}. \tag{2.3}$$

A simple integration of Lemma 1 then yields:

LEMMA 2. *Suppose $R_1 \in \mathcal{G}_R$. Then*

$$S_R^2(\varphi_{R_1}) \leqq c(r(R_1, R))^2 S_{R_i}^2(\varphi)$$

where c is any constant depending only on $\|\psi\|_\infty$, $\|\psi'\|_\infty$.

(i) $\Longrightarrow$ (iii). Assume (i) holds, and let $\|\varphi\|_*$ denote the $\mathrm{BMO}_{(\mathrm{a})}$ norm of φ, i.e., assume

$$\left\|\sum\nolimits_{R \subset \Omega} \varphi_R\right\|_2^2 \leqq \|\varphi\|_*^2 |\Omega| \quad \text{for all open sets} \quad \Omega \subset R^2\,. \tag{2.4}$$

Fix an open set Ω_0, let $\Omega = \bigcup_{R \subset \Omega_0} \tilde{R}$. For each integer $k \geqq 1$, let $\mathcal{G}_k = \{R_1\colon |\tilde{R}_1 \cap \Omega| \geqq (1/2^k)|\tilde{R}_1|\}$. Let $r_k = \mathcal{G}_k \backslash \mathcal{G}_{k-1}$, ($\mathcal{G}_0$ is the empty set), and $\Omega_k = \bigcup_{R_1 \in \mathcal{G}_k} R_1$. Finally, fix a big integer N which is to be chosen later, let $\varphi_1 = \sum_{R \subset \Omega_N} \varphi_R$, $\varphi_2 = \varphi - \varphi_1$. By our assumption (2.4), we have for φ_1,

$$\sum\nolimits_{R \subset \Omega_0} S_R^2(\varphi_1) \leqq \|\varphi_1\|_2^2 = \left\|\sum\nolimits_{R \subset \Omega_N} \varphi_R\right\|_2^2 \leqq \|\varphi\|_*^2 |\Omega_N| \leqq c_N \|\varphi\|_*^2 |\Omega| \tag{2.5}$$

where $c_N \sim 2^N \cdot N$ is a constant depending only on N.

To estimate φ_2, we then claim two things:

$$\sum_{R\subset\Omega_0} S_R^2(\varphi_2) \leqq c \sum_{R\subset\Omega_0} \sum_{R_1\in\mathcal{I}_R\setminus\mathcal{I}_N} r(R_1, R) S_{R_1}^2(\varphi)\,, \tag{2.6}$$

$$\sum_{R\subset\Omega_0} \sum_{R_1\in\mathcal{I}_R\setminus\mathcal{I}_N} r(R_1, R) S_{R_1}^2(\varphi) \leqq c \sum_{k=N+1}^{\infty} \left(\frac{1}{2^k}\right)^{3/2} \sum_{R_1\in r_k} S_{R_1}^2(\varphi) \tag{2.7}$$

where the c's are constants depending only on $\|\psi\|_\infty$, $\|\psi'\|_\infty$.

Since $\varphi_2 = \varphi - \varphi_1 = \sum_{R\subset\Omega_N} \varphi_R$, for each fixed $R \subset \Omega_0$, we have $\varphi_2(t, y) = \sum_{R_1\in\mathcal{I}_R\setminus\mathcal{I}_N} \varphi_{R_1}(t, y)$ when $(t, y) \in R_+$. Hence

$$\begin{aligned} S_R^2(\varphi_2) &\leqq \sum\sum_{R_1,R_2\in\mathcal{I}_R\setminus\mathcal{I}_N} S_R(\varphi_{R_1}) S_R(\varphi_{R_2}) \\ &\leqq c\sum\sum_{R_1,R_2\in\mathcal{I}_R\setminus\mathcal{I}_N} S_{R_1}(\varphi) S_{R_2}(\varphi) r(R_1, R) r(R_2, R) \\ &\leqq c\sum\sum_{R_1,R_2\in\mathcal{I}_R\setminus\mathcal{I}_N} (S_{R_1}^2(\varphi) + S_{R_2}^2(\varphi)) r(R_1, R) r(R_2, R) \\ &\leqq c\sum\sum_{R_1\in\mathcal{I}_R\setminus\mathcal{I}_N} S_{R_1}^2(\varphi) r(R_1, R) \sum_{R_2\in\mathcal{I}_R\setminus\mathcal{I}_N} r(R_2, R) \\ &\leqq c\sum_{R_1\in\mathcal{I}_R\setminus\mathcal{I}_N} S_{R_1}^2(\varphi) r(R_1, R)\,. \end{aligned} \tag{2.8}$$

The second step in the above inequalities follows from Lemma 2. The last step follows from the observation $\sum_{R_1\in\mathcal{I}_R} r(R_1, R) < \infty$. Adding up all the terms $R \subset \Omega_0$ over the estimate (2.8), we obtain (2.6).

If we rewrite

$$\sum_{R\subset\Omega_0} \sum_{R_1\in\mathcal{I}_R\setminus\mathcal{I}_N} r(R_1, R) S_{R_1}^2(\varphi) = \sum_{k=N+1}^{\infty} \sum_{R_1\in\mathcal{I}_R\cap r_k,\ R\subset\Omega_0} r(R_1, R) S_{R_1}^2(\varphi)\,,$$

then it is clear that (2.7) would follow from (2.9) below.

$$\text{If} \quad R_1 \in r_k \quad \text{then} \quad \sum_{R\subset\Omega_0;\ R_1\in\mathcal{I}_R} r(R_1, R) \leqq c\cdot\left(\frac{1}{2^k}\right)^{3/2}. \tag{2.9}$$

Inequality (2.9) is an easy consequence of the following simple geometrical arguments. Suppose $R_1 = I_1 \times J_1$, $R_1 \in r_k$; then among those $R \subset \Omega_0$, $R = I \times J$, such that $R_1 \in \mathcal{I}_R$, there are four types:

Type (1): $|I_1| \geqq |I|$, $|J_1| \leqq |J|$. Then

$$\frac{|I|}{3|I_1|}|\tilde{R}_1| \leqq |\tilde{R} \cap \tilde{R}_1| \leqq |\tilde{R}_1 \cap \Omega| \leqq \frac{1}{2^{k-1}}|\tilde{R}_1|\,.$$

Thus $|I_1| = 2^{k-3+n}|I|$ for some $n \geqq 0$. Yet for each fixed n, the number of such I's must be $\leqq 5\cdot 2^n$. As for $|J| = 2^m|J_1|$ for some $m \geqq 0$, for each fixed m, $\tilde{J} \cap \tilde{J}_1 \neq \varnothing$ implies that the number of such J is less than 5. Thus

$$\begin{aligned} \sum_{R\in\text{type (1)}} r(R_1, R) &\leqq \sum_{n,m\geqq 0} \left(\frac{1}{2^{n+m+k-3}}\right)^{3/2} \cdot 2^n \cdot 5^2 \\ &\leqq c\left(\frac{1}{2^k}\right)^{3/2}. \end{aligned}$$

Type (2): $|I_1| \leqq |I|$, $|J_1| \geqq |J|$. This could be handled similarly to type (1).

Type (3): $|I_1| \geqq |I|$, $|J_1| \geqq |J|$. Then

$$\frac{|I||J|}{9|I_1||J_1|}|\tilde{R}_1| \leqq |\tilde{R} \cap \tilde{R}_1| \leqq |\Omega \cap \tilde{R}_1| \leqq \frac{1}{2^{k-1}}|\tilde{R}_1| .$$

Hence $|R_1| = 2^{k-5+l}|R|$ for some $l \geqq 0$. Also for each fixed l, the number of such R's must be $\leqq 5 \cdot 2^l$. Thus we get the same estimate for $\sum_{R \in \text{type (3)}} r(R, R_1)$ as in type (1).

Type (4): $|I_1| \leqq |I|$, $|J_1| \leqq |J|$. Similar considerations as in type (3) give $|R| = 2^{k-5+l}|R_1|$ for some $l \geqq 0$. Also for each fixed l, the number of such R's is $\leqq 5^2$. Hence the result follows.

Combining our estimates in (2.5), (2.6), (2.7), and using $S_R^2(\varphi) \leqq 2(S_R^2(\varphi_1) + S_R^2(\varphi_2))$, we obtain

$$\text{(2.10)} \qquad \begin{aligned} \sum_{R \subset \Omega_0} S_R^2(\varphi) &\leqq c_N ||\varphi||_*^2 |\Omega_0| + c_0 \sum_{k=N+1}^{\infty} \left(\frac{1}{2^k}\right)^{3/2} \sum_{R \in r_k} S_R^2(\varphi) \\ &\leqq c_N ||\varphi||_*^2 |\Omega_0| + c_0 \sum_{k=N+1}^{\infty} \left(\frac{1}{2^k}\right)^{3/2} \sum_{R \subset \Omega_k} S_R^2(\varphi) , \end{aligned}$$

where c_N is a constant depending only on N, and c_0 is an absolute constant depending only on $||\psi||_\infty$, $||\psi'||_\infty$.

If we apply (2.10) to the open set Ω_k, and define, for each integer m, open sets $\Omega_{k,m}$ so that the relation of $\Omega_{k,m}$ to Ω_k is the same as Ω_m to Ω_0, then since $|\Omega_k| \leqq c_1 \cdot k \cdot 2^k |\Omega_0|$ with c_1 an absolute constant, we have

$$\text{(2.10)}' \qquad \begin{aligned} \sum_{R \subset \Omega_k} S_R^2(\varphi) \leqq\ & c_N ||\varphi||_*^2 \cdot c_1 \cdot k \cdot 2^k |\Omega_0| \\ &+ c_0 \sum_{m=N+1}^{\infty} \left(\frac{1}{2^m}\right)^{3/2} \sum_{R \subset \Omega_{k,m}} S_R^2(\varphi) . \end{aligned}$$

Substituting (2.10)′ back into (2.10), we get

$$\text{(2.11)} \qquad \begin{aligned} \sum_{R \subset \Omega_0} S_R^2(\varphi) \leqq\ & c_N ||\varphi||_*^2 |\Omega_0| \left(1 + c_0 c_1 \sum_{k=N+1}^{\infty} \left(\frac{1}{2^k}\right)^{1/2} \cdot k\right) \\ &+ c_0^2 \sum_{k,m=N+1}^{\infty} \left(\frac{1}{2^{k+m}}\right)^{3/2} \sum_{R \subset \Omega_{k,m}} S_R^2(\varphi) . \end{aligned}$$

Observe that for each k, m, $|\Omega_{k,m}| \leqq c_1 m 2^m |\Omega_k| \leqq c_1^2 k m 2^{k+m} |\Omega_0|$. Thus if we let $r = c_0 c_1 \sum_{k=N+1}^{\infty} (1/2^k)^{1/2} \cdot k$, and choose N sufficiently large so that $r < 1$, it is clear we can repeat the process (2.11) with respect to $\Omega_{k,m}$, $\Omega_{k,m,l}$, etc. recursively and obtain

$$\begin{aligned} \sum_{R \subset \Omega_0} S_R^2(\varphi) &\leqq c_N ||\varphi||_*^2 |\Omega_0| \left(\sum_{n=0}^{\infty} r^n\right) \\ &\leqq C ||\varphi||_*^2 |\Omega_0| \end{aligned}$$

for some constant C independent of Ω_0. This establishes (iii).

