VARIATIONS ON THE THEME OF JOURNÉ'S LEMMA

CARLOS CABRELLI, MICHAEL T. LACEY, URSULA MOLTER, AND JILL C. PIPHER

ABSTRACT. Journé's Lemma [11] is a critical component of many questions related to the product BMO theory of S.-Y. Chang and R. Fefferman. This article presents several different variants of the Lemma, in two and higher parameters, some known, some implicit in the literature, and some new.

1. Introduction, Journé's Lemma

We begin the discussion in two dimensions. Let M denote the strong maximal function in the plane. Let $\mathcal{U}$ denote a collection dyadic rectangles of the plane, whose union $\operatorname{sh}(\mathcal{U})$ is a set of finite measure, and set $\operatorname{Enl}(\mathcal{U}) \stackrel{\text{def}}{=} \{\mathrm{M}\,\mathbf{1}_{\operatorname{sh}(\mathcal{U})} > \frac{1}{2}\}$. For a dyadic rectangle $R = R_{(1)} \times R_{(2)} \in \mathcal{U}$, set

$$\operatorname{emb}(R;\mathcal{U}) \stackrel{\text{def}}{=} \sup\{\mu > 1 \,:\, (\mu R_{(1)}) \times R_{(2)} \subset \operatorname{Enl}(\mathcal{U})\}$$

In this display, and throughout this paper, we use the notation λR to denote the set that has the same center as R but is dilated by an amount λ. (Section 1.1 has a comprehensive list of notations and conventions.)

The subject of this paper is the result of J.-L. Journé from 1987 [11].

Lemma 1.1. *[Journé's Lemma] For all $\epsilon > 0$, and any collection $\mathcal{U}' \subset \mathcal{U}$ of pairwise incomparable dyadic rectangles contained in $\mathcal{U}$, we have the inequality*

$$\sum_{R \in \mathcal{U}'} \operatorname{emb}(R,\mathcal{U})^{-\epsilon} |R| \lesssim |\operatorname{sh}(\mathcal{U}')|. \tag{1.2}$$

The implied constant depends only upon $\epsilon > 0$.

This Lemma has proven to be an invaluable aid in those problems associated with Carleson measures in product setting. In particular, we have been careful to state (1.2) as an inequality that is uniform over all choices of subset $\mathcal{U}' \subset \mathcal{U}$. We insist on this formulation so that the Lemmas will more readily apply to the setting of Carleson measures in the product domain. The estimation of the norm of subject objects can be quite complicated, and the Journé Lemma permits an upper bound

This research is supported in part by an NSF Grant, a Guggenheim Fellowship, and the Erwin Schrödinger Institute, Vienna Austria.

The research of the first and third authors is partially supported by grants UBACyT X058, X108, and CONICET PIP 456/98.

in terms of simpler norms. See Corollary 2.16. We comment in more detail on the context of this Lemma in the next section.

By 'product setting' we mean that range of questions which are concerned with issues of harmonic analysis that are invariant with respect to a family of dilations with at least two free parameters.

Despite the appearance of this Lemma close to twenty years ago, one cannot yet describe the precise role that this Lemma plays in the product theory, especially when confronting issues related to the induction on the number of free parameters. Indeed, this role will be understood by further developments in what seems to be a still nascent product theory. Following the work of Chang and Fefferman, see Journé [12], Carberry and Seeger [1], Fefferman and Pipher [8], and the more recent results of Muscalu, Pipher, Tao and Thiele [14, 15], among other papers. We intend this paper to be a source book for ideas associated with the Lemma, with a description of what is known, recent innovations, as well as some refinements, that as of yet, have not found applications.

There are three themes to the refinements. First, the Lemma does not appear to admit a completely trivial extension to higher parameters. The point that simplifies the analysis in two parameters is that if R and R' are distinct, intersecting rectangles, then it is the case that two sides of the rectangles are in an inverse relation. On the other hand, in three parameters, the different faces of the two rectangles can have a number of relations. See Figure 1 for the situation in the plane. In fact, the best methods to pass to higher numbers of parameters probably has not as yet been discovered. And there are also versions of the Lemma, for the higher parameter setting, in which rectangles are replaced by more complicated sets. These constructions, which are taken up in Section 4 for instance, may in applications, permit one to get at more directly the particular manner in which e.g. the BMO space of three parameters is built up from that of two parameters.

Second, the "embeddedness term" $\operatorname{emb}(R,\mathcal{U})$ above, is the new element required in two and higher parameters. There is interest in having different measures of the embeddedness term that are essentially smaller than that given above. Now, the power of ϵ in (1.2) is used to cancel out terms that are logarithmic in $\operatorname{emb}(R,\mathcal{U})$. Any proof of the Lemma must account for the fact that a subcollection $\mathcal{U}'$ in which $\operatorname{emb}(R,\mathcal{U}) \simeq \mu$, for all $R \in \mathcal{U}'$, one has that $\mathcal{U}'$ is the union of $O(\log \mu)$ subcollections in which the rectangles are essentially disjoint.

If we decrease the embeddedness term, we expect the combinatorial difficulties to multiply, and the logarithmic terms to increase. We maintain the term $\operatorname{emb}(R,\mathcal{U})^{-\epsilon}$, and do not keep track of how quickly the logarithmic terms increase.

In typical applications of Journé's Lemma, one obtains, from say "Schwartz tails" arguments, a rapid decrease in terms of the embeddedness quantity.

Third, there are specific instances in which the "enlarged set" $\operatorname{Enl}(\mathcal{U})$ plays a important role. As phrased above, one has $|\operatorname{Enl}(\mathcal{U})| \le K|\operatorname{sh}(\mathcal{U})|$, with constant K strictly bigger than one. In a paper of Lacey and Ferguson, [9], it turns out to be essential that, for arbitrary $\delta > 0$, one can select $\operatorname{Enl}(\mathcal{U})$ such that $|\operatorname{Enl}(\mathcal{U})| \le (1+\delta)|\operatorname{sh}(\mathcal{U})|$. In this regard, also see Lacey and Terwilleger [13]. We investigate other examples where this can be obtained.

From time to time, we will refer to the set $\operatorname{Enl}(\mathcal{U})$ as V. It is interesting to note that the conclusion of Journé's Lemma implies the formally stronger conclusion that

$$\Big\| \sum_{R \in \mathcal{U}} \operatorname{emb}(R, \mathcal{U})^{-\epsilon} \mathbf{1}_R \Big\|_p \lesssim |\operatorname{sh}(\mathcal{U})|^{1/p}, \qquad 1 < p < \infty.$$

This is an immediate consequence of the John Nirenberg inequality, Lemma 2.12, in the product BMO setting.

1.1. **Notations and Conventions.** The dyadic intervals in $\mathbb{R}$ are

$$\mathcal{D} \stackrel{\text{def}}{=} \{[j2^k, (j+1)2^k) \,:\, j, k \in \mathbb{Z}\}.$$

This collection of intervals has the *grid property*, namely that for any two intervals $I, I' \in \mathcal{D}$ it is the case that $I \cap I' \in \{\emptyset, I, I'\}$. We return to the this property below.

The set $\mathcal{D}^d$ is then the set of dyadic rectangles in d dimensional space. Such a rectangle is a product $R = \prod_{j=1}^d R_{(j)}$, where $R_{(j)}$ is the product in the jth coordinate. $\mathcal{U}$ denotes a generic subset of $\mathcal{D}^d$. The *shadow* of $\mathcal{U}$ is

$$\operatorname{sh}(\mathcal{U}) \stackrel{\text{def}}{=} \bigcup_{R \in \mathcal{U}} R.$$

For a rectangle R and $\lambda > 0$ we set λR to be the rectangle with the same center as R, and whose dimension in each coordinate are to be λ times the corresponding dimension of R. It will be useful to have a more versatile notion of dilations. Thus, for a vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$, set

$$\operatorname{Dil}_{\vec{\lambda}} R \stackrel{\text{def}}{=} \otimes_{j=1}^d \lambda_j R_{(j)} \tag{1.3}$$

The dyadic rectangles are ordered by inclusion; *maximal elements* of $\mathcal{U}$ refer to rectangles that are maximal with respect to inclusion. This is quite a good partial order in one dimension of course: Two dyadic intervals intersect iff they are related under the partial order. It is less effective in higher dimensions, though a distinguishing feature of two dimensions is that if two non equal rectangles intersect and are not comparable, then the two sides of the rectangles must be in reverse order with respect to inclusion. This fails three parameters, and explains in part the difficulty in moving from two to three parameters in some of our arguments.

A set of dyadic rectangle $\mathcal{U}$ has *scales separated by* μ iff for any two rectangles $R, R' \in \mathcal{U}$, and for any j, if $|R_{(j)}| < |R'_{(j)}|$ then $\mu|R_{(j)}| < |R'_{(j)}|$. Any set $\mathcal{U}$ is a union of $\lesssim (\log \mu)^d$ subsets which have scales separated by μ. An example fact we shall rely upon is this. If I and J are intervals, with $|I| < |J|$ and $I \cap J \neq \emptyset$, then it is the case that

$$I \subset (1 + \tfrac{|I|}{|J|})J \subset \{\mathrm{M}\, \mathbf{1}_J > (1 + 2\tfrac{|I|}{|J|})^{-1}\} \tag{1.4}$$

where M is the maximal function in one dimension. We will be applying this with I and J dyadic intervals, and with scales separated by some large amount.

The strong maximal function is

$$\mathrm{M}\, f(x) = \sup_{x \ni R} \fint_R |f(y)|\, dy,$$

where the supremum is taken over all (non–dyadic) rectangles R in $\mathbb{R}^d$. In addition, we use the notation

$$\fint_A f\, dx = |A|^{-1} \int_A f\, dx.$$

We use without comment the L^p inequalities known for the strong maximal function. For intersecting rectangles R and R' we have

$$R \subset \prod_{j=1}^d \gamma_j R'_{(j)} \subset \Big\{\mathrm{M}\, \mathbf{1}_{R'} > \prod_{j=1}^d \gamma_j^{-1}\Big\}$$

where $\gamma_j \overset{\text{def}}{=} 1 + |R_{(j)}||R'_{(j)}|^{-1}$.

It is known, see e.g. the work of Melas[1] [16, 17], that even in one dimension, the maximal function maps L^1 into $L^{1,\infty}$ with norm strictly bigger than one. The dyadic maximal function however maps L^1 into $L^{1,\infty}$ with norm 1. We shall have need of a variant of this well–known fact.

