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WEIGHTED EXPONENTIAL APPROXIMATION AND NON-CLASSICAL ORTHOGONAL SPECTRAL MEASURES

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To Vladimir Marchenko with admiration

Abstract. A long-standing open problem in harmonic analysis is: *given a non-negative measure μ on $\mathbb{R}$, find the infimal width of frequencies needed to approximate any function in $L^2(\mu)$.* We consider this problem in the "perturbative regime", and characterize asymptotic smallness of perturbations of measures which do not change that infimal width. Then we apply this result to show that there are no local restrictions on the structure of orthogonal spectral measures of one-dimensional Schrödinger operators on a finite interval. This answers a question raised by V. A. Marchenko.

1. Introduction and main results

1.1. **The type problem.** We say that a non-negative measure μ on $\mathbb{R}$ has at most polynomial growth if, for some $s<\infty$,

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(\lambda)}{1+|\lambda|^{2s}} < \infty\,. \tag{1}$$

For a measure μ of at most polynomial growth, we define its *type* $T(\mu)$ as follows. Let $\mathcal{E}(a)=\mathfrak{F}C_0^\infty(-a,a)$ be the Fourier image of the space of the C^∞-smooth complex-valued functions compactly supported by $(-a,a)$. This is the linear space of entire functions of exponential type less than a that decay on the real axis faster than any negative power of $|\lambda|$. Then $\mathcal{E}(a)\subset L^2(\mu)$, and

$$T(\mu) \stackrel{\rm def}{=} \inf\left\{a\colon \mathcal{E}(a) \text{ is dense in } L^2(\mu)\right\} = \sup\left\{a\colon \mathcal{E}(a) \text{ isn't dense in } L^2(\mu)\right\},$$

which is one half of the infimal width of the spectrum needed to approximate any function in $L^2(\mu)$.

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The definition of the type is not sensitive to the choice of a "natural" linear space $\mathcal{E}(a)$ of entire functions of exponential type at most a. For instance, it is not difficult to check that if the Paley–Wiener space $\mathfrak{F}L^2(-a,a)$ is contained in $L^2(\mu)$, then without affecting the definition of the type, one can replace therein the linear space $\mathcal{E}(a)$ by $\mathfrak{F}L^2(-a,a)$. If the measure μ is finite, then one can replace $\mathcal{E}(a)$ by the space of the finite linear combinations of exponential functions $\lambda \mapsto e^{\mathrm{i}t\lambda}$ with $-a<t<a$.

The range of $T(\mu)$ is $[0,\infty]$ with both ends included. If the tails of the measure μ decay so fast that the polynomials belong to the space $L^2(\mu)$ and are dense therein, then it is easy to see that $T(\mu)=0$. Another instance of the zero type occurs when the support of the measure μ has long gaps. On the other hand, Lebesgue measure m on the real axis has infinite type. An intermediate case occurs for the sum of point masses δ at arithmetic progression: for $0<\ell<\infty$, we have $T\bigl(\sum_{\lambda\in\ell\mathbb{Z}}\delta_\lambda\bigr)=\pi\ell^{-1}$. These are toy models for other situations when the type $T(\mu)$ can be explicitly computed. For reader's orientation, we bring a short summary of what is known in Appendix A.

For any $g\in L^2(\mu)$, the Fourier transform of the measure $g\,\mathrm{d}\mu$ is a tempered distribution. If $\mathcal{E}(a)$ is not dense in $L^2(\mu)$ and $g\in L^2(\mu)$ is orthogonal to $\mathcal{E}(a)$, then the Fourier transform of $g\mathrm{d}\mu$ vanishes on $(-a,a)$. In this case, we say that the interval $(-a,a)$ is a spectral gap of g. Therefore, *the type $T(\mu)$ coincides with one half of the supremal length of spectral gaps of functions in $L^2(\mu)$.*

1.2. **Relations with other classical problems of analysis.** The problem of *effective computation of the type $T(\mu)$*, for brevity, the *type problem*, is intimately related to other classical problems in analysis. It originates from the works of Kolmogorov and Wiener on the prediction of Gaussian stationary processes, see [17], [11, § 3.7 and Chapter 4]. Then Gelfand and Levitan [13, §8] and Krein [18, Theorem 4] and [19] discovered a deep relation between the type problem and the spectral theory of the second order ordinary differential operators which we discuss in Section 1.4.

The type problem is one of the central problems in the de Branges theory of Hilbert spaces of entire functions. By this theory, given a non-negative measure μ satisfying the property

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(\lambda)}{1+\lambda^2} < \infty\,,$$

there exists a unique chain of de Branges Hilbert spaces of entire functions $\mathcal{H}(E_t)$ such that

(i) the entire functions E_t are of Cartwright class[1],

(ii) $\mathcal{H}(E_{t_1})$ is contained isometrically in $\mathcal{H}(E_{t_2})$ for $t_1<t_2$,

[1] That is, the functions E_t have at most exponential type and convergent logarithmic integral $\int_{\mathbb{R}} \frac{\log^+|E_t(x)|}{1+x^2}\,\mathrm{d}x<\infty$.

(iii) each space $\mathcal{H}(E_t)$ is contained isometrically in $L^2(\mu)$,
and

(iv) $\bigcup_t \mathcal{H}(E_t)$ is dense in $L^2(\mu)$.

Then it is not difficult to show that the supremum of exponential types of the functions E_t coincides with $T(\mu)$. There is a remarkable formula due to Krein [19] and de Branges [8, Theorem 39] that expresses the type $T(\mu)$ via the coefficients of the second order canonical system describing evolution along the chain of the spaces $\mathcal{H}(E_t)$. Note that de Branges' book [8] contains a wealth of results (Theorems 61–68) related to the type problem.

It is also worth mentioning that the type problem is a part of the general Bernstein weighted approximation problem, cf. Dym [10], Pitt [31], Koosis [15, Chapters VI-VII], and Levin [20]. The methods developed for solving the Bernstein problem will be used in this work.

At last, the type problem is closely connected with fundamental results of Beurling and Malliavin on multipliers and the radius of completeness, see Koosis [16], and recent works of Mitkovski and Poltoratski [28] and of Poltoratski [32]. The papers by Mitkovski and Poltoratski suggest a novel approach to the type problem based on injectivity of certain Toeplitz operators.

1.3. **Perturbations of measures.** Since we do not know how to compute the type, it is natural to ask which perturbations of positive measures *preserve* their types? We prove that exponentially small perturbations of measures do not change their types and then we show that this result is sharp.

Given $\delta > 0$ and $x \in \mathbb{R}$, we denote

$$I_x = I_{x,\delta} = [x - e^{-\delta|x|}, x + e^{-\delta|x|}]. \tag{2}$$

By $kI_x = kI_{x,\delta}$ we denote the closed interval centered at x with length k times that of I_x.