(iii) $\Rightarrow$ (ii). Suppose φ satisfies $\sup_\Omega 1/|\Omega| \sum_{R \subset \Omega} S_R^2(\varphi) \leqq M$. To prove

$\varphi \in \mathrm{BMO}_{(b)}$, we write $\varphi = \sum \varphi_R$, and for each open set $\Omega \subset \mathbf{R}^2$, choose $\tilde{\varphi}_\Omega = \sum \tilde{\varphi}_R$, where $\tilde{\varphi}_R = \varphi_R$ for each R such that $|\tilde{R} \cap \Omega| < (1/2)|\tilde{R}|$. Then as verified in Section 1, $\tilde{\varphi}_R$ satisfies conditions (a) and (b) in the definition of $\mathrm{BMO}_{(b)}$ where $C_R = cS_R(\varphi)$. Condition (c) can also be verified immediately as in the following:

$$\sum_{|R\cap\Omega|\sim(1/2^k)|\Omega|} C_R^2 \leqq c \sum_{|R\cap\Omega|\sim(1/2^k)|\Omega|} S_R^2(\varphi) \leqq c \sum_{R\subset\Omega_k} S_R^2(\varphi) \\ \leqq c \cdot M|\Omega_k| \leqq c \cdot M \cdot 2^k \cdot k|\Omega|$$

where $\Omega_k = \{x \mid M_s(\Omega) > 1/2^k\}$. ($M_s$ denotes the strong maximal function in $\mathbf{R}^2$.) Furthermore, we have

$$\int_\Omega |\varphi - \tilde{\varphi}_\Omega|^2 dt = \int_\Omega |\sum_{|\tilde{R}\cap\Omega| \geqq (1/2)|\tilde{R}|} \varphi_R|^2 dt \, . \tag{2.12}$$

Now to finish the proof of this step, we will first verify the easy step (iii) $\Rightarrow$ (i) along the way.

(iii) $\Rightarrow$ (i). Let $\mathcal{G}$ denote some collection of dyadic rectangles. Then

$$\begin{aligned} &\left|\int_{\mathbf{R}^2} \sum_{R\in\mathcal{G}} \varphi_R(t) \cdot g(t)\, dt\right| \\ &= \left|\sum_{R\in\mathcal{G}} \int_{\mathbf{R}^2} \varphi_R(t) \cdot g(t) dt\right| \\ &= \left|\sum_{R\in\mathcal{G}} \int_{\mathbf{R}^2} \iint_{R_+} \varphi(t, y)\psi_y(t - x) \frac{dtdy}{y_1 y_2}\, g(t) dt\right| \\ &= \left|\sum_{R\in\mathcal{G}} \iint_{R_+} \varphi(t, y) g(t, y) \frac{dtdy}{y_1 y_2}\right| \\ &\leqq \left(\sum_{R\in\mathcal{G}} S_R^2(\varphi)\right)^{1/2} \left(\sum_R S_R^2(g)\right)^{1/2} \\ &= \left(\sum_{R\in\mathcal{G}} S_R^2(\varphi)\right)^{1/2} \|g\|_2 \, . \end{aligned} \tag{2.13}$$

Thus letting g run through all $g \in L^2$, we obtain $\|\sum_{R\in\mathcal{G}} \varphi_R\|_2^2 \leqq \sum_{R\in\mathcal{G}} S_R^2(\varphi)$. Letting $\mathcal{G}$ be the class of all $R \subset \Omega$, we then obtain (i) from (iii).

We continue the estimate in (2.12) using (2.13) and obtain

$$\begin{aligned} \int_\Omega |\varphi - \tilde{\varphi}_\Omega|^2 dx &\leqq \|\sum_{|\tilde{R}\cap\Omega| \geqq (1/2)|\tilde{R}|} \varphi_R\|^2 \\ &\leqq \sum_{|\tilde{R}\cap\Omega| \geqq (1/2)|\tilde{R}|} S_R^2(\varphi) \\ &\leqq \sum_{R\subset\tilde{\Omega}_1} S_R^2(\varphi) \leqq M|\tilde{\Omega}_1| \leqq cM|\Omega| \, . \end{aligned} \tag{2.12$'$}$$

We have completed the proof that $\varphi \in \mathrm{BMO}_{(b)}$.

(ii) $\Rightarrow$ (iv). Suppose $\varphi \in \mathrm{BMO}_{(b)}$. To prove it is in the dual of $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, it suffices to verify that $\left|\int_{\mathbf{R}^2} a\varphi dx\right| \leqq cM$ for each atom a of $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ defined in the sense of the atomic decomposition in Section I. Suppose a is supported in the open set Ω, satisfying all the atomic properties as in the definition with respect to Ω, and is divided into elementary particles a_R with the corresponding constant c_R. We may, by a dilation invariance argument, assume that $|\Omega| = 1$. Then

$$(2.14)\qquad \begin{aligned} \left|\int_{\mathbf{R}^2} a\varphi dx\right| &= \left|\int_\Omega a\varphi dx\right| \leqq \left|\int_\Omega a(\varphi - \tilde{\varphi}_\Omega)dx\right| + \left|\int_\Omega a\tilde{\varphi}_\Omega dx\right| \\ &\leqq \|a\|^2\left(\int_\Omega |\varphi - \tilde{\varphi}_\Omega|^2 dx\right)^{1/2} + \left|\int a\tilde{\varphi}_\Omega dx\right| \\ &\leqq M + \left|\int \sum a_R \cdot \sum \tilde{\varphi}_i dx\right| \end{aligned}$$

where $\tilde{\varphi}_\Omega$, $\tilde{\varphi}_i$ are chosen as in the definition of $\mathrm{BMO}_{(b)}$ with respect to the open set Ω. The rest of the proof is similar to the proof of the step (i) $\Rightarrow$ (iii) with the following lemma replacing the role of Lemma 2.