Define a *grid* to be a collection $\mathcal{I}$ of intervals in the real line for which for all $I, I' \in \mathcal{I}$, $I \cap I' \in \{\emptyset, I, I'\}$. For a collection of intervals $\mathcal{I}$, not necessarily a grid, set

$$\mathrm{M}^{\mathcal{I}}\, f(x) \overset{\text{def}}{=} \sup_{I \in \mathcal{I}} \mathbf{1}_I(x) \fint_I f(y)\, dy. \tag{1.5}$$

Then, for any grid $\mathcal{I}$, $\mathrm{M}^{\mathcal{I}}$ maps $L^1(\mathbb{R})$ into into $L^{1,\infty}(\mathbb{R})$ with norm one. This, in particular, is true for the dyadic grid $\mathcal{D}$.

This fact we prove here, for the sake of completeness. For a non negative integrable f, and $\lambda > 0$, the set $\{\mathrm{M}^{\mathcal{I}}\, f > \lambda\}$ is union of intervals in the grid. Hence is a disjoint union of intervals in $\mathcal{I}' \subset \mathcal{I}$. For each interval $I \in \mathcal{I}'$, we must have

$$\fint_I f\, dx \geq \lambda.$$

[1] In fact, Melas computes the exact constant in the weak type inequality.

Hence,

$$\lambda|\{\mathrm{M}^{\mathcal{I}} f > \lambda\}| = \lambda \sum_{I \in \mathcal{I}'} |I| \le \sum_{I \in \mathcal{I}'} \int_I f \, dx \le \|f\|_1.$$

We shall have need of a notion of *shifted dyadic grids*, due to M. Christ, defined as follows. The definition of the grids depends upon a choice of integer d, and set $\delta = (2^{\mathsf{d}}+1)^{-1}$ for integer d. For integers $0 \le b < \mathsf{d}$, and $\alpha \in \{\pm(2^{\mathsf{d}}+1)^{-1}\}$, let

$$\begin{aligned} \mathcal{D}_{\mathsf{d},b,\alpha} &\stackrel{\text{def}}{=} \{2^{k\mathsf{d}+b}((0,1) + j + (-1)^k \alpha) \,:\, k \in \mathbb{Z},\ j \in \mathbb{Z}\}. \\ \mathcal{D}_{\mathsf{d}} &\stackrel{\text{def}}{=} \bigcup_{\alpha} \bigcup_{b=0}^{\mathsf{d}-1} \mathcal{D}_{\mathsf{d},b,\alpha}. \end{aligned} \tag{1.6}$$

One checks that $\mathcal{D}_{\mathsf{d},b,\alpha}$ is a grid. Indeed, it suffices to assume $\alpha = (2^{\mathsf{d}}+1)^{-1}$, and that $b = 0$. Checking the grid structure can be done by induction. And it suffices to check that the intervals in $\mathcal{D}_{\mathsf{d},0,\alpha}$ of length one are a union of intervals in $\mathcal{D}_{\mathsf{d},0,\alpha}$ of length $2^{-\mathsf{d}}$. One need only check this for the interval $(0,1)+\alpha$. But certainly

$$\begin{aligned} (0,1) + (2^{\mathsf{d}}+1) &= \bigcup_{j=0}^{2^{\mathsf{d}}-1} (0, 2^{-d}) + \frac{j}{2^{\mathsf{d}}} + (2^{\mathsf{d}}+1) \\ &= \bigcup_{j=0}^{2^{\mathsf{d}}-1} (0, 2^{-d}) + \frac{j+1}{2^{\mathsf{d}}} + 2^{\mathsf{d}}(2^{\mathsf{d}}+1) \end{aligned}$$

And this proves the claim.

What is just as important concerns the collections $\mathcal{D}_{\mathsf{d}}$. For each dyadic interval $I \in \mathcal{D}$, $I \pm \delta|I| \in \mathcal{D}_{\mathsf{d}}$.[2] Moreover, the maximal function $\mathrm{M}^{\mathcal{D}_{\mathsf{d}}}$ maps L^1 into $L^{1,\infty}$[3] with norm at most $2\mathsf{d} \simeq |\log \delta|$. In fact we need the finer estimate

$$|\{\mathrm{M}^{\mathcal{D}_{\mathsf{d}}} \mathbf{1}_{\mathrm{sh}(U)} > 1 - \delta\}| \le (1 + K\delta\mathsf{d})|\mathrm{sh}(U)|, \tag{1.7}$$

for all subset U of the real line of finite measure, and some constant K. This is an effective estimate since $\delta\mathsf{d} \simeq \delta|\log\delta| \longrightarrow 0$, as $\delta \longrightarrow 0$.

To see this estimate, note that

$$\begin{aligned} |\{\mathrm{M}^{\mathcal{D}_{\mathsf{d}}} \mathbf{1}_{\mathrm{sh}(U)} > 1-\delta\}| &\le |\mathrm{sh}(U)| + \sum_{b=0}^{d-1} \sum_{\alpha \in \{\pm(2^{\mathsf{d}}+1)^{-1}\}} |U^c \cap \{\mathrm{M}^{\mathcal{D}_{\mathsf{d},b,\alpha}} \mathbf{1}_{\mathrm{sh}(U)} > 1-\delta\}| \\ &\le (1 + 2\mathsf{d}[(1-\delta)^{-1} - 1])|\mathrm{sh}(U)|. \end{aligned}$$

[2]The problem we are avoiding here is that the dyadic grid distinguishes dyadic rational points. At the point 0, for instance, observe that for all integers k, $(1+\delta)(0,1) \not\subset (0, 2^k)$, regardless of how big k is.

[3]In fact, taking $\mathsf{d} = 1$, it is routine to check that $\mathrm{M}^{\mathcal{D}_1}$ dominates an absolute multiple of the usual maximal function. Thus, proving that it satisfies the weak type inequality.

In statements of Journé's Lemma, $\mathcal{U}$ will denote a generic collection of rectangles of $\mathbb{R}^d$, whose shadow is of finite measure. The statement of the Lemma will depend upon a particular choice of *enlarged set* which we will always define in terms of some maximal function. It will be denoted as $\operatorname{Enl}(\mathcal{U})$, or more simply as V. At times, this definition will be iterated. In this case, we denote the enlarged set as $\operatorname{Enl}_j(U)$, the subscript j denoting the number of times the definition is iterated.[4]

Journé's Lemma also depends upon a notion of *embeddedness* of a rectangle $R \in \mathcal{U}$, relative to the enlarged set $\operatorname{Enl}(\mathcal{U})$. If there is no ambiguity about the enlarged set, we use the notation $\operatorname{emb}(R,\mathcal{U})$. Otherwise, the notation $\operatorname{emb}(R, \operatorname{Enl}_j(U))$ is used. The definition of the enlarged set, and the notion of embeddedness will vary from section from section, but the notation will not.

Many factors, arising in most instances from combinatorial considerations, are increasing like a power of $\log \operatorname{emb}(R,\mathcal{U})$. These terms are considered to be inconsequential. One instance of this which frequently arises is as follows. Let $\mu > 1$ and let $\mathcal{U}$ be a collection of rectangles with a shadow of finite measure. Let $\mathcal{U}' \subset \mathcal{U}$ satisfy $\mu \le \operatorname{emb}(R,\mathcal{U}) \le 2\mu$ for all $R \in \mathcal{U}'$, and the scales of $\mathcal{U}'$ are separated by $10^{3d}\mu$. Then to prove Journé's Lemma, it suffices to show that

$$\sum_{R\in\mathcal{U}'} |R| \lesssim |\operatorname{sh}(\mathcal{U}')|. \tag{1.8}$$

This we will refer to as the *standard reduction.*

This last inequality obviously holds if the rectangles in $\mathcal{U}$ are *essentially disjoint.* That is, there is a choice of absolute constant c, and there are sets $E(R) \subset R$ so that $\{E(R) : R \in \mathcal{U}\}$ are pairwise disjoint sets. And that $|E(R)| \ge c|R|$. Obtaining this, or a property similar to it, is an obvious strategy for proving (1.8) in a manner that is uniform with respect to $\mathcal{U}' \subset \mathcal{U}$.

We write $A \lesssim B$ if there is an (unimportant) absolute constant K (permitted to depend upon parameters as specified in e.g. the statement of a Proposition) such that $A \le KB$. $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Hardy Space, BMO, and Carleson Measures in the Product Theory

The realm of application of Journé's Lemma is to the product BMO theory. Functions in this class are described by their Carleson measures. We survey these subjects, beginning with the Carleson measures, and including explicit definitions and Lemmas that follow from the more purely geometric versions of Journé's Lemma that are in other parts of this paper.

[4]In some statements of Journé's Lemma, the role of the enlarged set is suppressed, and only the "embeddedness" terms are used. In this paper, we are of course concerned with the selection of the "enlargement" and some of the enlargement's properties.

2.1. **Carleson Measures.** Journé's Lemma is most directly applied to the control of Carleson measures in the product setting. And we first address this implication, following up with connections to the product Hardy space theory.

For a map $\alpha : \mathcal{D}^d \longrightarrow \mathbb{R}_+$, set

$$\|\alpha\|_{CM} \stackrel{\text{def}}{=} \sup_{\mathcal{U}} |\text{sh}(\mathcal{U})|^{-1} \sum_{R\in\mathcal{U}} \alpha(R). \tag{2.9}$$

"CM" is for Carleson measure. What is most essential here is that the supremum is taken over all subsets $U \subset \mathbb{R}^d$ of finite measure. In one dimension, a small additional argument permits one to restrict the supremum to intervals.

This definition is confusing, as there are no measures present. In Section 2.2 we recall the more classical definition of a Carleson measure.

In more parameters, it is natural to suppose that one should be able to restrict the supremum above to rectangles. While this is not the case,[5] this supremum does play a distinguished role in the theory, and we denote this supremum by $\|\alpha\|_{CM(\text{rec})}$.

In particular, in dimensions 2 and higher, for all $\epsilon > 0$, there are Carleson measures α with $\|\alpha\|_{CM} = 1$ and $\|\alpha\|_{CM(\text{rec})} < \epsilon$. The main application of Journé's Lemma is to show that despite this general difficulty, we can in some instances use the rectangular norm to control the general norm.

Corollary 2.10. *For all $\epsilon > 0$, all $\mu > 1$, and collections of rectangles $\mathcal{U}$ whose shadow has finite area in the plane, let $\mathcal{U}_\mu \subset \mathcal{U}$ be a collection of rectangles with* $\text{emb}(R,\mathcal{U}) \simeq \mu$. *Then,*

$$\|\alpha|_{\mathcal{U}_\mu}\|_{CM} \lesssim \mu^\epsilon \|\alpha\|_{CM(\text{rec})}$$

It is to be stressed that this Lemma, as stated, is restricted to the plane. With more than two parameters, we need to either take more care with the definition of embeddedness, or with the definition of the "rectangular" norm.