Definition 1.1 (majorization in mean with exponentially small error)**.** *We write* $\mu \preccurlyeq \widetilde{\mu}$ *if there exist constants* $\delta > 0$, $C > 0$, *and* $n \geqslant 0$, *such that, for all* $x \in \mathbb{R}$,

$$\mu(I_{x,\delta}) \leqslant C(1+|x|)^n \left(\widetilde{\mu}(2I_{x,\delta}) + e^{-2\delta|x|}\right).$$

Definition 1.2 (stable density)**.** *We say that* $\mathcal{E}(a)$ *is stably dense in* $L^2(\mu)$ *if for each* $t \geqslant 0$, $\mathcal{E}(a)$ *is dense in* $L^2(\mu_t)$ *with* $\mathrm{d}\mu_t(\lambda) = (1+|\lambda|)^t\,\mathrm{d}\mu(\lambda)$.

It is not difficult to show (see Appendix B) that $\mathcal{E}(a)$ is stably dense in $L^2(\mu)$ if and only if, for any finite set of points $\lambda_1, ..., \lambda_N \in \mathbb{R}$, $\mathcal{E}(a)$ is dense in $L^2(\widetilde{\mu})$, where $\widetilde{\mu} = \mu + \sum_k \delta_{\lambda_k}$. Here and everywhere below, δ_λ is a unit point mass at λ.

Non-stable density is quite non-generic, though it often occurs in various applications. It is known that if $\mathcal{E}(a)$ is dense but not stably dense, then the measure μ is supported by the zero set of an entire function of exponential type a, see Lemma B.3 in Appendix B for a more precise statement.

Theorem 1.3. *Let $\widetilde{\mu}$ be a non-negative measure of at most polynomial growth, and let $\mu \preccurlyeq \widetilde{\mu}$. If $\mathcal{E}(a)$ is stably dense in $L^2(\widetilde{\mu})$, then $\mathcal{E}(a)$ is dense in $L^2(\mu)$.*

By the remark made before the theorem, if $\mathcal{E}(a)$ is dense in $L^2(\mu)$, then for each $a' > a$, $\mathcal{E}(a')$ is stably dense in $L^2(\mu)$. Hence,

Corollary 1.4. *Let $\widetilde{\mu}$ be a non-negative measure of at most polynomial growth, and let $\mu \preccurlyeq \widetilde{\mu}$. Then $T(\mu) \leqslant T(\widetilde{\mu})$.*

We call the measures $\widetilde{\mu}$ and μ *weakly equivalent* if $\widetilde{\mu} \preccurlyeq \mu$ and $\mu \preccurlyeq \widetilde{\mu}$. Note that if two positive measures coincide outside of a finite interval, then they are weakly equivalent.

Corollary 1.5. *Weakly equivalent measures have equal types.*

We note that Theorem 1.3 has a counterpart, which deals with polynomial density in $L^2(\mu)$, see Section 3.1.

The following result shows that the statements of Corollaries 1.4 and 1.5 are sufficiently sharp:

Theorem 1.6. *Given a positive function ε such that $\varepsilon(r) \to 0$, $r \to \infty$,*
(i) *there exists a function φ such that $T\big(\varphi(x)\,\mathrm{d}x\big) = 0$ while $T\big((\varphi(x) + e^{-\varepsilon(|x|)|x|})\,\mathrm{d}x\big) = \infty$.*
(ii) *there exist two sequences of points $\{x_n\}_{n\in\mathbb{Z}}$, $\{y_n\}_{n\in\mathbb{Z}}$, $|y_n - x_n| \leqslant e^{-\varepsilon(|x_n|)|x_n|}$, such that $T\big(\sum_{n\in\mathbb{Z}} \delta_{x_n}\big) = \pi$ while $T\big(\sum_{n\in\mathbb{Z}}(\delta_{x_n} + \delta_{y_n})\big) = 2\pi$.*

We note that the situation changes if we consider perturbations of sufficiently regular measures. The types of such measures are more stable, see a corollary to a classical result of Duffin and Schaeffer cited in Appendix A.2 (perturbations of Lebesgue measure), and Benedicks [4, Theorems 8] (perturbations of the sum of point masses at arithmetic progression).

1.4. **Spectral theory of one-dimensional Schrödinger operators.** First, we recall some classical facts pertaining to the spectral theory of one-dimensional Schrödinger operators. Below, we follow the first two chapters of [25] (see also [22, Chapter 1] and [13, 18]).

1.4.1. *A piece of Weyl's spectral theory.* Given a, $0 < a \leqslant \infty$, consider the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda^2 y\,, \qquad 0 \leqslant x < a, \tag{3}$$

with a real-valued potential $q \in C[0, a)$. Note that we use the "momentum" λ (not the "energy" λ^2) as the spectral parameter, and that we do not impose any restrictions on q at the right end-point $x = a$.

Take the solution $\omega(\lambda, x)$ satisfying the boundary condition

$$y(0) = 1, \qquad y'(0) = h\,. \tag{4}$$

For each $x \in [0, a)$, this is an entire function of λ. It satisfies the integral Sturm–Liouville equation

$$\omega(\lambda, x) = \cos \lambda x + h\,\frac{\sin \lambda x}{\lambda} + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} q(t)\omega(\lambda, t)\,\mathrm{d}t\,,$$

which easily yields the estimate

$$\bigl|\omega(\lambda, x) - \cos \lambda x\bigr| \leqslant e^{x|\mathrm{Im}\lambda|}\,\frac{Q(x) + |h|}{|\lambda| - Q(x)}\,,$$

with $Q(x) = \int_0^x |q|$ and $|\lambda| > Q(x)$. Thus, given x, the function $\lambda \mapsto \omega(\lambda, x)$ is an entire function of exponential type x bounded on the real axis.

Consider the Weyl transform

$$\mathfrak{W}f(\lambda) = \int_0^a f(x)\omega(\lambda, x)\,\mathrm{d}x\,.$$

This transform is well-defined on the subspace $L^2_0(0, a)$ of $L^2(0, a)$ consisting of the functions that vanish on a neighbourhood of the end point $x = a$. Note that if $f = -u'' + qu$, and

$$u'(0) = hu(0), \tag{5}$$

then $\mathfrak{W}f(\lambda) = \lambda^2\mathfrak{W}u(\lambda)$. A celebrated theorem of Weyl says that

- *there exists a measure μ supported by $\mathbb{R} \cup \mathrm{i}\mathbb{R}$ and symmetric with respect to the origin, such that*

$$\|f\|_{L^2(0,a)} = \|\mathfrak{W}f\|_{L^2(\mu)}\,, \qquad f \in L^2_0(0, a).$$

The measure μ is called a *spectral measure* of the Sturm–Liouville problem (3)–(4). The map $\mathfrak{W}$ extends to the isometry $L^2(0, a) \to L^2(\mu)$, and the inverse map, defined by

$$f(x) = \int_{\mathbb{R}\cup\mathrm{i}\mathbb{R}} \mathfrak{W}f(\lambda)\,\omega(\lambda, x)\,\mathrm{d}\mu(\lambda)\,,$$

is called the eigenfunction expansion associated with the Sturm–Lioville problem (3)–(4). If the image $\mathfrak{W}L^2(0, a)$ spans the closed subspace $L^2_{\mathsf{e}}(\mu)$ of even functions in $L^2(\mu)$, then the spectral measure μ is called *orthogonal* (or sometimes, *principal*). It is known that each Sturm–Liouville problem has orthogonal spectral measures.