LEMMA 3. *Given two functions φ_1 and φ_2 supported in $\tilde{R}_1$, $\tilde{R}_2$ respectively of dyadic rectangles $R_i = I_i \times J_i$, suppose $\varphi_i \in C^2(\mathbf{R}^2)$ with the following conditions:*

(a) *The φ_i have mean value zero over each vertical and horizontal segment of $\tilde{R}_i$.*

$$\text{(b)}\qquad \|\varphi_i\|_\infty \leqq \frac{c_i}{|R_i|^{1/2}}, \qquad \left\|\frac{\partial\varphi_i}{\partial x_1}\right\|_\infty \leqq \frac{c_i}{|R_i|^{1/2}|I_i|},$$

$$\left\|\frac{\partial\varphi_i}{\partial x_2}\right\|_\infty \leqq \frac{c_i}{|R_i|^{1/2}|J_i|}, \qquad \left\|\frac{\partial^2\varphi_i}{\partial x_1\partial x_2}\right\|_\infty \leqq \frac{c_i}{|R_i|^{3/2}}, \qquad i = 1, 2 .$$

Then

$$(2.15)\qquad \left|\int_{\mathbf{R}^2} \varphi_1\varphi_2 dx\right| \leqq c_1c_2 r(R_1, R_2)$$

where $r(R_1, R_2)$ is as defined in (2.3).

Proof of Lemma 3. In the case $|I_1| \geqq |I_2|$ and $|J_2| \geqq |J_1|$, or $|I_2| \geqq |I_1|$ and $|J_1| \geqq |J_2|$ we may apply twice a dilation argument similar to the one we used in the proof of Lemma 1 to obtain the result we want. In the case $|I_1| \geqq |I_2|$ and $|J_1| \geqq |J_2|$ (or $|I_2| \geqq |I_1|$ and $|J_2| \geqq |J_1|$), we write

$$\begin{aligned} &\left|\int_{\mathbf{R}^2} \varphi_1\varphi_2 dx\right| \\ &= \left|\int_{\tilde{R}_1\cap\tilde{R}_2} (\varphi_1(x_1, x_2) - \varphi_1(m_1, x_2) - \varphi_1(x_1, m_2) + \varphi_1(m_1, m_2))\varphi_2(x_1, x_2)dx_1dx_2\right| \\ &\leqq \left\|\frac{\partial^2\varphi_1}{\partial x_1\partial x_2}\right\|_\infty |\tilde{R}_1 \cap \tilde{R}_2||\tilde{I}_1 \cap \tilde{I}_2||\tilde{J}_1 \cap \tilde{J}_2| \cdot \frac{c_2}{|R_2|^{1/2}} \\ &\leqq cc_1c_2\frac{1}{|R_1|^{3/2}} \cdot \frac{|R_2|^2}{|R_2|^{1/2}} \\ &= cc_1c_2 r(R_1, R_2) . \end{aligned}$$

Applying (2.15) to the term $\int \sum a_R \cdot \sum \tilde{\varphi}_i dx$, we have

$$\left|\int \sum a_R \cdot \sum \tilde{\varphi}_i\right|$$
$$\leqq \sum_{R\subset\Omega} \sum_{|\tilde{R}_1\cap\Omega|<(1/2)|\tilde{R}_1|} C_R \cdot C_{R_1} r(R_1, R)$$
$$\leqq \sum_{R\subset\Omega} \sum_{|\tilde{R}_1\cap\Omega|<(1/2)|\tilde{R}_1|} (C_R^2 + C_{R_1}^2) r(R_1, R)$$
$$\leqq \sum_{R\subset\Omega} C_R^2\Big(\sum_{|\tilde{R}_1\cap\Omega|<(1/2)|\tilde{R}_1|} r(R_1, R)\Big) + \sum_{R\subset\Omega} \sum_{R_1\in\mathscr{S}_R} C_{R_1}^1 r(R_1, R) .$$

Comparing this expression to (2.7) and using (2.9), we have

$$(2.16) \qquad \left|\int \sum a_R \cdot \sum \tilde{\varphi}_i\right| \leqq c \sum_{R\subset\Omega} C_R^2 + c \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^{3/2} \sum_{R_1\in r_k} C_{R_1}^2$$
$$\leqq \frac{cA}{|\Omega|} + c \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^{3/2} \cdot k \cdot 2^k |\Omega|$$
$$\leqq c , \qquad \text{since we assume that } |\Omega| = 1 .$$

Here $r_k = \{R_1: (1/2^k)|\tilde{R}_1| \leqq |\tilde{R}_1 \cap \Omega| < (1/2^{k-1})|\tilde{R}_1|\}$. The second inequality follows from our assumption on the properties of the atom a and the function $\tilde{\varphi}_\Omega$. Adding (2.16) to (2.14), we have finished the proof.

(iv) $\Rightarrow$ (iii). Suppose φ is in the dual of $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$; then by a routine duality argument, there exists some $f_i \in L^\infty$ $(1 \leqq i \leqq 4)$ such that $\varphi = f_1 + H_{x_1} f_2 + H_{x_2} f_3 + H_{x_1} H_{x_2} f_4$, where H_{x_i} is the Hilbert transform in the direction x_i. Thus to prove (iv) $\Rightarrow$ (iii), it suffices to verify (iii) for bounded functions, and Hilbert transforms of bounded functions.

First, if $\varphi \in L^\infty$, then for each open set Ω, we have $\varphi(t, y) = (\varphi\chi_{\tilde{\Omega}})(t, y)$ for each $(t, y) \in R_+$, where R is some dyadic rectangle contained in Ω and where $\tilde{\Omega} = \bigcup_{R\subset\Omega} \tilde{R}$. Hence

$$\sum_{R\subset\Omega} S_R^2(\varphi) = \sum_{R\subset\Omega} \iint_{R_+} |\varphi(t, y)|^2 \frac{dtdy}{y_1 y_2} = \sum_{R\subset\Omega} \iint_{R_+} |\varphi\chi_{\tilde{\Omega}}(t, y)|^2 \frac{dtdy}{y_1 y_2}$$
$$\leqq \|\varphi\chi_{\tilde{\Omega}}\|_2^2 \leqq c|\Omega| \|\varphi\|_\infty^2 .$$

To verify (iii) for Hilbert transforms of bounded functions, we will actually prove something stronger. Suppose K is an odd kernel defined on $\mathbf{R}$, which satisfies $\|f * K\|_2 \leqq c\|f\|_2$ for each $f \in L^2(\mathbf{R})$. For each $f \in L^2(\mathbf{R}^2)$, we will still use $f * K$ to denote the function

$$(f * K)(x_1, x_2) = \iint_{\mathbf{R}^2} f(x_1 - \alpha_1, x_2 - \alpha_2) K(\alpha_1) K(\alpha_2) d\alpha_1 d\alpha_2 , \qquad (x_1, x_2) \in \mathbf{R}^2 .$$

Then $\|f * K\|_2 \leqq c|f\|_2$ holds also on $\mathbf{R}^2$. We will verify (iii) for $\varphi * K$ for all bounded functions φ. For this purpose, we need the following lemma.