Proof. We should see that for all sets $\mathcal{V} \subset \mathcal{U}_\mu$, we have

$$\sum_{R\in\mathcal{V}} \alpha(R) \lesssim \mu^\epsilon \|\alpha\|_{CM(\text{rec})} |\text{sh}(\mathcal{V})|.$$

Let $\mathcal{V}'$ be the maximal dyadic rectangles in $\mathcal{V}$. Then,

$$\begin{aligned}\sum_{R'\in\mathcal{V}'} \sum_{\substack{R\in\mathcal{V}\\ R\subset R'}} \alpha(R) &\le \|\alpha\|_{CM(\text{rec})} \sum_{R'\in\mathcal{V}'} |R'| \\ &\lesssim \mu^\epsilon \|\alpha\|_{CM(\text{rec})} |\text{sh}(\mathcal{V})|\end{aligned}$$

[5]Historically, these examples did not arrive in this way, but where phrased in the language of the Hardy space H^1, and it's dual. We comment in more detail below.

In the top line we have used the definition of the rectangular Carleson measure norm, and in the bottom Journé's Lemma, as stated in Lemma 1.1 say. □

In higher parameters, one can continue to use the rectangular norm, using instead of the planar version of Journé's Lemma, the form as stated in Lemma 5.38. What is more interesting is to define a notion of Carleson measure norms that uses Lemma 4.33. Towards this end, let us say that a collection $\mathcal{U}$ of rectangles in $\mathbb{R}^d$ has ℓ parameters iff there is a subset $L \subset \{1, \dots, d\}$ with $|L| = \ell$, so that for any two rectangles $R, R' \in \mathcal{U}$ we have $R_{(j)} = R'_{(j)}$ for all $j \notin L$. Let us set

$$\|\alpha\|_{CM(\ell)} = \sup_{\mathcal{U}\ \ell \text{ parameters}} |\mathrm{sh}(\mathcal{U})|^{-1} \sum_{R \in \mathcal{U}} \alpha(R).$$

Notice that the $CM(1)$ norm reduces to essentially the most natural extension of the rectangular norm to higher parameters. These norms will increase in ℓ. In general, one cannot control the $CM(\ell)$ norm by the $CM(\ell - 1)$ norm, except through devices like Journé's Lemma.

This definition goes someway towards capturing the subtle way that Carleson measures of d parameters are built up from those of $d-1$ parameters. In particular, we have the following Lemma, in which we use the notations of (4.31) and (4.32). We only state this Lemma in the case of $\ell = d-1$ as it is the only case that has found application to date.

Proposition 2.11. *For all $\delta > 0$ the following holds. Let $\mathcal{U}$ be a collection of rectangles in $\mathbb{R}^d$ whose shadow has finite measure, and for $\mu > 1$ set*

$$\mathcal{U}_\mu \stackrel{\text{def}}{=} \{R \in \mathcal{U} \,:\, \mu \le \mathrm{emb}(R, \mathrm{sh}(\mathcal{U})) \le 2\mu\}.$$

Then, we have

$$\|\alpha|_{\mathcal{U}_\mu}\|_{CM(d)} \lesssim \mu^\epsilon \|\alpha\|_{CM(d-1)}$$

The implied constant depends upon $\epsilon > 0$.

These concepts, and this lemma are used in Lacey and Terwilleger [13].

An important aspect of the subject is the connection of the definition of Carleson measures to a John–Nirenberg inequality.

Lemma 2.12. *We have the inequality below, valid for all collections of rectangles $\mathcal{U}$ whose shadows have finite measure.*

$$\Big\| \sum_{R \in \mathcal{U}} \frac{\alpha(R)}{|R|} \mathbf{1}_R \Big\|_p \lesssim \|\alpha\|_{CM} |\mathrm{sh}(\mathcal{U})|^{1/p}, \qquad 1 < p < \infty.$$

Proof. This is the proof by duality from [4]. Let $\|\alpha\|_{CM} = 1$. Define

$$F_V \stackrel{\text{def}}{=} \sum_{R \subset V} \frac{\alpha(R)}{|R|} \mathbf{1}_R$$

We shall show that for all $\mathcal{U}$, there is a set V satisfying $|V| < 2|\mathrm{sh}(\mathcal{U})|$ for which

$$\|F_{\mathrm{sh}(\mathcal{U})}\|_p \lesssim |\mathrm{sh}(\mathcal{U})|^{1/p} + \|F_V\|_p \tag{2.13}$$

Clearly, inductive application of this inequality will prove our Lemma.

The argument for (2.13) is by duality. Thus, for a given $1 < p < \infty$, and conjugate index p', take $g \in L^{p'}$ of norm one so that $\|F_U\|_p = \langle F_U, g\rangle$. Set

$$V = \{\mathrm{M}\, g > K|\mathrm{sh}(\mathcal{U})|^{-1/p'}\}$$

where M is the strong maximal function and K is sufficiently large so that $|V| < 2|\mathrm{sh}(\mathcal{U})|$. Then,

$$\langle F_{\mathrm{sh}(\mathcal{U})}, g\rangle = \sum_{\substack{R\in\mathcal{U}\\ R\not\subset V}} \alpha(R) \fint_R g\, dx + \langle F_V, g\rangle$$

The second term is at most $\|F_V\|_p$ by Hölder's inequality. For the first term, note that the average of g over R can be at most $K|\mathrm{sh}(\mathcal{U})|^{-1/p'}$. So by the definition of Carleson measure norm, it is at most

$$\sum_{\substack{R\in\mathcal{U}\\ R\not\subset V}} \alpha(R) \fint_R g\, dx \lesssim |\mathrm{sh}(\mathcal{U})|^{-1/p'} \sum_{R\in\mathcal{U}} \alpha(R) \lesssim |\mathrm{sh}(\mathcal{U})|^{1/p},$$

as required by (2.13).

□

2.2. **Classical Definition, Carleson Embedding Theorem.** Our use of the the term "Carleson measure" is not the standard one. Given a function $\alpha : \mathcal{D}^d \longrightarrow \mathbb{R}_+$, define a measure on $\mathbb{R}^d \times \mathbb{R}^d_+$ by

$$\mu_\alpha = \sum_{R\in\mathcal{D}^d} \alpha(R)\delta_{R\times\|R\|}$$

where $\|R\| = (|R_{(1)}|, \ldots, |R_{(d)}|)$. In the instance that $\alpha(R) = |R|^{-1}|\langle f, h_R\rangle|^2$, the measure μ_α is of the type associated with the area integral of f. (Indeed, in this setting both the continuous and discrete formulations are equivalent.)

For a set $U \subset \mathbb{R}^d$, define an associated set $\mathrm{Tent}(U) \subset \mathbb{R}^d \times \mathbb{R}^d_+$ by

$$\mathrm{Tent}(U) \stackrel{\mathrm{def}}{=} \bigcup_{\substack{R\in\mathcal{D}^d\\ R\subset U}} R \times [0, |R_{(1)}|] \times \cdots \times [0, |R_{(d)}|]$$

This is the *tent over* U. Then, the substance of the Carleson measure condition is the inequality

$$\mu_\alpha(\mathrm{Tent}(U)) \le \|\alpha\|_{CM}|U|,$$

for all sets $U \subset \mathbb{R}^d$ of finite measure. Notice that the left hand side concerns objects of $2d$ dimensions, while the right hand side has only dimension d.

The importance of the Carleson measure condition arises from the Carleson Embedding Theorem, which we again state in a discrete form. Given a function $\alpha : \mathcal{D}^d \mapsto [0,\infty)$, define an operator

$$\mathrm{T}_\alpha f \stackrel{\text{def}}{=} \sum_{I\in\mathcal{D}^d} \alpha(R)\mathbf{1}_R \fint_R f(y)\,dy$$

Theorem 2.14. *We have the equivalence below, valid for all $1<p<\infty$.*

$$\|\mathrm{T}_\alpha\|_p \simeq \|\alpha\|_{CM}$$

Proof. The inequality $\|T_\alpha\|_p \gtrsim \|\alpha\|_{CM}$ follows by testing the operator T_α against a function $f=\mathbf{1}_U$. Thus,

$$\begin{aligned}\Big\|\sum_{R\subset U}\alpha(R)\mathbf{1}_R\Big\|_p &\le \|\mathrm{T}_\alpha\,\mathbf{1}_U\|_p\\ &\lesssim \|\mathrm{T}_\alpha\|_p|U|^{1/p}.\end{aligned}$$

This condition appears stronger than the definition of the Carleson measure norm, but the John–Nirenberg inequality of course implies that it is equivalent to this definition.

And we shall find the John Nirenberg inequality essential for the proof of the reverse inequality. We do not prove the strong type inequality directly, but rather prove the weak type inequality

$$|\{\mathrm{T}_\alpha f>\lambda\}| \lesssim \|\alpha\|_{CM}^p\lambda^{-p}\|f\|_p^p, \qquad 1<p<\infty.$$

To prove this inequality, let us observe that the definition of the Carleson measure norm, and that of the operators T_α is invariant under dilations. Namely, letting μ be any power of 2, and setting

$$\beta(R) \stackrel{\text{def}}{=} \alpha(\mathrm{Dil}_{(\mu,\cdots,\mu)}\,R)$$

we have $\|\beta\|_{CM}=\|\alpha\|_{CM}$. And,

$$\mathrm{T}_\beta = \mathrm{T}_\alpha\,\mathrm{Dil}_{(\mu,\cdots,\mu)}\,1$$

Thus, it suffices to prove a single instance of the weak type inequality. Namely, that for $1<p<\infty$, there is a constant K_p so that for all α with Carleson measure norm 1, and all functions $f\in L^p$ of norm one, we have

$$|\{\mathrm{T}_\alpha f>1\}| \le K_p. \tag{2.15}$$

We inductively decompose the collection of dyadic rectangles. In the base step, take

$$\mathcal{U}_0 \stackrel{\text{def}}{=} \{R\in\mathcal{D}^d : \fint_R f(y)\,dy \ge 1\}$$

Set Stock $\stackrel{\text{def}}{=} \mathcal{D}^d - \mathcal{U}_0$. In the inductive stage, given $\mathcal{U}_0, \cdots, \mathcal{U}_k$, to construct $\mathcal{U}_{k+1}$, we set

$$\mathcal{U}_{k+1} \stackrel{\text{def}}{=} \{R \in \text{Stock} : \fint_R f(y)\, dy \geq 2^{-k+1}\}.$$

Then, update Stock $\stackrel{\text{def}}{=}$ Stock $-\mathcal{U}_{k+1}$.