For the reader's orientation, we mention that there is a one-to-one correspondence between orthogonal spectral measures μ and self-adjoint extensions to a dense subset of $L^2(0, a)$ of the operator $-y'' + q(x)y$ with boundary condition (5). Each self-adjoint extension of this type is unitarily equivalent to the operator of multiplication by λ^2 in $L^2_{\mathsf{e}}(\mu)$. In the limit-point case at the end point $x = a$, when the self-adjoint extension is unique, the operator has only one spectral measure, and it is orthogonal. In the limit-circle case, there are many spectral measures and some of them are orthogonal, while the others correspond to self-adjoint operators defined on an extension of the space $L^2(0,a)$. See Akhiezer and Glazman [2, Appendix II] for the details of this correspondence.

Note that it follows from Weyl's theory that

$$\int_{\mathbb{R}\cup\mathrm{i}\mathbb{R}} \frac{\mathrm{d}\mu(\lambda)}{1+|\lambda|^2} < \infty\,. \tag{6}$$

1.4.2. *A piece of the theory developed by Gelfand–Levitan, Krein, and Marchenko.* Weyl proved his theorem in 1909–10. Forty years later, Gelfand–Levitan, Krein, and Marchenko developed a beautiful theory that fully describes spectral measures of the one-dimensional Schrödinger operator, and tells how to recover the potential q from the spectral measure μ; see Marchenko [26] for a very illuminating account of the development of this theory.

Given a measure μ supported by $\mathbb{R}\cup\mathrm{i}\mathbb{R}$, symmetric with respect to the origin and satisfying the growth condition (6), we define the function

$$\Phi(x) = \Phi[\mu](x) = \int_{\mathbb{R}\cup\mathrm{i}\mathbb{R}} \frac{1-\cos\lambda x}{\lambda^2}\,\mathrm{d}\mu(\lambda)\,. \tag{7}$$

This transform was introduced and studied by Povzner and Krein in the 1940-s; for its basic properties see [1, items 10-12 in Addenda to Chapter V][2].

Theorem 1.7 (Gelfand–Levitan). *The measure μ is a spectral measure of the Sturm–Liouville boundary problem* (3)–(4) *with a continuous potential q if and only if*

$$\Phi\in C^3[0,2a) \quad \text{and} \quad \Phi'(+0)=1,\ \Phi''(+0)=-h\,. \tag{8}$$

Moreover, the potential q has the same number of continuous derivatives on $[0,a)$ *as* Φ''' *has on* $[0,2a)$.

Following [25, Chapter 2, §4], we rewrite the Gelfand–Levitan condition (8) in a different form replacing the Φ-transform by the Fourier transform. Let $\mu=\mu_{\mathbb{R}}+\mu_{\mathrm{i}\mathbb{R}}$, where the measure $\mu_{\mathbb{R}}$ is supported by $\mathbb{R}$, and the measure $\mu_{\mathrm{i}\mathbb{R}}$ is supported by $\mathrm{i}\mathbb{R}\setminus\{0\}$. The rôles of these measures are very different. The measure $\mu_{\mathbb{R}}$ is close in some sense to Lebesgue measure $\frac{1}{\pi}m$, while the tails of the measure $\mu_{\mathrm{i}\mathbb{R}}$ decay exponentially and $\mu_{\mathrm{i}\mathbb{R}}$ can be considered as a "perturbation" of the measure $\mu_{\mathbb{R}}$. More precisely, the Gelfand–Levitan condition (8) is *equivalent* to the following two conditions:

[2] Therein, the transform is written in the form

$$\int_{\mathbb{R}} \frac{1-\cos(x\sqrt{s})}{s}\,\mathrm{d}\rho(s)\,, \qquad \text{with} \quad \mathrm{d}\rho(s)=2\mathrm{d}\mu(\sqrt{s})\,.$$

(GL–**i**) there exists an even function $M \in C^1[0,2a)$ such that the restriction of the distributional Fourier transform[3] $\widehat{\mu_{\mathbb R}}$ to $(-2a,2a)$ is equal to $2\delta_0 + M$, that is,

$$\int_{\mathbb R} \widehat{f}(\lambda)\,\mathrm{d}\mu(\lambda) = 2f(0) + \int_{-2a}^{2a} f(x)M(x)\,\mathrm{d}x \tag{9}$$

for each $f \in C_0^\infty(-2a,2a)$;

(GL–**ii**) the tails of the measure $\mu_{\mathrm{i}\mathbb R}$ decay exponentially fast: for each $x \in (0,2a)$,

$$\int_0^\infty e^{x\lambda}\,\mathrm{d}\mu_{\mathrm{i}\mathbb R}(\mathrm{i}\lambda) < \infty\,.$$

If conditions (GL–**i**) and (GL–**ii**) are fulfilled, then $h = -\widehat{\mu}(+0)$ (in other words, $M(0) + \mu_{\mathrm{i}\mathbb R}(\mathrm{i}\mathbb R) = -h$).

For instance, in the case of the zero potential on the semi axis ($q=0$, $a=\infty$ and $h=0$), we have

$$\mu_{\mathbb R} = \tfrac{1}{\pi} m, \qquad \mu_{\mathrm{i}\mathbb R} = 0, \qquad \widehat{\mu} = 2\delta_0,$$

where m is Lebesgue measure on $\mathbb R$, and $\Phi(x) = |x|$, $M(x) = 0$, $x \in \mathbb R$. In the case of the zero potential on a finite interval $[0,a)$ and $h=0$, we have

$$\mu_{\mathbb R} = \tfrac{1}{a}\sum_{n\in\mathbb Z}\delta_{\pi n/a}, \qquad \mu_{\mathrm{i}\mathbb R} = 0, \qquad \widehat{\mu} = 2\sum_{n\in\mathbb Z}\delta_{2na},$$

and $\Phi(x) = |x|$, $M(x) = 0$ for $|x| < 2a$.

The original statement of Theorem 1.7 in the paper by Gelfand and Levitan [13] contained a gap in one derivative between the necessary and sufficient conditions which was removed by Krein in [18]. Also note that sometimes the Gelfand–Levitan theorem is formulated with an additional assumption on the measure μ (e.g., see [25, Theorem 2.3.1]):

(GL–**iii**) for any $0 < b < a$, and for any even $f \in L^2(-b,b)$, we have $\int |\widehat{f}|^2 \mathrm{d}\mu > 0$ unless $f = 0$.

However, Yavryan [36] showed that this additional assumption (GL–**iii**) follows[4] from condition (GL–**i**).

[3] We always use the following normalization

$$\widehat{f}(\lambda) = \int_{\mathbb R} f(x) e^{-\mathrm{i}\lambda x}\,\mathrm{d}x\,.$$

for the Fourier transform.

[4] Indeed, suppose that $\int |\widehat{f}|^2 \mathrm{d}\mu = 0$. Take any even continuous function $\widetilde{M}$ with support in $(-2a,2a)$ coinciding with M on $[-2b,2b]$, and denote by $\widetilde{\mu}$ a measure supported by $\mathbb R$ whose distributional Fourier transform coincides with $2\delta_0 + \widetilde{M}$. Then it is not difficult to see that still $\int |\widehat{f}|^2 \mathrm{d}\widetilde{\mu} = 0$. This is impossible unless $f = 0$ since now the set $\mathbb R \setminus \operatorname{supp}\widetilde{\mu}$ is at most finite (and $\widehat{f}$ is an entire function).