LEMMA 4. *Suppose R_1, R_2 are dyadic rectangles with $R_2 \in \mathscr{S}_{R_1}$; then*

$$S_{R_1}^2(\varphi_{R_2} * K) = cr^2(R_1, R_2)S_{R_2}^2(\varphi) .$$

Assuming Lemma 4 for the moment, we can then finish the proof of our assertion following the same pattern as (i) $\Rightarrow$ (iii) (with Lemma 4 replacing Lemma 2). Actually suppose $f = \varphi * K$ with $\varphi \in L^\infty$. For a fixed open set Ω_0, and $\Omega = \bigcup_{R\subset\Omega_0} \tilde{R}$, let $\varphi_1 = \sum_{R\subset\Omega} \varphi_R$, $\varphi_2 = \varphi - \varphi_1$. We have

$$\sum_{R\subset\Omega_0} S_R^2(\varphi_1 * K) \leqq \|\varphi_1 * K\|_2^2 \leqq c\|\varphi_1\|_2^2 \leqq c\|\varphi\|_\infty^2 |\Omega_0| . \tag{2.17}$$

Also applying Lemma 4 and following the same line of proof as in (2.6) and (2.7), we get

$$\begin{aligned} \sum_{R\subset\Omega_0} S_R^2(\varphi_2 * K) &\leqq c\sum_{k=1}^\infty \left(\frac{1}{2^k}\right)^{3/2} \sum_{R_1\subset\Omega_k} S_{R_1}^2(\varphi) \\ &\leqq c\sum_{k=1}^\infty \left(\frac{1}{2^k}\right)^{3/2} \cdot 2^k \cdot k|\Omega_0|\|\varphi\|_\infty^2 . \end{aligned} \tag{2.18}$$

Adding (2.17) and (2.18), we obtain the desired estimate:

$$\sum_{R\subset\Omega_0} S_R^2(\varphi * K) \leqq c\|\varphi\|_\infty^2 |\Omega_0| .$$

It remains to prove Lemma 4. The method of proof indicated below is an easy application of Lemma 1 plus some careful changing of the order of integration. We will first state two one-variable versions of it.

SUBLEMMA 1. *If I_1, I_2 are dyadic intervals, $\tilde{I}_1 \cap \tilde{I}_2 \neq \varnothing$, $|I_1| \leqq |I_2|$ and φ is defined on $\mathbf{R}$, then*

$$\iint_{(I_1)_+} (\varphi_{I_2} * K)^2(t, y)\frac{dtdy}{y} \leqq c\left|\frac{I_1}{I_2}\right|^3 \left(\iint_{(I_2)_+} |\varphi(r, z)|^2 \frac{drdz}{z}\right) ,$$

where for each interval I,

$$I_+ = \left\{(t, y) \in \mathbf{R}_2^+ \,|\, t \in I, \frac{|I|}{2} < y \leqq |I|\right\} .$$

Proof.

$$(\varphi_{I_2} * K)(t, y) = \iint_{(I_2)_+} \varphi(r, z)\left(\int K(\beta)\Phi_{y,z,t}(r + \beta)d\beta\right)\frac{drdz}{z}$$

where

$$\Phi_{y,z,t}(x) = \int \psi_z(\alpha - x)\psi_y(t - \alpha)d\alpha .$$

A careful examination of the domains of ψ_y, ψ_z indicates that for $(t, y) \in (I_1)_+$, $(r, z) \in (I_2)_+$ with $|I_1| \leqq |I_2|$, we have $\Phi_{y,z,t}(r + \beta) = 0$ unless $r + \beta \in I_2'$ where I_2' is some interval with the same center as I_2 with some fixed enlarged size. Thus

$$(\varphi_{I_2} * K)(t, y) = \iint_{(I_2)_+} \varphi(r, z)(K * \chi_{I_2'}\Phi_{y,z,t})(r) \frac{drdz}{z} .$$

Notice that

$$\int_{I_2} |K * \chi_{I_2'}\Phi_{y,z,t}|^2(r)dr \leqq \int_{I_2'} |\Phi_{y,z,t}|^2(r)dr$$

by the kernel property of K. We can then make a routine application of Lemma 1 and the Schwarz inequality to finish the rest of the proof.

SUBLEMMA 2. *If I_1, I_2 are dyadic intervals, $\tilde{I}_1 \cap \tilde{I}_2 \neq \emptyset$. If $|I_2| \leqq |I_1|$ and φ is defined on* $\mathbf{R}$ *then*

$$\iint_{(I_1)_+} |(\varphi_{I_2} * K)(t, y)|^2 \frac{dtdy}{y} \leqq c\left(\frac{|I_2|}{|I_1|}\right)^3 \left(\iint_{(I_2)_+} |\varphi(r, z)|^2 \frac{drdz}{z}\right).$$

Proof. Since $(\varphi_{I_2} * K)(t, y) = (\varphi_{I_2}(\cdot, y) * K)(t)$, we have

$$\begin{aligned}\int_{I_1} |\varphi_{I_2} * K|^2(t, y)dt &\leqq \int_{\mathbf{R}} |\varphi_{I_2} * K|^2(t, y)dt \\ &= \int_{\mathbf{R}} |\varphi_{I_2}(\cdot, y) * K|^2(t)dt \\ &\leqq \int_{\mathbf{R}} |\varphi_{I_2}(x, y)|^2 dx .\end{aligned}$$