By the strong Maximal Function estimate, we have

$$|\mathrm{sh}(\mathcal{U}_k)| \lesssim 2^{kp}, \qquad k \geq 0.$$

Thus, we shall not even estimate $\mathrm{T}_\alpha f$ on the set $\mathrm{sh}(\mathcal{U}_0)$.

For the collections $\mathcal{U}_k$ for $k \geq 1$, we have an upper bound on the average of f over those rectangles $R \in \mathcal{U}_k$. This, with the John–Nirenberg, will give us a favorable estimate in L^s norm, for a choice of $s > p$.

$$\begin{aligned}\Big\| \sum_{R \in \mathcal{U}_k} \alpha(R) \fint_R f(y)\, dy \mathbf{1}_R \Big\|_s &\leq 2^{-k+1} \Big\| \sum_{R \in \mathcal{U}_k} \alpha(R) \mathbf{1}_R \Big\|_s \\ &\lesssim 2^{-k} |\mathrm{sh}(\mathcal{U}_k)|^{1/s} \\ &\lesssim 2^{-k(1-p/s)}.\end{aligned}$$

This is summable over $k \geq 1$, and so easily completes the proof of (2.15).

□

2.3. **The Product Hardy Theory.** We turn to the product Hardy space theory, as developed by S.-Y. Chang and R. Fefferman [3–7]. $\mathrm{H}^1(\mathbb{C}_+^d)$ will denote the real d–fold product Hardy space. This space consists of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$. $\mathbb{R}^d$ is viewed as the boundary of

$$\mathbb{C}_+^d = \prod_{j=1}^d \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

And we require that there is a function $F : \mathbb{C}_+^d \longrightarrow \mathbb{C}$ that is holomorphic in each variable separately, and

$$f(x) = \lim_{\|y\| \to 0} \mathrm{Re}(F(x_1 + iy_1, \ldots, x_d + iy_d)).$$

The norm of f is taken to be

$$\|f\|_{\mathrm{H}^1} = \lim_{y_1 \downarrow 0} \cdots \lim_{y_d \downarrow 0} \|F(x_1 + y_1, \ldots, x_d + y_d)\|_{L^1(\mathbb{R}^d)}$$

The dual of this space is $\mathrm{Re}\,\mathrm{H}^1(\mathbb{C}_+^d)^* = \mathrm{BMO}(\mathbb{C}_+^d)$, the d–fold product BMO space. It is a Theorem of S.-Y. Chang and R. Fefferman [4] that this space has a characterization in terms of the product Carleson measure introduced above. We need the product Haar basis. Thus, set

$$h(x) = -\mathbf{1}_{[-\frac{1}{2},0]}(x) + \mathbf{1}_{[0,\frac{1}{2}]}(x), \qquad h_I(x) = h\Big(\frac{x-c(I)}{|I|}\Big), \qquad I \in \mathcal{D}.$$

The functions $\{h_I \,:\, I \in \mathcal{D}\}$ are the Haar basis for $L^2(\mathbb{R})$, which is closely associated with the analysis of singular integrals. For a rectangle $R = \prod_{j=1}^d R_{(j)} \in \mathcal{D}^d$ set

$$h_R(x_1, \dots, x_d) = \prod_{j=1}^d h_{R_{(j)}}(x_j).$$

The basis $\{h_R \,:\, R \in \mathcal{D}^d\}$ is the d–fold tensor product of the Haar basis. Then it is the Theorem of Chang and Fefferman that the product BMO space has the equivalent norm

$$\begin{aligned}\|b\|_{\mathrm{BMO}}^2 &= \sup_{\mathcal{U}\subset\mathbb{R}^d} |\mathrm{sh}(\mathcal{U})|^{-1} \sum_{R\in\mathcal{U}} |\langle b, h_R\rangle|^2 \\ &= \|R \longrightarrow |\langle b, h_R\rangle|^2\|_{CM}\end{aligned}$$

The next comments are specific to the two parameter case, namely $\mathrm{H}^1(\mathbb{C}_+^2)$. The space BMO(rec) has the definition

$$\|b\|_{\mathrm{BMO(rec)}}^2 = \|R \longrightarrow |\langle b, h_R\rangle|^2\|_{CM(\mathrm{rec})}$$

It was at first, natural supposition that this space is the dual to H^1. This stems in part from the fact that the rectangular BMO norm has an equivalent formulation in terms that look quite familiar:

$$\|b\|_{\mathrm{BMO(rec)}}^2 =$$
$$\sup_{I\times J} \fint_{I\times J} \Big| b(x,y) - \fint_I f(x,y)\, dx - \fint_J f(x,y)\, dy + \fint_{I\times J} f(x,y)\, dx, dx \Big|^2 \, dx\, dy.$$

This of course looks like the familiar intrinsic definition of BMO in terms of bounded mean oscillation over intervals in the real line. But an example of Carleson [2] consisted of a class of functions which acted as linear functionals on H^1 with norm one, yet had arbitrarily small BMO(rec) norm. This example is recounted at the beginning of R. Fefferman's article [7].

Parallel to Corollary 2.10, we have this Corollary to Journé's Lemma.

Corollary 2.16. *For all $\epsilon > 0$, all $\mu > 1$, collections $\mathcal{U}$ of rectangles in the plane whose shadow has finite measure, let $\mathcal{U}_\mu$ be a collection of rectangles with* $\mathrm{emb}(R; U) \simeq \mu$. *Then,*

$$\Big\| \sum_{R\in\mathcal{U}_\mu} \langle f, h_R\rangle h_R \Big\|_{\mathrm{BMO}} \lesssim \mu^\epsilon \|b\|_{\mathrm{BMO(rec)}}$$

There is a corresponding notion of a BMO$(d-1)$ norm, and a Lemma that is parallel to Proposition 2.11, but we will not state it explicitly.

3. Journé's Lemma in Two Parameters

We state and prove different versions of Journé's Lemma in the two parameter setting. We shall be explicit about the definition of the expanded set, and somewhat flagrant with logarithms of $\operatorname{emb}(R,\mathcal{U})$. This is in contrast to the original references, which give slightly more precise estimates for the sum in the Lemma than we do.

3.1. **The Original Formulation.** Two proofs of the Lemma in its original formulation, namely Lemma 1.1, are given.

3.1.1. *The First Proof.* We define, as above, $\operatorname{Enl}(\mathcal{U}) \overset{\text{def}}{=} \{\mathrm{M}\,\mathbf{1}_{\operatorname{sh}(\mathcal{U})} > \frac{1}{2}\}$, and

$$\operatorname{emb}(R,\mathcal{U}) \overset{\text{def}}{=} \sup\{\mu \geq 1 \,:\, \operatorname{Dil}_{(\mu,1)} R \subset \operatorname{Enl}(\mathcal{U})\} \tag{3.17}$$

In particular we only expand R in it's first coordinate. We are to prove Lemma 1.1.

We pass to the standard reduction,[6] see (1.8). In particular we will use the "essentially disjoint" argument mentioned immediately below (1.8). Say that $R <_1 R'$ if $R \cap R' \neq \emptyset$ and $|R_{(1)}| < |R'_{(1)}|$. Consider the collection Bad of rectangles $R \in \mathcal{U}$ for which there are $R^1, \ldots, R^J$ in $\mathcal{U}$ with $R <_1 R^j$ and finally, that

$$\Big| R \cap \bigcup_{j=1}^{J} R^j \Big| \geq \tfrac{7}{8}|R|.$$

Observe that the collection Bad is empty. Indeed, if $R \in \text{Bad}$, then it must be the case that $\operatorname{emb}(R,\mathcal{U}) \geq 10\mu$, which is a contradiction. This is a straightforward consequence of (1.4) and the fact that we defined the enlarged set in terms of the strong maximal function. Thus, there is at least $\frac{1}{8}$ of each rectangle $R \in \mathcal{U}$ that is disjoint from all other rectangles in $\mathcal{U}$. And so the rectangles in $\mathcal{U}$ are essentially disjoint, and we have completed the proof.

3.1.2. *The Second Proof.* The second proof begins with a key new definition for dyadic intervals I and integers $k \geq 0$. For a subcollection $\mathcal{U}' \subset \mathcal{U}$ that is fixed, set

$$\mathcal{E}(I,k) \overset{\text{def}}{=} \{I \times J \in \mathcal{U}' \,:\, \operatorname{Dil}_{(2^k,1)} I \times J \subset \operatorname{sh}(\mathcal{U})\}.$$

We will suppress the dependence on the choice of $\mathcal{U}'$. There are two points to observe. First, due to the maximality of the rectangles in $\mathcal{U}$, we have

$$\sum_{R \in \mathcal{E}(I,k)} |R| \lesssim 2|\operatorname{sh}(\mathcal{E}(I,k))|. \tag{3.18}$$

[6] It will be clear that in this instance we need only separate scales in first coordinate, not both as we have defined the standard reduction.

Second, we consider two dyadic intervals $I \subset I'$, with $2^n|I| \le |I'|$. If it is the case that

$$|I \times J \cap \operatorname{sh}(\mathcal{E}(I',k))| \ge \tfrac{1}{2}|I \times J|, \tag{3.19}$$

then it must be that $\operatorname{emb}(I \times J, \mathcal{U}) \ge 2^{k+n-1}$.

Thus, if we take $\mathcal{U}'$ to be a subset of rectangles $R \in \mathcal{U}$, with $\operatorname{emb}(R,\mathcal{U}) \le 2^{k_0}$ for all rectangles. It follows from (3.19) that

$$\sum_{I \in \mathcal{D}} |\operatorname{sh}(\mathcal{E}(I,k))| \le k_0 |\operatorname{sh}(\mathcal{U})|, \qquad k \le k_0.$$

And for $k > k_0$, we have $\mathcal{E}(I,k) = \emptyset$ for all I. By (3.18) this completes the proof.