1.4.3. *Orthogonal spectral measures.* Now, we turn to the orthogonality condition. Denote by $\mathfrak{C}L^2_0(0,a)$ the image of the linear space $L^2_0(0,a)$ under the cosine-transform

$$\mathfrak{C}f(\lambda) = \int_0^a f(x)\cos\lambda x\,\mathrm{d}x\,, \qquad f\in L^2_0(0,a)\,.$$

By the Paley–Wiener theorem, $\mathfrak{C}L^2_0(0,a)$ coincides with the linear space of even entire functions of exponential type less than a which belong to the space $L^2(\mathbb{R})$. The images of $L^2_0(0,a)$ under the cosine-transform and the Weyl transform coincide; i.e., $\mathfrak{W}L^2_0(0,a)=\mathfrak{C}L^2_0(0,a)$ as linear spaces of entire functions. This follows from the classical equations

$$\omega(\lambda,x) = \cos\lambda x + \int_0^x K(x,t;h)\cos\lambda t\mathrm{d}t,$$
$$\cos\lambda x = \omega(\lambda,x) + \int_0^x L(x,t;h)\omega(\lambda,t)\mathrm{d}t,$$

with the kernels K and L continuous for $0\leqslant x,t<a$, cf. [25, Chapter 2, §2]. Since $L^2_0(0,a)$ is dense in $L^2(0,a)$ and since $\mathfrak{W}$ is an isometry between $L^2(0,a)$ and $L^2(\mu)$, we conclude that

- *the spectral measure μ is orthogonal if and only if $\mathfrak{C}L^2_0(0,a)$ is dense in the subspace of even functions $L^2_{\mathsf{e}}(\mu)$.*

The latter condition follows from the density of $\mathcal{E}(a)$ in the whole space $L^2(\mu)$. Hence, *the spectral measure μ is orthogonal provided that $\mathcal{E}(a)$ is dense in $L^2(\mu)$.* This relates orthogonal spectral measures are to a weighted exponential approximation problem, though a peculiar one: now, μ is a symmetric measure supported by $\mathbb{R}\cup \mathrm{i}\mathbb{R}$.

1.5. **Non-classical orthogonal spectral measures.** If $a<\infty$ and the potential q is continuous at the right end point $x=a$, we arrive at the classical Sturm–Liouville problem with two regular end points. In this case, the measure μ is discrete and has a well-known asymptotic behavior. For instance, if $a=\pi$, then $\mu_{\mathbb{R}}=\sum_{n\in\mathbb{Z}}\alpha_n\delta_{\lambda_n}$ with $\lambda_n=n+O\bigl(\frac1n\bigr)$, $\lambda_{-n}=\lambda_n$, and $\alpha_n=\frac1\pi+O\bigl(\frac1n\bigr)$, $\alpha_{-n}=\alpha_n$, while $\mu_{\mathrm{i}\mathbb{R}}$ may consist only of finitely many atoms. However, in the general case, when $x=a$ is a finite singular end point, the situation is more tangled, and not much is known about orthogonal spectral measures. Even the most basic question: *what are the restrictions imposed on the local structure of spectral measures of Schrödinger operators on a finite interval by the orthogonality condition?* remained open[5] since the 1950-s, see Marchenko [26]. The

[5] In the case $a=\infty$ there are *no* restrictions on the local structure of orthogonal spectral measures, see Gelfand and Levitan [13, § 8]. In this respect, the case of the infinite interval $[0,\infty)$ is much simpler since in that case *any* spectral measure supported by $\mathbb{R}$ is automatically orthogonal. This follows from the density of the space $\bigcup_{a<\infty}\mathfrak{C}L^2_0(0,a)$ of even entire functions

only result in this direction we are aware of is a delicate construction by Pearson [29]. He builds a potential q on a finite interval for which the orthogonal spectral measure μ is absolutely continuous on a finite interval $[-\lambda_0, \lambda_0]$, the restriction of μ to $\mathbb{R} \setminus [-\lambda_0, \lambda_0]$ is discrete, and the restriction of μ to $\mathrm{i}\mathbb{R} \setminus \{0\}$ is at most the sum of finitely many atoms. Pearson writes: "The existence of an absolutely continuous spectrum for a Schrödinger operator in a finite interval may be regarded as an exceptional phenomenon, and we have to work quite hard to achieve it" [30, p.495]. Curiously enough, Theorem 1.3 tells us that from the point of view of the inverse spectral theory, this phenomenon is not exceptional.

Definition 1.8 (stably orthogonal spectral measures)**.** *We call a spectral measure μ of the Sturm–Liouville problem* (3)–(4) *on a finite interval* $[0, a)$ *stably orthogonal if $\mathcal{E}(a)$ is stably dense in $L^2(\mu_{\mathbb{R}})$.*

The orthogonal spectral measures corresponding to the classical Sturm-Liouville problems with two regular end-points, or more generally, to the problems with the limit-circle case at $x = a$, are not stable. In Appendix C we bring an explicit construction of a rather wide class of discrete spectral measures, and in Appendix D we show that some of them are stably orthogonal. We perturb these measures, proving that there are *no* local restrictions on the structure of orthogonal spectral measures of the Sturm–Liouville problem on a finite interval:

Theorem 1.9. *Suppose that μ_0 is a stably orthogonal spectral measure of a Sturm–Liouville problem* (3)–(4) *on a finite interval* $[0, a)$*, and that the measure μ_0 is supported by the real axis. Suppose that $\mu = \mu_{\mathbb{R}} + \mu_{\mathrm{i}\mathbb{R}}$ is a non-negative symmetric measure on $\mathbb{R} \cup \mathrm{i}\mathbb{R}$ such that*

(i) *the support of the measure $\mu_{\mathbb{R}}$ does not coincide with the zero set of an entire function of exponential type $\leqslant a$;*

(ii) *the integral* $\displaystyle\int_0^\infty \mathrm{d}(\mu_{\mathbb{R}} - \mu_0)$ *converges, and for some $\delta > 0$ we have*

$$\int_0^\infty e^{\delta\lambda} \left| \int_\lambda^\infty \mathrm{d}(\mu_{\mathbb{R}} - \mu_0) \right| \mathrm{d}\lambda < \infty \, , \tag{10}$$

and

$$\int_0^\infty e^{\delta\lambda^2} \, \mathrm{d}\mu_{\mathrm{i}\mathbb{R}}(\mathrm{i}\lambda) < \infty \, . \tag{11}$$

Then the measure μ is an orthogonal spectral measure of a Sturm–Liouville problem (3)–(4) *on the same interval* $[0, a)$*, with the potential q of the same class of smoothness on* $[0, a)$ *as that of the potential q_0 that corresponds to μ_0.*

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of exponential type that belong to $L^2(\mathbb{R})$ in the space of even functions $L^2_{\mathsf{e}}(\mu)$ where μ is a measure supported by $\mathbb{R}$ and satisfying (1).