Since φ_{I_2} is supported on $\tilde{I}_2$, $|I_2| \leqq |I_1|$, an examination indicates that $\varphi_{I_2}(\cdot, y)$ is supported on I_1', if $|I_1|/2 < y \leqq |I_1|$. Here I_1' is an interval with the same center, but fixed enlarged size of I_1. Say $I_1' \subset \bigcup_{k=1}^n J_k$ where each J_k is a dyadic interval of the same size as I_1; then

$$\begin{aligned}\iint_{(I_1)_+} |\varphi_{I_2} * K|^2(t, y) \frac{dtdy}{y} &\leqq \int_{|I_1|/2}^{|I_1|} \left(\int_{I_1'} |\varphi_{I_2}(t, y)|^2 dt\right)\frac{dy}{y} \\ &\leqq \sum_k \left(\frac{|I_2|}{|J_k|}\right)^3 \iint_{(I_2)_+} |\varphi(r, z)|^2 \frac{drdz}{z} \\ &\leqq c\left(\frac{|I_2|}{|I_1|}\right)^2 \iint_{(I_2)_+} |\varphi(r, z)|^2 \frac{drdz}{z} .\end{aligned}$$

The second step in the above estimate follows from the one-dimension version of Lemma 2.

Combining Sublemmas 1 and 2 above, we could finish the proof of Lemma 4 in the following way.

Assume $R_1 = I_1 \times J_1$, $R_2 = I_2 \times J_2$ are two dyadic rectangles with $R_1 \in \mathscr{J}_R$. Then there are four possibilities.

(1) $|I_1| \leqq |I_2|$, $|J_1| \leqq |J_2|$, (2) $|I_1| \leqq |I_2|$, $|J_1| \geqq |J_2|$,
(3) $|I_1| \geqq |I_2|$, $|J_1| \leqq |J_2|$, (4) $|I_1| \geqq |I_2|$, $|J_1| \geqq |J_2|$.

We will indicate the proof of possibility (3) here. From the proof it is clear that case (1) could be proved by applying Sublemma 1 twice, while case (4) could be proved by applying Sublemma 2 twice. It is also clear case (2) could be proved similarly to case (3).

Assume $|I_1| \geqq |I_2|$, $|J_1| \leqq |J_2|$; then for each $x = (x_1, x_2) \in \mathbf{R}^2$,

$$(2.19)\quad \varphi_{R_2}(x) = \iint_{(R_2)_+} \varphi(r, z)\psi_z(x-r)\frac{drdz}{z}$$
$$= \iint_{(I_2)_+}\left(\iint_{(J_2)_+} \varphi(r_1, r_2, z_1, z_2)\psi_{z_2}(x_2-r_2)\frac{dr_2dz_2}{z_2}\right)\psi_{z_1}(x_1-r_1)\frac{dr_1dz_1}{z_1}$$
$$= \iint_{(I_2)_+} \varphi_{J_2}(r_1, z_1, x_2)\psi_{z_1}(x_1-r_1)\frac{dr_1dz_1}{z_1}$$

where we define $\varphi_{J_2}(x_1, x_2) = \iint_{(J_2)_+}(\varphi(x_1, \cdot) * \psi_{z_2})(r_2)\psi_{z_2}(x_2 - r_2)(dr_2dz_2/z_2)$ and $\varphi_{J_2}(r_1, z_1, x_2) = (\varphi_{J_2}(\cdot, x_2) * \psi_{z_1})(r_1)$. With this notation, for each fixed $(t_2, y_2) \in (J_2)_+$, let $g_{t_2,y_2}(x_1) = ((\varphi_{J_2}(x_1, \cdot) * \psi_{y_2}) * K)(t_2)$. Then by (2.19),

$$(\varphi_{R_2} * K)(t, y)$$
$$= \iint_{\mathbf{R}^2}\left(\iint_{(I_2)_+} g_{t_2,y_2}(r_1, z_1)\psi_{z_1}(\alpha_1-\beta_1-r_1)\frac{dr_1dz_1}{z_1}\right)K(\beta_1)\psi_{y_1}(t_1-\alpha_1)d\alpha_1 d\beta_1$$
$$= ((g_{t_2,y_2})_{I_2} * K)(t_1, y_1)\ .$$

Thus we have

$$S^2_{R_1}(\varphi_{R_2} * K)^2 = \iint_{(R_1)_+} |(\varphi_{R_2}) * K|^2(t, y)\frac{dtdy}{y}$$
$$= \iint_{(J_1)_+}\left(\iint_{(I_1)_+} |(g_{t_2,y_2})_{I_2} * K|^2(t_1, y_1)\frac{dt_1dy_1}{y_1}\right)\frac{dt_2dy_2}{y_2}$$

by Sublemma 2

$$\leqq c\iint_{(J_1)_+}\left|\frac{I_2}{I_1}\right|^3\left(\iint_{(I_2)_+} |g_{t_2,y_2}|^2(r_1, z_1)\frac{dr_1dz_1}{z_1}\right)\frac{dt_2dy_2}{y_2}$$
$$= c\left|\frac{I_2}{I_1}\right|^3\iint_{(I_2)_+}\left(\iint_{(J_1)_+} |g_{t_2,y_2}|^2(r_1, z_1)\frac{dt_2dy_2}{y_2}\right)\frac{dr_1dz_1}{z_1}$$
$$= c\left|\frac{I_2}{I_1}\right|^3\iint_{(J_2)_+}\left(\iint_{(J_1)_+} |\varphi_{J_2}(r_1, z_1, \cdot) * K|^2(t_2, y_2)\frac{dt_2dy_2}{y_2}\right)\frac{dr_1dz_1}{z_1}$$

by Sublemma 1

$$= c\left|\frac{I_2}{I_1}\right|^3\iint_{(I_2)_+}\left|\frac{J_1}{J_2}\right|^3\left(\iint_{(J_2)_+} |\varphi(r_1, z_1, r_2, z_2)|^2\frac{dr_2dz_2}{z_2}\right)\frac{dr_1dz_1}{z_1}$$
$$\leqq cr^2(R_1, R_2)S^2_{R_2}(\varphi)\ .$$

We have finished the proof of Lemma 4, hence completing the proof of the theorem.