3.2. **Uniform Embeddedness in Two Parameters.** We define the notion of embeddedness by simultaneously expanding all sides of the rectangle. Let U be a subset of $\mathbb{R}^2$ of finite measure. We inductively define a sequence of enlarged sets associated to U by

$$\operatorname{Enl}_2(\mathcal{U}) \stackrel{\text{def}}{=} \{M\mathbf{1}_{\operatorname{sh}(\mathcal{U})} > \tfrac{1}{16}\}, \tag{3.20}$$

$$\operatorname{Enl}_{j+1}(\mathcal{U}) \stackrel{\text{def}}{=} \operatorname{Enl}_2(\operatorname{Enl}_j(U)) \quad j > 2. \tag{3.21}$$

Given a dyadic rectangle $R \in \mathcal{U}$, we give measures of how deeply embedded this rectangle is inside of U by

$$\operatorname{emb}(R, \operatorname{Enl}_j(\mathcal{U})) \stackrel{\text{def}}{=} \sup\{\mu \ge 1 \,:\, \mu R \subset \operatorname{Enl}_j(\mathcal{U})\}, \quad j \ge 2. \tag{3.22}$$

One can construct examples in which for many rectangles, this measure of embeddedness is essentially smaller than the measure used above.

Lemma 3.23. *For all $\epsilon > 0$, for all collections of rectangles $\mathcal{U}$, whose shadow has finite measure in the plane, we have the inequality*

$$\sum_{R \in \mathcal{U}'} \operatorname{emb}(R, \operatorname{Enl}_2(\mathcal{U}))^{-\epsilon} |R| \lesssim |\operatorname{sh}(\mathcal{U}')|.$$

The implied constant depends only on ϵ, and holds uniformly over all collections $\mathcal{U}' \subset \mathcal{U}$.

This form of the Journé Lemma was first proved in Ferguson and Lacey [9].

3.2.1. *The First Proof.* We rely very much on the version of Journé's Lemma that we have already established. Indeed, we will need a variant of this Lemma, one in which the standard dyadic grid is replaced by a shifted dyadic grid, as defined in Section 1.1. See in particular (1.6).

Apply Lemma 1.1, to a collection of rectangles $\mathcal{U}$. For an integer k, we consider a collection of rectangles $R \in \mathcal{U}$ such that $\operatorname{emb}(R,\mathcal{U}) \le 2^k$, where the embeddedness quantity is defined as in (3.17). Call this collection $\mathcal{U}'$.

We now define a new collection $\mathcal{V}$ of rectangles. These rectangles will be a product of $\mathcal{D}_1$ and a dyadic interval. Recall that for the collection of intervals $\mathcal{D}_1$, for any interval $K \subset \mathbb{R}$, we can find $I, I' \in \mathcal{D}_1$ so that

$$\tfrac{1}{4}I \subset K \subset I' \subset 2K.$$

Now, for each $I \times J \in \mathcal{U}'$, take $\widetilde{I} \in \mathcal{D}_1$ to be the maximal element such that (i) $I \subset \widetilde{I}$ and (ii)

$$\widetilde{I} \times \tfrac{|\widetilde{I}|}{|I|} J \subset \operatorname{Enl}_2(\mathcal{U}).$$

Set $\mathcal{V} \stackrel{\text{def}}{=} \{\widetilde{I} \times J \,:\, I \times J \in \mathcal{U}'\}$. Certainly, we have by the Journé Lemma,

$$\begin{aligned} &|\operatorname{sh}(\mathcal{V})| \lesssim |\operatorname{sh}(\mathcal{U})|, \\ &\sum_{R \in \mathcal{U}'} |R| \lesssim 2^{\epsilon k} |\operatorname{sh}(\mathcal{U})|. \end{aligned}$$

In the second line, $\epsilon > 0$ is an arbitrary positive constant, and the implied constant depends upon ϵ.

Clearly, we want to apply the Journé Lemma to the collection $\mathcal{V}$ in the second coordinate. This is not quite straight forward to do, as the collection of rectangles $\mathcal{V}$ may not consist exclusively of pairwise incomparable rectangles. Yet, if we have two rectangles $\widetilde{I} \times J \subset \widetilde{I}' \times J'$, with both rectangles in the collection $\mathcal{V}$, and in addition we have

$$8|\widetilde{I}| \le |\widetilde{I}'|, \qquad 2^{k+2}|J| \le |J'|.$$

then, it would be the case that $\operatorname{emb}(I \times J, \mathcal{U}) > 2^k$, which is a contradiction. Therefore, we see that $\mathcal{V}$ is a union of at most $O(k)$ subcollections $\mathcal{V}'$, each of which consists only of pairwise incomparable rectangles. Thus, we deduce from Lemma 1.1 that

$$\sum_{R \in \mathcal{V}'} |R| \lesssim 2^{\epsilon k} |\operatorname{sh}(\mathcal{V})| \lesssim 2^{\epsilon k} |\operatorname{sh}(\mathcal{U})|.$$

Therefore, the proof is complete.

3.2.2. *The Second Proof.* We employ the standard reduction (1.8), and use the "essentially disjoint" argument to prove the Lemma.

The main construction of the proof is this inductive procedure. We construct a decomposition of $\mathcal{U}$ into "good" $\mathcal{G}(\mathcal{U})$ and "bad" $\mathcal{B}_j(\mathcal{U})$ parts, with $j = 1, 2$. Initialize

$$\text{Stock} \stackrel{\text{def}}{=} \mathcal{U}, \quad \mathcal{G} = \emptyset, \quad \mathcal{B}_j = \emptyset,\ j = 1, 2.$$

If Stock $= \emptyset$ we return $\mathcal{G}(\mathcal{U}) = \mathcal{G}$, $\mathcal{B}_j(\mathcal{U}) \stackrel{\text{def}}{=} \mathcal{B}_j$, for $j = 1, 2$.

While Stock is non–empty, select any $R \in$ Stock, and update

$$\text{Stock} = \text{Stock} - \{R\}, \quad \mathcal{G} = \mathcal{G} \cup \{R\}.$$

Continuing, for $j = 1, 2$, while there is an $R' \in \text{Stock}$ so that there are $R_1, R_2, \dots, R_N \in \mathcal{G}$ such that the R_n are longer than R' in the jth coordinate, and

$$\Big|R' \cap \bigcup_{n=1}^{N} R_n\Big| > \tfrac{8}{9}|R'|, \tag{3.24}$$

update

$$\text{Stock} = \text{Stock} - \{R'\}, \quad \mathcal{B}_j = \mathcal{B}_j \cup \{R'\}.$$

By construction, the rectangles in $\mathcal{G}(\mathcal{U})$ are essentially disjoint. It suffices therefore to argue that for $j = 1, 2$, we have

$$\mathcal{B}_j(\mathcal{B}_j(\mathcal{U})) = \emptyset. \tag{3.25}$$

And it follows that inductively applying the decomposition into good and bad parts to each of $\mathcal{B}_j(\mathcal{U})$ will terminate after three rounds.

Suppose by way of contradiction, that there is an $R \in \mathcal{B}_1(\mathcal{B}_1(\mathcal{U}))$. Thus, there are $R_1, R_2, \dots, R_N \in \mathcal{B}_1(\mathcal{U})$ for which each R_n is longer in the first coordinate and (3.24) holds. Then, suppose that R_1 has first coordinate $R_{1(1)}$ that among all the R_n is shortest in the first coordinate. Since each R_n is in $\mathcal{B}_1(\mathcal{U})$ each of these rectangles are themselves nearly covered by rectangles in $\mathcal{U}$ that are longer in the first coordinate. By the standard reduction, these rectangles are themselves much longer than $R_{1(1)}$. Hence, we take I to be the dyadic interval of length $10\mu|R_{1(1)}| \leq |I| < 20\mu|R_{1(1)}|$ that contains $R_{1(1)}$. Let J be the second coordinate of R. Then, it is necessarily the case that

$$|I \times J \cap \text{sh}(\mathcal{U}')| \geq (\tfrac{8}{9})^2|I \times J|.$$

But then, $\frac{9}{8}(I \times J) \subset \text{Enl}_2(\mathcal{U})$.

I is much larger than R_1 in the first coordinate, as we have separated scales. For the same reason, J is much longer than than R_1 in the second coordinate. Hence, we see that $3\mu R_1 \subset \frac{9}{8}(I \times J)$. But this contradicts the assumption that $\text{emb}(R, \text{Enl}_2(\mathcal{U})) \leq 2\mu$, and so completes the proof of the Lemma.

3.3. **Uniform Embeddedness with Small Enlargement in Two Parameters.**

In this section, our emphasis shifts to the enlarged sets. Specifically, we permit the enlarged set $\text{Enl}(\mathcal{U})$ to be only slightly bigger than $\text{sh}(\mathcal{U})$ itself, no more $|\text{Enl}(\mathcal{U})| \leq (1 + \delta)|\text{sh}(\mathcal{U})|$, where $\delta > 0$ is arbitrarily small.

We shall see that as δ decreases, the method by which we have to select it changes considerably. So let us emphasize that $U \subset V$, and that we shall define

$$\text{emb}(R, V) \stackrel{\text{def}}{=} \sup\{\mu \geq 1 \,:\, \mu R \subset V\}, \qquad R \in \mathcal{U}.$$

The fact we wish to explain is the Lemma from the Appendix of [9].[7]

Proposition 3.26. *For each $0 < \delta, \epsilon < 1$ there is a constant $K_{\delta,\epsilon}$, so that for all for all collections of rectangles $\mathcal{U}$ whose shadow has finite measure in the plane, there is a set $V \supset \operatorname{sh}(\mathcal{U})$ for which $|V| < (1+\delta)|\operatorname{sh}(\mathcal{U})|$, so that for any collection $\mathcal{U}' \subset \mathcal{U}$ we have the inequality*

$$\sum_{R \in \mathcal{U}'} \operatorname{emb}(R, V)^{-\epsilon} |R| \lesssim |\operatorname{sh}(\mathcal{U}')|. \tag{3.27}$$

The implied constant depends only on $\epsilon, \delta > 0$.

We define V. Recall the definition and properties of shifted dyadic grids, (1.6). For a collection of intervals $\mathcal{I}$ and $j = 1, 2$, set $\mathrm{M}_j^{\mathcal{I}}$ to be the maximal function associated to $\mathcal{I}$, computed in the coordinate j. Initially, we use only the dyadic grids, setting $\delta = (1 + 2^{\mathsf{d}})^{-1}$ and

$$\operatorname{Enl}_0(\mathcal{U}) \overset{\text{def}}{=} \bigcup_{i \neq j} \{\mathrm{M}_i^{\mathcal{D}} \, \mathbf{1}\{\mathrm{M}_j \, \mathbf{1}_{\operatorname{sh}(\mathcal{U})} > 1 - \delta\} > 1 - \delta\}.$$

It is clear that $|\operatorname{Enl}_0(\mathcal{U})| < (1 + K\delta)|\operatorname{sh}(\mathcal{U})|$. Invoking the collections $\mathcal{D}_{\mathsf{d}}$, set

$$\operatorname{Enl}(\mathcal{U}) \overset{\text{def}}{=} \bigcup_{i \neq j} \{\mathrm{M}_i^{\mathcal{D}_{\mathsf{d}}} \, \mathbf{1}\{\mathrm{M}_j^{\mathcal{D}_{\mathsf{d}}} \, \mathbf{1}_{V_0} > 1 - \delta\} > 1 - \delta\}. \tag{3.28}$$

Then $|\operatorname{Enl}(\mathcal{U})| < (1 + K\delta \log \delta^{-1})|\operatorname{sh}(\mathcal{U})|$, and we will work with this choice of V. This is the set V of the Lemma.