Let us comment on the rôle of the technical condition (i) in Theorem 1.9. Condition (10) allows us to apply Theorem 1.3 and to conclude that $\mathcal{E}(a)$ is dense in $L^2(\mu_{\mathbb{R}})$. Then condition (i) will allow us to apply Lemma B.3 from Appendix B and conclude that $\mathcal{E}(a)$ is stably dense in $L^2(\mu_{\mathbb{R}})$. We need this conclusion in order to further perturb $\mu_{\mathbb{R}}$ by a measure $\mu_{i\mathbb{R}}$ supported by the imaginary axis.

1.6. **A brief reader's guide.** The rest of the paper consists of three parts, mostly independent of each other, and four appendices. In the first part (Sections 2 and 3), we prove Theorem 1.3 which says that exponentially small perturbations of measures do not change their type. In Section 2, we recall necessary preliminaries, and the proof itself occupies Section 3. In the second part (Section 4), we construct examples that show how sharp is Theorem 1.3. In the third part (Section 5) we turn to perturbations of stably orthogonal spectral measures and prove Theorem 1.9. At the end of the paper we give information on the cases when the type can be explicitly calculated (Appendix A), on unstable weighted approximation (Appendix B), on Nazarov's construction of spectral measures based on a "distorted Poisson formula" (Appendix C), and on how to get stably orthogonal spectral measures (Appendix D).

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2. Weighted exponential approximation on $\mathbb{R}$

Here, we bring several facts from the well-developed theory of weighted exponential approximation on the real axis.

Definition 2.1. *A lower semi-continuous function $W\colon \mathbb{R} \to (0,\infty]$ is called a weight if W is not equal to ∞ identically, and for some $s \in \mathbb{R}$ we have*

$$\lim_{|x|\to\infty} (1+|x|)^s W(x) = \infty. \tag{12}$$

Definition 2.2. *$C_0(W)$ is the semi-normed space of the functions f continuous on the real line and such that*

$$\lim_{|x|\to\infty} \frac{f(x)}{W(x)} = 0,$$

$$\|f\|_{C_0(W)} = \sup_{\mathbb{R}} \frac{|f|}{W}.$$

Theorem 2.3 (M. Riesz–Mergelyan)**.** *Let μ be a non-negative measure on $\mathbb{R}$ having at most polynomial growth* (1)*, and let W be a weight. Suppose that $a > 0$ and that $X = C_0(W)$ or $X = L^2(\mu)$. The set $\mathcal{E}(a)$ is dense in X if and only if there exists a non-zero function $f \in \mathcal{E}(a)$ with $f(\mathrm{i}) \neq 0$ such that $f(x)(x-\mathrm{i})^{-1} \in \mathrm{clos}_X\, \mathcal{E}(a)$.*

This is a counterpart of a classical theorem of M. Riesz and Mergelyan (see, for instance, [27]). Both of them considered the polynomial approximation, M. Riesz in $L^2(\mu)$, while Mergelyan in $C_0(W)$. Rather general versions of their results can be found in Pitt [31, Theorem 3.1] and in Levin [20, Section I.1].

Proof of Theorem 2.3: The necessity part is evident. To verify the sufficiency part, we fix a function $f \in \mathcal{E}(a)$ such that $f(\mathrm{i}) \neq 0$ and $f(x)(x-\mathrm{i})^{-1} \in \mathrm{clos}_X\, \mathcal{E}(a)$. Without loss of generality, we assume that f does not vanish on $\mathbb{R}$. Otherwise, we take any function $g \in \mathcal{E}(a)$ that vanishes at $z = \mathrm{i}$ and does not vanish on $\mathbb{R}$, and replace f by the function $f - cg$ with an appropriate constant c (such a constant exists since the function f/g is analytic in a neighborhood of the real line). Next, we use that

$$h \in \mathcal{E}(a) \implies \frac{f(\mathrm{i})h - h(\mathrm{i})f}{\cdot - \mathrm{i}} \in \mathcal{E}(a).$$

Claim 2.4. *Let $f^*(z) = \overline{f(\overline{z})}$. Then $f^*(x)(x-\mathrm{i})^{-1} \in \mathrm{clos}_X\, \mathcal{E}(a)$.*

Proof: We have

$$\frac{f^*(x)}{x-\mathrm{i}} = \frac{f^*(\mathrm{i})}{f(\mathrm{i})} \cdot \frac{f(x)}{x-\mathrm{i}} + \frac{f^*(x)f(\mathrm{i}) - f(x)f^*(\mathrm{i})}{f(\mathrm{i})(x-\mathrm{i})}\,.$$

The first term on the right-hand side belongs to $\mathrm{clos}_X\, \mathcal{E}(a)$, while the second term lies in $\mathcal{E}(a)$. We are done. $\square$

Claim 2.5. *For each $n \geqslant 1$ we have*

$$f(x)(x \pm \mathrm{i})^{-n} \in \mathrm{clos}_X\, \mathcal{E}(a)\,.$$

Proof: Let $h_k \in \mathcal{E}(a)$, $h_k \xrightarrow{X} f(x)(x-\mathrm{i})^{-1}$. Then

$$\Big\| \frac{h_k(x)}{x-\mathrm{i}} - f(x)(x-\mathrm{i})^{-2} \Big\|_X \leqslant \big\| h_k(x) - f(x)(x-\mathrm{i})^{-1} \big\|_X \to 0\,,$$

and the sequence of functions

$$\frac{h_k(x)}{x-\mathrm{i}} = \frac{h_k(x)f(\mathrm{i}) - h_k(\mathrm{i})f(x)}{f(\mathrm{i})(x-\mathrm{i})} + \frac{h_k(\mathrm{i})f(x)}{f(\mathrm{i})(x-\mathrm{i})}$$

is contained in $\mathrm{clos}_X\, \mathcal{E}(a)$. We obtain that $f(x)(x-\mathrm{i})^{-2} \in \mathrm{clos}_X\, \mathcal{E}(a)$. In the same way, $f(x)(x-\mathrm{i})^{-n} \in \mathrm{clos}_X\, \mathcal{E}(a)$ for each $n \geqslant 1$. Furthermore, using

Claim 2.4, in the same way we obtain that $f^*(x)(x-\mathrm{i})^{-n} \in \operatorname{clos}_X \mathcal{E}(a)$ for each $n \geqslant 1$. Noting that

$$f(x)(x+\mathrm{i})^{-n} = \overline{f^*(x)(x-\mathrm{i})^{-n}}, \qquad x \in \mathbb{R}\,, n \geqslant 1\,,$$

and that $\operatorname{clos}_X \mathcal{E}(a)$ is closed with respect to the conjugation, we obtain that $f(x)(x+\mathrm{i})^{-n} \in \operatorname{clos}_X \mathcal{E}(a)$ for each $n \geqslant 1$. We are done. □

Let $V_s(x) = (1+|x|)^{-s}$, $C_s(\mathbb{R}) = C_0(V_s)$. For large s, convergence in $C_s(\mathbb{R})$ implies convergence in X.

Claim 2.6. *The linear span of the functions* $\bigl\{f(x)(x \pm \mathrm{i})^{-n}\bigr\}_{n\geqslant 1}$ *is dense in* $C_s(\mathbb{R})$.