Remark. If we are willing to assume more smoothness for the function ψ (e.g., $\psi \in C^\infty$ would be enough), the step (iv) $\Rightarrow$ (iii) could be proved more easily. The following proof was indicated to us by R. Coifman and Y. Meyer.

Assume ψ is a function supported on $[-1, 1]$, satisfying $\hat{\psi}(0) = 0$, and $\int_0^\infty (|\hat{\psi}(x)|^2/x)dx < \infty$ as before. To indicate the dependence on ψ, write $\varphi_\psi(t, y) = (\varphi * \psi_y)(t)$. Then for $\varphi \in L^\infty$ we have for each open set Ω

$$\sum_{R\subset\Omega} \iint_{R_+} |\varphi_\psi(t, y)|^2 \frac{dtdy}{y_1 y_2} \leqq \iint_{(\mathbf{R}_+^2)^2} |(\varphi_\Omega)_\psi(t, y)|^2 \frac{dtdy}{y}$$
$$\leqq c \cdot |\Omega| \, \|\varphi\|_\infty^2 \left(\int_0^\infty \frac{|\hat{\psi}(x)|^2}{x} dx\right)^2$$

where $\varphi_\Omega = \varphi\chi_{\tilde{\Omega}}$, and $\tilde{\Omega} = \bigcup_{R\subset\Omega} \tilde{R}$.

Now suppose all the assumptions for ψ hold except that $\operatorname{supp}\psi \subseteqq [-2^k, 2^k] \times [-2^j, 2^j]$ for some integer k, j. Define the open set $\Omega_{k,j}$ as follows: For rectangle $R \subset \Omega$, let R_{kj} be the rectangle with the same center as R, but 2^k times length in the x_1-direction and 2^j times length in the x_2-direction. Let $\Omega_{k,j} = \bigcup_{R\subset\Omega} R_{k,j}$. Then

$$\text{(2.20)} \qquad \sum_{R\subset\Omega} \iint_{R_+} |\varphi_\psi(t, y)|^2 \frac{dtdy}{y} = \sum_{R\subset\Omega} \iint_{R_+} |(\varphi_{\Omega_{k,j}})_\psi(t, y)|^2 \frac{dtdy}{y}$$
$$\leqq c\|\varphi\|_\infty^2 |\Omega_{k,j}| \left(\int_0^\infty \frac{|\hat{\psi}(x)|^2}{x} dx\right)^2$$
$$\leqq c2^{k+j} |\Omega| \, \|\varphi\|_\infty^2 \left(\int_0^\infty \frac{|\hat{\psi}(x)|^2}{x} dx\right)^2.$$

To estimate $S_R(\Phi)$ for $\Phi = H_{x_1}H_{x_2}$ (H denotes the Hilbert transform), notice that

$$\Phi_\psi(\cdot, y) = (\Phi * \psi_y)(\cdot) = \varphi * H_{x_1}H_{x_2}(\psi_y)$$
$$= \varphi * H_{x_1}(\psi_{y_1})H_{x_2}(\psi_{y_2}) = \varphi * H_{x_1}(\psi)_{y_1}H_{x_2}(\psi)_{y_2}$$
$$= \varphi_\Psi(\cdot, y)$$

where $\Psi = H_{x_1}H_{x_2}(\psi)$.

So if we decompose $\Psi = \sum_{k,j} \Psi_{k,j}$, where each $\Psi_{k,j}$ has support contained in $[-2^k, 2^k] \times [-2^j, 2^j]$ and has mean value zero, then we may apply (2.20) to each Ψ_{kj}. Since the map $\psi \to (1/|\Omega| \sum_{R\subset\Omega} S_R^2(\varphi_\psi))^{1/2}$ is subadditive, we can sum over our result for each (k, j) if $\int_0^\infty (|\hat{\Psi}_{kj}(x)|^2/x)dx$ decreases fast enough (say if it is dominated by 2^{k+j}). In our case, we can actually choose $\Psi_{kj}(x_1, x_2) = \Psi_k(x_1)\Psi_j(x_2)$ where both Ψ_k, Ψ_j have mean value zero, are supported in $[-2^k, 2^k]$, $[-2^j, 2^j]$ respectively, and have the rapid decrease property $\int_0^\infty |\hat{\Psi}_k(x)|^2(dx/x) \leqq 2^k$, which is possible when $\Psi \in C^\infty$ and has mean value zero and compact support.

III. The John-Nirenberg inequality for BMO

In this section we shall prove that functions in the class BMO are locally in all the L^p classes, and, in fact, in the exponential square-root class $\left(\text{i.e., } \int e^{c\sqrt{|\varphi|}} < \infty\right)$. (In Section II we showed that several possible definitions of BMO are equivalent.)