The additional important property that $\operatorname{Enl}(\mathcal{U})$ has can be formulated this way. For all dyadic rectangles $R = R_1 \times R_2 \subset \operatorname{Enl}_0(\mathcal{U})$, the four rectangles

$$(R_1 \pm \delta|R_1|) \times (R_2 \pm \delta|R_2|) \subset \operatorname{Enl}(\mathcal{U}) \tag{3.29}$$

This follows immediately from the construction of the shifted dyadic grids. The first stage of the proof is complete.

The remainder of the argument is as in Section 3.2. We impose the standard reduction, with the additional stipulation that the scales in $\mathcal{U}$ be separated by $10^6 \mu \delta^{-1}$. And we use the essentially disjoint proof strategy. There is a "bad" class of rectangles $\mathcal{B} = \mathcal{B}(\mathcal{U})$ to consider, defined as follows. For $j = 1, 2$, let $\mathcal{B}_j(\mathcal{U})$ be those rectangles R for which there are rectangles

$$R^1, R^2, \dots, R^K \in \mathcal{U} - \{R\},$$

so that for each $1 \le k \le K$, $|R_j^k| > |R_j|$, and

$$\left| R \cap \bigcup_{k=1}^{K} R^k \right| > (1 - \tfrac{\delta}{10})|R|.$$

[7] It seems likely one could also use the embeddedness in Section 3.4, but we do not pursue that here.

Thus $R \in \mathcal{B}_j$ if it is nearly completely covered by dyadic rectangles in the jth direction of the plane. Set $\mathcal{B}(\mathcal{U}) = \mathcal{B}_1(\mathcal{U}) \cup \mathcal{B}_2(\mathcal{U})$. It follows that if $R \notin \mathcal{B}(\mathcal{U})$, it is not covered in both the vertical and horizontal directions, hence

$$\Bigl| R \cap \bigcap_{R' \in \mathcal{U} - \{R\}} (R')^c \Bigr| \geq \tfrac{\delta^2}{100} |R|.$$

And so

$$\sum_{R \in \mathcal{U} - \mathcal{B}(\mathcal{U})} |R| \leq 100 \delta^{-2} |\mathrm{sh}(\mathcal{U})|.$$

Thus, it remains to consider the set of rectangles $\mathcal{B}_1(\mathcal{U})$ and $\mathcal{B}_2(\mathcal{U})$. Observe that for any collection $\mathcal{U}'$, $\mathcal{B}_j(\mathcal{U}') \subset \mathcal{U}'$ as follows immediately from the definition. Hence $\mathcal{B}_1(\mathcal{B}_2(\mathcal{B}_1(\mathcal{U}))) \subset \mathcal{B}_1(\mathcal{B}_1(\mathcal{U}))$. And we argue that this last set is empty. As our definition of $V \stackrel{\mathrm{def}}{=} \mathrm{Enl}(\mathcal{U})$ and $\mathrm{emb}(R, \mathcal{U})$ is symmetric with respect to the coordinate axes, this is enough to finish the proof.

We argue that $\mathcal{B}_1(\mathcal{B}_1(\mathcal{U}))$ is empty by contradiction. Assume that R is in this collection. Consider those rectangles R' in $\mathcal{B}_1(\mathcal{U})$ for which (i) $|R_1'| > |R_1|$ and (ii) $R' \cap R \neq \emptyset$. Then

$$\Bigl| R \cap \bigcup_{R' \in \mathcal{B}_1(\mathcal{U})} R' \Bigr| \geq (1 - \tfrac{\delta}{10}) |R|.$$

Fix a one of these rectangles R' with $|R_1'|$ being minimal. We then claim that $8\mu R' \subset \mathrm{Enl}(\mathcal{U})$, which contradicts the assumption that $\mathrm{emb}(R', \mathcal{U})$ is no more than 2μ.

Indeed, all the rectangles in $\mathcal{B}_1(\mathcal{U})$ are themselves covered by dyadic rectangles in the first coordinate axis. We see that the the set $\{M_2^{\mathcal{D}} \mathbf{1}_{\mathrm{sh}(\mathcal{U})} > 1 - \delta\}$ contains the dyadic rectangle $R_1'' \times R_2$, in which R_2 is the second coordinate interval for the rectangle R and R_1'' is the dyadic interval that contains R_1' and has measure $8\mu\delta^{-1}|R_1'| \leq |R_1''| < 16\mu\delta^{-1}|R_1'|$.

That is $R_1'' \times R_2$ is contained in $\mathrm{Enl}_0(\mathcal{U})$. And the dimensions of this rectangle are very much bigger than those of R. Applying (3.29), the rectangles $(R_1'' \pm |R_1''|) \times R_2 \pm \delta|R_2|$ are contained in $\mathrm{Enl}(\mathcal{U})$. And since $8\mu R'$ is contained in one of these last four rectangles, we have contradicted the assumption that $\mathrm{emb}(R', \mathcal{U}) < 2\mu$.

3.4. **Uniform Embeddedness Redux.** The previous notion of embeddedness expanded all directions in an equal amount. We propose here an alternate method, in which non diagonal dilations are used.[8] We continue with the definitions of (3.20). For a vector of positive numbers (μ_1, μ_2), set

$$\mathrm{emb}(R, \mathcal{U}) \stackrel{\mathrm{def}}{=} \sup\{\mu_1\mu_2 \,:\, \mathrm{Dil}_{(\mu_1,\mu_2)} R \subset \mathrm{Enl}_2(\mathcal{U}),\ \mu_1,\ \mu_2 \geq 1\}.$$

[8]Unlike the other formulations of Journé's Lemma in this paper, this one has not as of yet found application in the literature.

This definition of embeddedness can be essentially smaller than the form studied in Section 3.2.

Lemma 3.30. *In the case $d = 2$, for any $\epsilon > 0$, any collection of rectangles $\mathcal{U}$ in the plane, whose shadow has finite measure, and all $\mathcal{U}' \subset \mathcal{U}$ of rectangles which are maximal, we have*

$$\sum_{R \in \mathcal{U}'} \operatorname{emb}(R,\mathcal{U}')^{-\epsilon}|R| \lesssim |\operatorname{sh}(\mathcal{U}')|.$$

The implied constant depends only on $\epsilon > 0$.

To prove the Journé Lemma, we assume that $\mathcal{U}$ satisfies the standard reduction. We should further refine this reduction. Fix (μ_1, μ_2) with $\mu \leq \prod_{j=1}^{2} \mu_j \leq 2\mu$ and each $\mu_j \geq 1$. Assume that for each $R \in \mathcal{U}$ we have

$$\operatorname{Dil}_{(\mu_1/2,\mu_2/2)} R \subset \operatorname{Enl}_2(\mathcal{U}) \qquad \operatorname{Dil}_{2(\mu_1,\mu_2)} R \not\subset \operatorname{Enl}_2(U).$$

It suffices to consider $\lesssim (\log \mu)^3$ such collections. The argument of Section 3.2 proceeds with only modest changes.

4. High Parameter Case with Unions of Rectangles

We introduce a variant of Journé Lemma, in parameters three and higher, which can be found in J. Pipher's paper [18]. We measure embeddedness in only one coordinate, but then must form the sum over sets more general than rectangles.

The necessity of this can be seen by considering a set in $\mathbb{R}^3$ of the form $U = [0,1] \times U_2$, for a set $U_2 \subset \mathbb{R}^2$. Each rectangle $R = [0,1] \times R_2 \subset U$ has a measure of embeddedness in the first coordinate of 1. But the rectangles are certainly not disjoint in general in the second and third coordinates.

4.1. With Large Enlargement.

Let U be a subset of $\mathbb{R}^d$ with finite measure. Our Lemma makes sense in two parameters, but is primarily of interest in parameters $d \geq 3$. Let $\mathcal{U}$ be a set of maximal dyadic rectangles contained in U. And define the enlarged set, and embeddedness by

$$\operatorname{Enl}(\mathcal{U}) \stackrel{\text{def}}{=} \{\mathrm{M}\, \mathbf{1}_{\operatorname{sh}(\mathcal{U})} > \tfrac{1}{2}\}. \tag{4.31}$$

$$\operatorname{emb}(R,\mathcal{U}) \stackrel{\text{def}}{=} \sup\{\mu \geq 1 \,:\, \operatorname{Dil}_{(\mu,1,\dots,1)} R \subset \operatorname{Enl}(\mathcal{U})\}. \tag{4.32}$$

In the top line, the first maximal function M_1 is applied in the first coordinate only, and the second maximal function M is the usual strong maximal function.[9] The sets which we sum over are specified by the choice of subcollection $\mathcal{U}' \subset \mathcal{U}$, a choice of $j \in \mathbb{N}$ and dyadic interval $I \in \mathcal{D}$.

$$F(I, j, \mathcal{U}') \stackrel{\text{def}}{=} \bigcup \{I \times R' \,:\, I \times R' \in \mathcal{U}',\ 2^{j-1} \leq \operatorname{emb}(I \times R', \mathcal{U}) < 2^j\}.$$

[9] The second maximal function could be restricted to the strong maximal function in all coordinates *except* the first coordinate.

Our Lemma is then[10]

Lemma 4.33. *For all $\epsilon > 0$, we have the estimate*

$$\sum_{j=1}^{\infty} \sum_{I \in \mathcal{D}} 2^{-\epsilon j} |F(I, j, \mathcal{U}')| \lesssim |\mathrm{sh}(\mathcal{U}')|.$$

In fact, we have the estimate below, valid for any integer $n > 1$, and choice of $1 < p < \infty$.

$$\Big\| \sum_{j=1}^{\infty} \sum_{I \in \mathcal{D}} 2^{-\epsilon j} (\mathrm{M}\, \mathbf{1}_{F(I,j,\mathcal{U}')})^n \Big\|_p \lesssim |\mathrm{sh}(\mathcal{U}')|^{1/p}.$$

These estimates hold for all collections of rectangles $\mathcal{U}$ whose shadow has finite measure, and all collections $\mathcal{U}' \subset \mathcal{U}$.