Proof: Otherwise, there is a non-zero finite complex-valued measure ν on $\mathbb{R}$ such

$$\int_{\mathbb{R}} \frac{f(x)(1+|x|)^s \,\mathrm{d}\nu(x)}{(x \pm \mathrm{i})^n} = 0\,, \qquad n \in \mathbb{N}\,,$$

whence

$$\int_{\mathbb{R}} \frac{f(x)(1+|x|)^s \,\mathrm{d}\nu(x)}{x-\zeta} = 0\,, \qquad \zeta \in \mathbb{C} \setminus \mathbb{R}\,,$$

which, in its turn, yields that the measure $f(x)(1+|x|)^s\,\mathrm{d}\nu(x) = 0$ vanishes. Since f does not vanish on $\mathbb{R}$, ν is the zero measure, which contradicts our assumption. □

Now, we easily complete the proof of sufficiency in Theorem 2.3. By Claim 2.6, each continuous function on $\mathbb{R}$ with compact support can be approximated in $C_s(\mathbb{R})$, and hence in X, by finite linear combinations of the functions $f(x)(x \pm \mathrm{i})^{-n}$. Then, by Claim 2.5, continuous functions with compact support belong to $\operatorname{clos}_X \mathcal{E}(a)$. It remains to recall that continuous functions with compact support are dense in X, completing the proof of Theorem 2.3. □

Next, we introduce a $C_0(W)$-counterpart of stable density, cf. Definition 1.2. Given a weight W, we set $W_t(x) = W(x)(1+|x|)^{-t}$.

Definition 2.7 (stable density in $C_0(W)$)**.** *We say that $\mathcal{E}(a)$ is stably dense in $C_0(W)$ if for each $t \geqslant 0$, $\mathcal{E}(a)$ is dense in $C_0(W_t)$.*

The following theorem is a version of a recent result of Bakan [3] who dealt with weighted polynomial approximation.

Theorem 2.8 (Bakan)**.** *Let μ be a non-negative measure on $\mathbb{R}$ satisfying the growth condition* (1)*, and let $a > 0$. The set $\mathcal{E}(a)$ is (stably) dense in $L^2(\mu)$ if and only if there exists a weight $W \in L^2(\mu)$ satisfying the growth condition* (12) *such that $\mathcal{E}(a)$ is (stably) dense in $C_0(W)$.*

Proof: We consider only the stable density case. The same argument works (with some simplifications) in the other case.

To verify the sufficiency part note that

$$\|f\|^2_{L^2(\mu_p)} = \int_{\mathbb{R}} |f(x)|^2(1+|x|)^p \mathrm{d}\mu(x)$$
$$= \int_{\mathbb{R}} \Big|\frac{f(x)}{W_{p/2}(x)}\Big|^2 W^2(x)\,\mathrm{d}\mu(x) \leqslant \|W\|^2_{L^2(\mu)}\,\|f\|^2_{C_0(W_{p/2})}.$$

Therefore, if there exists a function $f \in \mathcal{E}(a)$ with $f(\mathrm{i}) \neq 0$ such that

$$f(x)(x-\mathrm{i})^{-1} \in \mathrm{clos}_{C_0(W_{p/2})}\,\mathcal{E}(a)\,,$$

then

$$f(x)(x-\mathrm{i})^{-1} \in \mathrm{clos}_{L^2(\mu_p)}\,\mathcal{E}(a)\,.$$

It remains to apply Theorem 2.3.

To verify the necessity part, we choose n so big that the function $x \mapsto (1+|x|^n)^{-1}$ belongs to $L^2(\mu)$, and suppose that $\mathcal{E}(a)$ is dense in $L^2(\mu_p)$ for every $p<\infty$. Take any function $f \in \mathcal{E}(a)$ with $f(\mathrm{i}) \neq 0$, and choose functions $h_k \in \mathcal{E}(a)$ such that

$$\int_{\mathbb{R}} |h_k(x) - f(x)(x-\mathrm{i})^{-1}|^2(1+|x|)^{2k}\,\mathrm{d}\mu(x) < 8^{-k}\,.$$

We set

$$W(x) = \Big[(1+|x|^n)^{-1} + \sum_{k\geqslant 1} 4^k \Big|h_k(x) - \frac{f(x)}{x-\mathrm{i}}\Big|^2(1+|x|)^{2k}\Big]^{1/2}.$$

Then W is a lower semi-continuous function (since W^2 is the sum of a series with continuous non-negative terms) satisfying the growth condition (12), and $W \in L^2(\mu)$. Let $s \in \mathbb{R}$. For $k \geqslant s$ we have

$$\|h_k(x) - f(x)(x-\mathrm{i})^{-1}\|_{C_0(W_s)} = \sup_{\mathbb{R}} \frac{|h_k(x) - f(x)(x-\mathrm{i})^{-1}|\,(1+|x|)^s}{W(x)}$$
$$\leqslant \sup_{\mathbb{R}} 2^{-k}\frac{|h_k(x) - f(x)(x-\mathrm{i})^{-1}|\,(1+|x|)^s}{|h_k(x) - f(x)(x-\mathrm{i})^{-1}|\,(1+|x|)^k} \leqslant 2^{-k}.$$

Thus, by Theorem 2.3, $\mathcal{E}(a)$ is dense in $C_0(W_s)$. □

The last result in this section is a version of de Branges' classical theorem on weighted polynomial approximation [7], [8, Theorem 66], [15, Section VI.F].

Definition 2.9 (entire functions of Krein's class)**.** *Given $a>0$, we denote by $\mathcal{K}(a)$ the class of entire functions f of exponential type a with simple real zeros*

$\Lambda(f)$ such that $f(\mathbb{R})\subset\mathbb{R}$, and $1/f$ is represented as an absolutely convergent series:

$$\frac{1}{f(z)} = R(z) + \sum_{\lambda\in\Lambda(f)} \frac{1}{f'(\lambda)}\Big(\frac{1}{z-\lambda}+\frac{1}{\lambda}+\frac{z}{\lambda^2}+\ldots+\frac{z^N}{\lambda^{N+1}}\Big)\,, \tag{13}$$

with some $N\geqslant 0$ and with a polynomial R.

Given a weight W, we denote by $\mathcal{K}(a,W)$ the class of all functions $f\in\mathcal{K}(a)$ such that

$$\sum_{\lambda\in\Lambda(f)} \frac{W(\lambda)}{|f'(\lambda)|} < \infty. \tag{14}$$

By Krein's theorem, entire functions of the class $\mathcal{K}(a)$ belong to the Cartwright class [21, Theorem 3 in Lecture 16]. Using the Phragmén-Lindelöf principle, it is not difficult to verify that if an entire function f of Cartwright class maps $\mathbb{R}$ into $\mathbb{R}$, has simple real zeroes $\Lambda(f)$, and satisfies (14), then representation (13) is valid, cf. Kossis [15, Section VIF.4].