THEOREM (John-Nirenberg). *Let $\varphi \in \mathrm{BMO}$, and $\Omega \subset \mathbf{R}^2$ be an open set with $|\Omega| < \infty$. Then*

$$\|\textstyle\sum_{R\subset\Omega} \varphi_R\|_{L^p}^p \leqq C_p|\Omega| \qquad \textit{for} \quad p < \infty$$

where C_p depends only on the BMO *norm of φ and p.*

Proof. The proof proceeds by duality. Let $1/p + 1/q = 1$ and $\|g\|_q = 1$. Then

$$\begin{aligned}\left|\int_{\mathbf{R}^2} \textstyle\sum_{R\subset\Omega} \varphi_R \cdot g dx\right| &= \left|\textstyle\sum_{R\subset\Omega} \displaystyle\int_{\mathbf{R}^2} \iint_{R_+} \varphi(t, y)\psi_y(x - t) \frac{dtdy}{y_1y_2} g(x)dx\right| \\ &= \left|\textstyle\sum_{R\subset\Omega} \displaystyle\iint_{R_+} \varphi(t, y)g(t, y) \frac{dtdy}{y_1y_2}\right| \\ &\leqq \iint_{S(\Omega)} |\varphi(t, y)||g(t, y)| \frac{dtdy}{y_1y_2} .\end{aligned}$$

Now we shall construct a bounded, vector-valued function

$$\vec{F}: \mathbf{R}^2 \longmapsto \bigoplus_{x\in\mathbf{R}_2} L^2\left(\Gamma(x); \frac{dtdy}{y_1^2y_2^2}\right)$$

in such a way that $\vec{F}(x) \in L^2(\Gamma(x); dtdy/y_1^2y_2^2)$ and $\vec{F}(x)(t, y)$ will either be $\varphi(t, y)$ or 0.

To construct $\vec{F}$, let us define

$$\Gamma_\Omega(x) = \left\{(t, y) \in \mathbf{R}_+^2 \times \mathbf{R}_+^2 \,\middle|\, |t_i - x_i| < y_i \quad \text{and} \quad |R_{t,y} \cap \Omega| > \frac{1}{2}|R_{t,y}|\right\}$$

where $R_{t,y}$ is the rectangle in $\mathbf{R}^2$ centered at t with side lengths y_1 and y_2. Also let us put $S_\Omega^2(\varphi)(x) = \iint_{\Gamma_\Omega(x)} |\varphi(t, y)|^2(dtdy/y_1^2y_2^2)$. With these definitions, notice that

$$\int_{x\in\Omega} S_\Omega^2(\varphi)(x)dx_1dx_2 \leqq \iint_{S(\tilde{\Omega})} |\varphi(t, y)|^2 \frac{dtdy}{y_1y_2} \leqq A\|\varphi\|_{\mathrm{BMO}}^2|\Omega| .$$

(For convenience, assume $\|\varphi\|_{\mathrm{BMO}} = 1$, and $A = 1$.) Let

$$\Omega_1 = \{x \in \Omega \,|\, S_\Omega(\varphi)(x) > 10\} .$$

Then

$$|\Omega_1| < \frac{1}{100}|\Omega| .$$

Notice, again, that

$$\int_{\Omega_1} S^2_{\Omega_1}(\varphi)(x)dx_1dx_2 \leqq \iint_{S(\Omega_1)} |\varphi(t, y)|^2 \frac{dtdy}{y_1y_2} \leqq |\Omega_1| .$$

As before let $\Phi_2 = \{x \in \Omega_1 | S_{\Omega_1}(\varphi)(x) > 10\}$. Then $|\Omega_2| < (1/100)|\Omega_1|$.

Continuing in this way we get open sets $\Omega \supset \Omega_1 \supset \Omega_2 \supset \cdots$. If $x \in \Omega_k - \Omega_{k+1}$, $(t, y) \in \Gamma(x)$, then define $\vec{F}(x)(t, y) = \varphi(t, y)$ if $|R_{t,y} \cap \Omega_k| > (1/2)|R_{t,y}|$, and define $\vec{F}(x)(t, y) = 0$ if $|R_{t,y} \cap \Omega_k| \leqq (1/2)|R_{t,y}|$. Then, by construction, $|\vec{F}(x)| \leqq 10$ and we have

$$\iint_{S(\Omega)} |\varphi(t, y)||g(t, y)| \frac{dtdy}{y_1y_2} \leqq A \int_{x\in\Omega}\int_{\Gamma(x)} |\vec{F}(x)(t, y)||g(t, y)| \frac{dtdy}{y_1^2y_2^2} dx .$$

In fact this last inequality is clear, since if $(t, y) \in S(\Omega)$, there is an integer k so that $|R_{t,y} \cap \Omega_k| > (1/2)|R_{t,y}|$ but $|R_{t,y} \cap \Omega_{k+1}| \leqq (1/2)|R_{t,y}|$. A quick glance at the definition of $\vec{F}(x)$ reveals that if $x \in R_{t,y} - \Omega_{k+1}$, $\vec{F}(x)(t, y) = \varphi(t, y)$ so that

$$\begin{aligned}\int_{x\in\Omega}\int_{\Gamma(x)} &|\vec{F}(x)(t, y)||g(t, y)| \frac{dtdy}{y_1^2y_2^2} dx \\ &= \iint_{S(\Omega)} |\{x \in R_{t,y} | \vec{F}(x)(t, y) = \varphi(t, y)\}||\varphi(t, y)||g(t, y)| \frac{dtdy}{y_1^2y_2^2} \\ &\geqq \frac{1}{2}\iint_{S(\Omega)} |\varphi(t, y)||g(t, y)| \frac{dtdy}{y_1y_2} .\end{aligned}$$

We conclude the proof by noticing that

$$\begin{aligned}\int_{x\in\Omega}\int_{\Gamma(x)} &|\vec{F}(x)(t, y)||g(t, y)| \frac{dtdy}{y_1^2y_2^2} dx \\ &\leqq \int_{x\in\Omega}\left(\int_{\Gamma(x)} |\vec{F}(x)(t, y)|^2 \frac{dtdy}{y_1^2y_2^2}\right)^{1/2} S(g)(x)dx \\ &\leqq 10\|S(g)\|_{L^1(\Omega)} .\end{aligned}$$

Since $g \in L^q(R^2)$,

$$\|S(g)\|_{L^1(\Omega)} \leqq \|S(g)\|_{L^q(R^2)} |\Omega|^{1/p} \leqq A|\Omega|^{1/p} .$$

Note that our proof shows that $\sum_{R\subset\Omega} \varphi_R$ is in the Orlicz class $e^{\sqrt{L}}$ since we have shown that we can integrate $\sum \varphi_R$ against any function g whose S function is locally in L^1, i.e., for $g \in L(\log^2 L)$.

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