This estimate has the most power when one has a collection of rectangles $\mathcal{U}$ for which the rectangles are embedded in the enlarged set by a small amount in the first coordinate, but embedded in the other coordinates by a very large amount.

We begin the proof, which draws upon the arguments of J. Pipher [18]. Fix an integer j. We assume that $2^{j-1} \le \mathrm{emb}(R, \mathcal{U}) < 2^j$ for all $R \in \mathcal{U}$, and separate scales accordingly. Following a modification of the "essentially disjoint" proof strategy, we then identify a subset $H(I) \subset F(I, j, \mathcal{U}')$ for which

$$|H(I)| \gtrsim |F(I, j, \mathcal{U}')|, \qquad \mathrm{M}\, \mathbf{1}_{H(I)} \gtrsim \mathbf{1}_{F(I,j,\mathcal{U}')},$$

and these sets are disjoint as $I \in \mathcal{D}$ varies. The first claim of the Lemma is then clear, and the second claim follows from the Fefferman–Stein maximal function estimate.

Define

$$G(I) \stackrel{\mathrm{def}}{=} \bigcup_{I \subsetneq I'} F(I', j, \mathcal{U}').$$

Suppose that it is the case that for some $R \in \mathcal{U}$ with $R_{(1)} = I$, that we have

$$|R \cap G(I)| \ge \tfrac{3}{4}|R|.$$

Then, by separation of scales, we see that $\mathrm{emb}(R, \mathcal{U}) > 2^j$, a contradiction. Therefore, we take the set $H(I) \stackrel{\mathrm{def}}{=} F(I, j, \mathcal{U}') \cap G(I)^c$. These sets are clearly disjoint in I.

By construction, we must have that $|R \cap H(I)| \ge \frac{1}{4}|R|$, for all $R \in \mathcal{U}$ with $R_{(1)} = I$. Hence, applying the strong maximal function, we see that

$$|F(I, j, \mathcal{U}')| \le |\{\mathrm{M}\, \mathbf{1}_{H(I)} \ge \tfrac{1}{4}\}| \lesssim |H(I)|.$$

This completes the proof.

[10] We have stated the lemma in the formulation for the first coordinate to ease the burden of notation. In application, the role of the first coordinate is imposed on an arbitrary choice of coordinate.

4.2. **With Small Enlargement.** There is a version of the previous Lemma that employs an enlargement that is only slightly larger than the set U, as in Section 3.3.

Given a a collection of rectangles $\mathcal{U}$ whose shadow has finite measure, suppose that $V \supset U$, and define

$$\operatorname{emb}(R,V) \stackrel{\text{def}}{=} \sup\{\mu \geq 1 \,:\, \operatorname{Dil}_{(\mu,1,\dots,1)} R \subset V\}.$$

As before, define

$$F(I,j,\mathcal{U}') \stackrel{\text{def}}{=} \bigcup\{I\times R' \,:\, I\times R' \in \mathcal{U}',\ 2^{j-1} \leq \operatorname{emb}(I\times R', V) < 2^j\}.$$

Lemma 4.34. *For all $\delta, \epsilon > 0$, all $\mathcal{U}$ as above, we can select $V \supset \operatorname{sh}(\mathcal{U})$ with $|V| \leq (1+\delta)|\operatorname{sh}(\mathcal{U})|$, for which we have the estimate*

$$\sum_{j=1}^{\infty}\sum_{I\in\mathcal{D}} 2^{-\epsilon j}|F(I,j,\mathcal{U}')| \lesssim |\operatorname{sh}(\mathcal{U}')|.$$

This holds for all sets $\mathcal{U}' \subset \mathcal{U}$. In fact, we have the estimate below, valid for any integer $n > 1$, and choice of $1 < p < \infty$.

$$\Big\|\sum_{j=1}^{\infty}\sum_{I\in\mathcal{D}} 2^{-\epsilon j}(\mathrm{M}\,\mathbf{1}_{F(I,j,\mathcal{U}')})^n\Big\|_p \lesssim |\operatorname{sh}(\mathcal{U}')|^{1/p}.$$

The implied constants in these estimates depend only on dimensions and the choices of ϵ, δ.

We begin the proof. Recall the properties of shifted dyadic grids (1.6). We take $\delta = (1+2^{\mathsf{d}})^{-1}$, for integer d. We use the maximal function $\mathrm{M}^{\mathcal{D}_{\mathsf{d}}}$, which satisfies (1.7). Define

$$V \stackrel{\text{def}}{=} \{\mathrm{M}_1^{\mathcal{D}_{\mathsf{d}}}\,\mathbf{1}_{\operatorname{Enl}_1(\mathcal{U})} > 1-\delta\},$$

Then, it is the case that $|V| \leq (1+K\delta)|\operatorname{sh}(\mathcal{U})|$.

The remainder of the proof is much as in the previous section. We assume that $\mathcal{U}'$ is such that $2^j \leq \operatorname{emb}(R,V) < 2^{j+1}$, for all $R\in\mathcal{U}'$, and separate scales by $40\cdot 2^j$.

Define

$$G(I) \stackrel{\text{def}}{=} \sup_{I \subsetneq I'} F(I',j,\mathcal{U}').$$

Suppose that it is the case that for some $R\in\mathcal{U}'$ with $R_{(1)} = I$, that we have

$$|R\cap G(I)| \geq (1-\tfrac{\delta}{2})|R|.$$

Then, by separation of scales, we see that $\operatorname{emb}(R,\mathcal{U}) > 2^j$, a contradiction. Therefore, we take the set $H(I) \stackrel{\text{def}}{=} F(I,j,\mathcal{U}')\cap G(I)^c$. These sets are clearly disjoint in I.

By construction, we must have that $|R\cap H(I)| \geq \frac{\delta}{2}|R|$, for all $R\in\mathcal{U}$ with $R_{(1)} = I$. Hence, applying the strong maximal function, we see that

$$|F(I,j,\mathcal{U}')| \leq |\{\mathrm{M}\,\mathbf{1}_{H(I)} \geq \tfrac{\delta}{4}\}| \lesssim \delta^{-1}|H(I)|.$$

The construction of these sets proves the Lemma.

4.3. **With Uniform Embeddedness.** We list a version of Lemma 4.33 which has some advantages as we use a uniform notion of embeddedness.

Lemma 4.35. *For all $\epsilon > 0$, and all collections of rectangles $\mathcal{U}$ whose shadow has finite measure, there is a set $V \supset \mathrm{sh}(\mathcal{U})$ such that $|V| \lesssim |\mathrm{sh}(\mathcal{U})|$, and there is a map* $\mathrm{emb} : \mathcal{U} \mapsto [1,\infty)$*, and a map $\imath : \mathcal{U} \mapsto \{1,2,\ldots,d\}$ such that*

$$\mathrm{emb}(R) \cdot R \subset V, \qquad R \in \mathcal{U}$$

and for all collections $\mathcal{U}' \subset \mathcal{U}$,

$$\sum_{j=1}^{d} \sum_{v=0}^{\infty} \sum_{I \in \mathcal{D}} 2^{-(d+\epsilon)v} |F(I,j,v,\mathcal{U}')| \lesssim |\mathrm{sh}(\mathcal{U}')|,$$

where $$F(I,j,v,\mathcal{U}') \stackrel{\mathrm{def}}{=} \bigcup \{R \in \mathcal{U}' \,:\, 2^v < \mathrm{emb}(R) \le 2^{v+1},\ R_{(j)} = I,\ \imath(R) = j\}.$$

Notice that we have to have a substantially worse power on the embeddedness term, namely the power of embeddedness is strictly smaller than $-d$.

The method of proof requires that we apply Lemma 4.34, although we find it necessary to apply it both inductively and to a wide range of possible collections of rectangles. In fact, it is useful to us that Lemma 4.34 applies not just to collections of dyadic rectangles $\mathcal{U}$ such that the shadow of $\mathcal{U}$ is of finite measure. It also applies to collections of rectangles $\mathcal{U} \subset \otimes_{j=1}^{d} \mathcal{D}_1$, where we are referring to the d fold product of shifted dyadic intervals. And it moreover applies to all subcollections of $\mathcal{U}$.

We apply Lemma 4.34 to $\mathcal{U}^0 \stackrel{\mathrm{def}}{=} \mathcal{U}$. Thus, we get a set $V^1 \supset \mathrm{sh}(\mathcal{U}^0)$, with $|V^1| \lesssim |\mathrm{sh}(\mathcal{U}^0)|$, so that for

$$\mathrm{emb}^1(R, V^1) \stackrel{\mathrm{def}}{=} \sup\{\mu \ge 1 \,:\, \mu R_{(1)} \times R_{(2)} \times \cdots \times R_{(n)} \subset V^1\}.$$

we have the conclusion of Lemma 4.34 holding. We then construct $\mathcal{U}^1 \subset \otimes_{j=1}^{d} \mathcal{D}_1$. Set

$$\mathcal{U}^1 \stackrel{\mathrm{def}}{=} \{\Gamma \times \otimes_{j=2}^{n} R_{(j)} \,: R \in \mathcal{U},\ \Gamma \in \mathcal{S}, \\ (R_{(1)} \cup \tfrac{1}{4}\,\mathrm{emb}^1(R,V^1) R_{(1)}) \subset \Gamma \subset \mathrm{emb}^1(R,V^1) R_{(1)}\}.$$

Notice that we are relying on the structure of the shifted dyadic grids in this definition.