Theorem 2.10 (de Branges)**.** *Let W be a weight function, and let $a>0$. The linear space $\mathcal{E}(a)$ is not dense in $C_0(W)$ if and only if $\mathcal{K}(a,W)\neq\emptyset$.*

Proof: First, we assume that $\mathcal{K}(a,W)\neq\emptyset$. Let $B\in\mathcal{K}(a,W)$. Without loss of generality, assume that $B(0)\neq 0$. Then the measure

$$\mu_B = \sum_{\lambda\in\Lambda(B)} \frac{\delta_\lambda}{B'(\lambda)}$$

belongs to the dual space $C_0(W)^*$. The Lagrange interpolation formula shows that, for each $f\in\mathcal{E}(a)$ we have

$$zf(z) = \sum_{\lambda\in\Lambda(B)} \frac{\lambda f(\lambda)B(z)}{(z-\lambda)B'(\lambda)}\,.$$

Letting $z=0$, we see that the measure μ_B annihilates $\mathcal{E}(a)$. Hence, $\mathcal{E}(a)$ is not dense in $C_0(W)$.

The other implication is more deep. Here, we use a modification of the argument presented in [34] for the polynomial approximation problem.

Suppose that $\mathcal{E}(a)$ is not dense in $C_0(W)$, denote by $X=\mathrm{clos}_{C_0(W)}\,\mathcal{E}(a)$ its closure, and fix a function $\varphi\in\mathcal{E}(a)$, $\varphi(\mathrm{i})\neq 0$, such that $\varphi(t)/(t-\mathrm{i})$ does not belong to X.

First, we show that X consists of entire functions of exponential type at most a with convergent logarithmic integral,

$$\int_{\mathbb{R}} \frac{\log^+|f(x)|}{1+x^2}\,\mathrm{d}x < \infty\,, \qquad f\in X\,. \tag{15}$$

Indeed, if L is a bounded linear functional on $C_0(W)$ vanishing on $\mathcal{E}(a)$ such that $L\big[\varphi(t)/(t-\mathrm{i})\big]\neq 0$, then the function $F\colon w\mapsto L\big[\varphi/(\cdot-w)\big]$ is analytic in the upper half-plane, and

$$\int_{\mathbb{R}}\frac{\log^-|F(x+\mathrm{i})|}{1+x^2}\,\mathrm{d}x<\infty.$$

If $M(w)=\sup\{|f(w)|\colon f\in\mathcal{E}(a),\ \|f\|_{C_0(W)}\leqslant 1\}$, then the relation

$$f(w)L\Big[\frac{\varphi}{\cdot-w}\Big]=\varphi(w)L\Big[\frac{f}{\cdot-w}\Big],\qquad f\in\mathcal{E}(a),$$

yields that

$$\int_{\mathbb{R}}\frac{\log^+ M(x+\mathrm{i})}{1+x^2}\,\mathrm{d}x<\infty.$$

Then estimates for subharmonic functions in a half plane imply that

$$\int_{\mathbb{R}}\frac{\log^+ M(x)}{1+x^2}\,\mathrm{d}x<\infty,$$

and, hence, (15) holds. Furthermore, for every $\varepsilon>0$,

$$\log M(w)\leqslant(a+o(1))|\operatorname{Im}w|,\qquad \varepsilon<\arg w<\pi-\varepsilon,\ |w|\to\infty.$$

A Phragmén-Lindelöf type theorem completes our argument; see also [15, VI E].

Next, we claim that there is a function $\psi\in\mathcal{E}(a)$, real on $\mathbb{R}$, such that $h(t)=\psi(t)/(t^2+1)\notin X$. Indeed, let $\varphi=\varphi_1+\mathrm{i}\varphi_2$ where φ_1,φ_2 are functions in $\mathcal{E}(a)$ real on $\mathbb{R}$. Then

$$\frac{\varphi}{t-\mathrm{i}}=\big(\varphi_1+\mathrm{i}\varphi_2\big)\Big(\frac{t}{t^2+1}+\mathrm{i}\frac{1}{t^2+1}\Big)=\frac{t\varphi_1-\varphi_2}{t^2+1}+\mathrm{i}\frac{t\varphi_2+\varphi_1}{t^2+1}\,,$$

and at least one of the two summands on the right-hand side does not belong to X. Then we consider a Chebyshev-type extremal problem of the best approximation in $C_0(W)$ to h by elements of X. By a normal family argument, an extremal function f exists, and belongs to the Cartwright class, and by symmetry, we may assume that it is real on the real line.

Next we verify the following

Lemma 2.11. *The function $U=h-f$ is of exponential type a, has only simple real zeros, and between each pair of consecutive zeros there is a point λ such that $|U(\lambda)|=L\,W(\lambda)$, where $L=\|U\|_{C_0(W)}>0$.*

Proof: First, U is of exponential type a. Otherwise, considering the functions f_ε,

$$f_\varepsilon(x)=f(x)+\frac{x^2+1}{x^2}U(x)\sin^2(\varepsilon x),\qquad \varepsilon>0,$$

we obtain that f is not extremal. Note that in this situation the function $x\mapsto\frac{x^2+1}{x^2}U(x)\sin^2(\varepsilon x)$ belongs to X for small positive ε. Here, we use a classical fact that if an entire function of exponential type less than a belongs to $C_0(W)$, then it can be approximated in $C_0(W)$ by entire functions in $\mathcal{E}(a)$, cf. [15, Section VI H 1], [24, Section 2].

Next, U has only simple real zeros. Otherwise, if $\alpha, \bar\alpha$ is a pair of conjugate zeros of U, then, considering the functions f_ε,

$$f_\varepsilon(x) = f(x) + \varepsilon\,\frac{x^2+1}{(x-\alpha)(x-\bar\alpha)}U(x), \qquad \varepsilon > 0,$$

we obtain that f is not extremal. Once again, the function $x \mapsto \varepsilon\,\frac{x^2+1}{(x-\alpha)(x-\bar\alpha)}U(x)$ belongs to X.

Finally, if $\alpha < \beta$ are consecutive zeros of U such that $|U| < L\,W$ on (α,β), then by the lower semicontinuity of W, we have $\sup_{(\alpha,\beta)} |U|/W < L$. Considering the functions f_ε,

$$f_\varepsilon(x) = f(x) + \varepsilon\,\frac{x^2+1}{(x-\alpha)(x-\beta)}U(x), \qquad \varepsilon > 0,$$

we obtain that f is not extremal. This completes the proof of the lemma. $\square$

We return to the proof of de Branges theorem. Let $A(z) = (z^2+1)U(z)$, let $\Lambda = \Lambda(U) = \{x_j\}$, and let $\Lambda^* = \{x'_j\} \subset \{x : |U(x)| = L\,W(x)\}$, with interlacing x_j and x'_j: $x_j < x'_j < x_{j+1}$. Set

$$B(z) = A(z)\prod_j \frac{z-x'_j}{z-x_j}.$$

Then $\operatorname{Im}(A/B)$ does not change the sign in the upper half-plane, and hence,

$$\sum_{\lambda\in\Lambda^*} \frac{|A(\lambda)|}{|B'(\lambda)|(1+\lambda^2)} < \infty.$$

By the definition of Λ^*,

$$\sum_{\lambda\in\Lambda^*} \frac{W(\lambda)}{|B'(\lambda)|} < \infty.$$

Since B belongs to the Cartwright class, we conclude that $B \in \mathcal{K}(a, W)$. $\square$

3. Proof of Theorem 1.3

First, let us recall the definition (2) of the intervals I_x, kI_x. An elementary calculation shows that for $k_1 = k_1(\delta)$, we have the following property:

$$y \in I_x \implies 2I_x \subset k_1 I_y.$$

Lemma 3.1. *Suppose that $\mu \preccurlyeq \widetilde\mu$. Given $p > 0$ and a weight function $\widetilde W \in L^2(\widetilde\mu)$, we define a function W by*

$$W(x) = \min\Bigl[\inf_{k_1 I_x} \widetilde W_p, e^{\delta|x|/3}\Bigr]. \tag{16}$$

Then W is a weight function, and $W \in L^2(\mu)$, provided that p is big enough.