The inductive stage of the construction is this. For $2 \le m \le n$, given $\mathcal{U}^{m-1} \subset \otimes_{j=1}^{d} \mathcal{D}_1$, we apply Lemma 4.34 to get a set V^m satisfying

$$V^m \supset \mathrm{sh}(\mathcal{U}^{m-1}), \qquad |V^m| \lesssim |\mathrm{sh}(\mathcal{U}^{m-1})|.$$

The embedding function for rectangles $R \in \mathcal{U}^{m-1}$ is

$$\mathrm{emb}^m(R, V^m) \stackrel{\mathrm{def}}{=} \sup\{\mu \ge 1 \,: R_1 \times \cdots \times R_{m-1} \times \mu R_m \\ \times R_{(m+1)} \times \cdots \times R_{(n)} \subset V^m\}.$$

And the conclusion of Lemma 4.34 holds. The collection $\mathcal{U}^m$ is then taken to consist of all rectangles of the form

$$\otimes_{j=1}^{m-1} R_{(j)} \times \Gamma \times \otimes_{j=m+2}^{n} R_{(j)}$$

where $R \in \mathcal{U}^{m-1}$ and $\Gamma \in \mathcal{S}$ satisfies

$$(R_{(m)} \cup \tfrac{1}{4}\operatorname{emb}^m(R, V^m) R_{(m)}) \subset \Gamma \subset \operatorname{emb}^m(R, V^m) R_{(m)}.$$

To prove our Lemma, we take $V \stackrel{\text{def}}{=} V^n$. It is the case that

$$\begin{aligned} |V^n| &\lesssim |\operatorname{sh}(\mathcal{U}^{n-1})| \\ &\lesssim |V^{n-1}| \\ &\lesssim |\operatorname{sh}(\mathcal{U})|. \end{aligned}$$

The definition of the embedding function is not so straight forward. It is taken to be

$$\operatorname{emb}(R) = \tfrac{1}{16} \inf_{1 \le m \le n} \beta^m(R)$$

where $\beta^m(\cdot)$ are inductively defined below. The function $\imath(R)$ is taken to be the coordinate in which the infimum for the embedding function is achieved.

Set $\beta^1(R) \stackrel{\text{def}}{=} \operatorname{emb}^1(R, V^1)$. In the inductive step, for $2 \le m \le n$, set $\gamma_m(R) \stackrel{\text{def}}{=} \inf_{j<m} \beta^j(R)$. For $1 < \gamma < \gamma_m(R)$, let

$$\beta^m_\gamma(R) \stackrel{\text{def}}{=} \operatorname{emb}^m(\varphi^m_\gamma(R), V^m)$$

where $\varphi^m_\gamma(R) \in \mathcal{U}^{m-1}$ is the rectangle with $\varphi^m_\gamma(R)_j = R_{(j)}$ for $j \ge m$, and for $1 \le j < m$, $\varphi^m_\gamma(R)_{(j)}$ is the element of $\mathcal{D}_1$ of maximal length such that

$$(R_{(j)} \cup \tfrac{1}{4}\gamma R_{(j)}) \subset \varphi^m(R)_{(j)} \subset \gamma R_{(j)}.$$

Now, take $\overline{\gamma}$ to be the largest value of $1 \le \gamma \le \gamma_m(R)$ for which we have the inequality $\beta^m_\gamma(R) \ge \gamma$. Let us see that this definition of $\overline{\gamma}$ makes sense. This last inequality is strict for $\gamma = 1$, and as γ increases, $\beta^m_\gamma(R)$ decreases, so $\overline{\gamma}$ is a well defined quantity. Then define $\beta^m(R) \stackrel{\text{def}}{=} \beta^m_{\overline{\gamma}}(R)$, and for our use below, set $\varphi^m(R) \stackrel{\text{def}}{=} \varphi^m_{\overline{\gamma}}(R)$.

The choices above prove our lemma, as we show now. For each rectangle $R \in \mathcal{U}$, it is clear that $\operatorname{emb}(R)R \subset V$. Take $\mathcal{U}' \subset \mathcal{U}$. Considering the sets $F(I, k, m, \mathcal{U}')$, then, by Lemma 4.34 applied in the mth coordinate,

$$\sum_{I \in \mathcal{D}} |F(I, k, m, \mathcal{U}')| \le 2^{\epsilon k} |\operatorname{sh}(\varphi^m(\mathcal{U}'))|.$$

While we have a very good estimate for the shadow of $\varphi^m(\mathcal{U})$, a corresponding good estimate for an arbitrary subset $\mathcal{U}'$ seems very difficult to obtain. But it is a consequence of our construction that the rectangle $\varphi^m(R)$ is a rectangle which agrees with

R in the coordinates $j \geq m$ and, for coordinates $1 \leq j < m$, is expanded by at most $32\,\mathrm{emb}(R) \leq 2^{k+6}$. Hence, we have the estimate

$$\Big|\bigcup\{\varphi^m(R) \,:\, R \in \mathcal{U}'\}\Big| \lesssim 2^{dk}|\mathrm{sh}(\mathcal{U}')|.$$

This follows from the weak L^1 bound for the maximal function in one dimension, applied in each coordinate separately. It is in this last step that we lose the large power of the embeddedness. Our proof is complete.

4.4. **With Small Enlargement.** Continuing in this theme, there is a version of the previous Lemma in which one does not permit the enlarged set to be very big. We record

Lemma 4.36. *For all $\delta > 0$, and all $\epsilon > 0$, there is a constant $K_{\delta,\epsilon}$ so that for all collections of rectangles $\mathcal{U}$ whose shadow has finite measure, there is a set $V \supset \mathrm{sh}(\mathcal{U})$ such that $|V| \leq (1+\delta)|\mathrm{sh}(\mathcal{U})|$, and there is a map* $\mathrm{emb} : \mathcal{U} \mapsto [1,\infty)$*, and a map $\imath : \mathcal{U} \mapsto \{1,2,\ldots,d\}$ such that*

$$\mathrm{emb}(R)\cdot R \subset V, \qquad R \in \mathcal{U}$$

and for all collections $\mathcal{U}' \subset \mathcal{U}$,

$$\sum_{j=1}^{d}\sum_{v=0}^{\infty}\sum_{I\in\mathcal{D}} 2^{-(d+\epsilon)v}|F(I,j,v,\mathcal{U}')| \leq K_{\delta,\epsilon}|\mathrm{sh}(\mathcal{U}')|,$$

where $$F(I,j,v,\mathcal{U}') \stackrel{\mathrm{def}}{=} \bigcup\{R \in \mathcal{U}' \,:\, 2^v < \mathrm{emb}(R) \leq 2^{v+1},\ R_{(j)} = I,\ \imath(R) = j\}.$$

This has been applied in a paper of Lacey and Terwilleger [13], and we refer to that paper for the detailed proof.

5. The Higher Parameter Case, with Rectangles

The version of Journé's Lemma described by J. Pipher [18] requires a different notation. As before, we set $\mathrm{Enl}(\mathcal{U}) \stackrel{\mathrm{def}}{=} \{M\mathbf{1}_{\mathrm{sh}(\mathcal{U})} > \frac{1}{2d}\}$. For each integer $1 \leq j \leq d$, we set

$$\mathrm{emb}(j,R) \stackrel{\mathrm{def}}{=} \sup\{\mu \geq 1 \,:\, R_{(1)} \times \cdots \times \mu R_{(j)} \times \cdots \times R_{(d)} \subset \mathrm{Enl}(\mathcal{U})\} \tag{5.37}$$

That is, only the jth coordinate of R is expanded. While we have defined this for all coordinates j, we only use it for $1 \leq j < d$.

Lemma 5.38. *For each $d \geq 3$, and $0 < \epsilon < 1$, all subset U of $\mathbb{R}^d$ of finite measure, and collections $\mathcal{U}$ of pairwise incomparable dyadic rectangles $R \in \mathcal{U}$, we have*

$$\sum_{R\in\mathcal{U}'}|R|\prod_{j=1}^{d-1}\mathrm{emb}(j,R)^{-\epsilon} \lesssim |\mathrm{sh}(\mathcal{U}')|.$$

The inequality holds uniformly over all subsets $\mathcal{U}'$ of $\mathcal{U}$.

This formulation has the advantage that the sum on the left hand side is over simpler objects, namely rectangles. On the other hand, the embeddedness is now more complicated, in that it is a product of terms. In particular $|R|$ is essentially weighted by the *largest* embeddedness term.

We give two proofs of this result.

5.0.1. *The First Proof.* Let us define partial orders $<_j$ on dyadic rectangles by writing $R <_j R'$ iff $R \cap R' \neq \emptyset$ and $R_{(j)} \subset_{\neq} R'_{(j)}$.

The notion of the standard reduction is slightly different. Let us assume that for $1 \leq \mu_j$, $1 \leq j < d$, we have a collection of rectangles $\mathcal{U}$ as above, with $\mu_j \leq \operatorname{emb}(j, R) \leq 2\mu_j$ for $1 \leq j < d$. In addition, we assume that the scales of $\mathcal{U}$ are separated by $10 \max_j \mu_j$. We then follow the essentially disjoint proof stat egy.

Suppose that there is a rectangle $R \in \mathcal{U}$ so that

$$|R \cap \operatorname{sh}(\mathcal{U} - \{R\})| \geq \tfrac{7}{8}|R|.$$

Then, for some $1 \leq j \leq d$, we can choose $R^1, \ldots, R^K \in \mathcal{U} - \{R\}$ with $R <_j R^k$ for all k, and

$$\Big| R \cap \bigcup_{j=1}^{K} R^k \Big| \geq \tfrac{7}{8d}|R|.$$

This is a contradiction to $\operatorname{emb}(j, R) \simeq \mu_j$. Thus, the rectangles in $\mathcal{U}$ are essentially disjoint, and the proof is complete.

5.0.2. *The Second Proof.* We give the proof of J. Pipher [18]. It is convenient for us to restrict attention to the three parameter case, and comment on the higher parameter case briefly.

We can assume that $\mathcal{U}'$ is a collection of rectangles with $\operatorname{emb}(1, R) \simeq 2^k$ for some integer k. We modify slightly the notation of Lemma 4.33.

$$F(I) \stackrel{\text{def}}{=} \bigcup \{R \in \mathcal{U}' \,:\, R_{(1)} = I\}.$$

An essential point to observe is that if we hold the first coordinate of the rectangles R fixed, then the two parameter arguments will apply, in particular Lemma 1.1 applies. Doing so, will place Lemma 4.33 at our disposal.

$$\begin{aligned} \sum_{R \in \mathcal{U}'} |R| \operatorname{emb}(2, R)^{-\epsilon} &\lesssim \sum_{I \in \mathcal{D}} |F(I)| \\ &\lesssim 2^{\epsilon k} |\operatorname{sh}(\mathcal{U}')|. \end{aligned}$$

This completes the proof in three parameters.

In higher parameters, one can implement this proof, but one needs certain variants of Journé's Lemma that fall between the original formulation and Lemma 4.33. We omit the details.

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E-mail address: `cabrelli@dm.uba.ar`

Carlos Cabrelli, Departamento de Matemática, Fac. de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Capital Federal, Argentina

Michael T. Lacey, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 USA

E-mail address: lacey@math.gatech.edu

Ursula Molter, Departamento de Matemática, Fac. de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Capital Federal, Argentina

E-mail address: umolter@dm.uba.ar

Jill C. Pipher, Department of Mathematics, Brown University, Providence RI 02912 USA

E-mail address: jpipher@math.brown.edu