Here, as above, $\widetilde{W}_p(x) = (1+|x|)^{-p}\widetilde{W}(x)$.

Proof: It is immediately seen that W is a weight function. Next we choose x_j, $j \in \mathbb{Z}$, on $\mathbb{R}$ in such a way that the intervals I_{x_j} cover $\mathbb{R}$ with intersections only at endpoints. Then the intervals $2I_{x_j}$ cover $\mathbb{R}$ with multiplicity bounded by $C = C(\delta)$. Using that

$$\sup_{y\in I_x} W(y) \leqslant \sup_{y\in I_x} \inf_{t\in k_1 I_y} \widetilde{W}_p(t) \leqslant \sup_{y\in I_x} \inf_{t\in 2I_x} \widetilde{W}_p(t) = \inf_{t\in 2I_x} \widetilde{W}_p(t),$$

$$\mu(I_x) \leqslant C(1+|x|)^n \left(\widetilde{\mu}(2I_x) + e^{-2\delta|x|}\right),$$

we obtain that for $p \geqslant n$,

$$\begin{aligned}
\int_{I_x} W^2(x)\, d\mu(x) &\leqslant \sup_{I_x} W^2 \cdot \mu(I_x) \\
&\leqslant C(1+|x|)^{2n}\widetilde{\mu}(2I_x)\cdot \inf_{2I_x} \widetilde{W}_p^2 + C(1+|x|)^{2n}e^{-2\delta|x|}\cdot e^{2\delta|x|/3} \\
&\leqslant C(1+|x|)^{2n-2p}\int_{2I_x} \widetilde{W}^2(y)\, \mathrm{d}\widetilde{\mu}(y) + C\int_{2I_x} (1+|y|)^{2n}e^{-\delta|y|/3}\, \mathrm{d}y.
\end{aligned}$$

Summing up these inequalities for $x = x_j$, $j \in \mathbb{Z}$, we complete the proof. ☐

Lemma 3.2. *Let W be a weight, and let $B \in \mathcal{K}(a, W)$. Then for some $c > 0$ and $C < \infty$ we have*

$$|B(x)| + |B'(x)| + |B''(x)| < e^{\delta|x|/5}, \qquad |x| > C, \tag{17}$$

$$|\lambda - \lambda'| > ce^{-\delta|\lambda|/4}, \qquad \lambda, \lambda' \in \Lambda(B),\ \lambda \neq \lambda', \tag{18}$$

$$\sum_{\lambda\in\Lambda(B),\,\lambda\neq 0} \frac{1}{|\lambda|^2} < \infty. \tag{19}$$

Proof: Since B is of exponential type, we have (19). Denote by H_B the Phragmén-Lindelöf indicator function of the entire function B,

$$H_B(\theta) = \limsup_{r\to\infty} \frac{\log|B(re^{i\theta})|}{r}.$$

Since the function B belongs to the Cartwright class, we have $H_B(0) = H_B(\pi) = 0$ (see [21, Lecture 16]). As a consequence of Cauchy's formula for the derivative, the indicator of the derivative of an entire function cannot exceed the indicator of the function itself. Hence,

$$H_{B'}(0) \leqslant 0, \quad H_{B'}(\pi) \leqslant 0, \quad H_{B''}(0) \leqslant 0, \quad H_{B''}(\pi) \leqslant 0,$$

which implies (17).

Using that

$$\sum_{\lambda\in\Lambda(B)}\frac{W(\lambda)}{|B'(\lambda)|}<\infty$$

and that

$$\lim_{|x|\to\infty}(1+|x|)^sW(x)=\infty$$

for some $s\in\mathbb{R}$, we obtain

$$|B'(\lambda)|\geqslant\frac{c}{(1+|\lambda|)^s},\qquad \lambda\in\Lambda(B).$$

Since the signs of B' at consecutive zeros of B are opposite, this inequality together with estimate (17) on B'' gives (18). □

Lemma 3.3. *Let $\widetilde{W}$ be a weight. If $\mathcal{K}(a,W)\neq\emptyset$, where W is defined by (16), then $\mathcal{K}(a,\widetilde{W}_{p+\ell})\neq\emptyset$ provided that ℓ is big enough.*

Proof: Let $B\in\mathcal{K}(a,W)$. Since

$$\sum_{\lambda\in\Lambda(B)}\frac{W(\lambda)}{|B'(\lambda)|}<\infty,$$

for some $c>0$ we have

$$|B'(\lambda)|\geqslant c\cdot\min\bigl[\inf_{k_1I_\lambda}\widetilde{W}_p, e^{\delta|\lambda|/3}\bigr],\qquad \lambda\in\Lambda(B).$$

By (17),

$$|B'(\lambda)|=o(e^{\delta|\lambda|/3}),\qquad |\lambda|\to\infty,$$

and, hence, for some $C<\infty$,

$$|B'(\lambda)|\geqslant c\cdot\inf_{k_1I_\lambda}\widetilde{W}_p,\qquad \lambda\in\Lambda(B),\ |\lambda|>C.$$

Let D_λ be the disc centered at $\lambda\in\mathbb{R}$ of radius $e^{-\delta|\lambda|/3}$. For some $M<\infty$, by (18) we have the following implication:

$$\lambda,\lambda'\in\Lambda(B),\quad |\lambda|>M,\ |\lambda'|>M,\quad \lambda\neq\lambda'\implies D_\lambda\cap D_{\lambda'}=\emptyset.$$

and $k_1I_\lambda\subset D_\lambda$ for $\lambda\in\Lambda(B)$, $|\lambda|>M$.

Now, for some $c>0$ and for every $\lambda\in\Lambda(B)$ with $|\lambda|>M$ we find $\zeta_\lambda\in k_1I_\lambda$ such that

$$|B'(\lambda)|\geqslant c\widetilde{W}_p(\zeta_\lambda).$$

Without loss of generality assume that $B(0)=1$. Since B is of Cartwright class, we have

$$B(z)=\lim_{R\to\infty}\prod_{|\lambda|\leqslant R,\,\lambda\in\Lambda(B)}\Bigl(1-\frac{z}{\lambda}\Bigr).$$

For $z \in \partial D_\lambda$, $\lambda \in \Lambda(B)$, $|\lambda| > M$ we have

$$\Big|\Big(1-\frac{z}{\zeta_\lambda}\Big)\Big(1-\frac{z}{\lambda}\Big)^{-1}-1\Big| = \Big|\frac{z}{\zeta_\lambda}\cdot\frac{\lambda-\zeta_\lambda}{z-\lambda}\Big| \leqslant ce^{-2\delta|\lambda|/3}, \tag{20}$$

and by the maximum principle, we have the same estimate for all $z \in \mathbb{C} \setminus D_\lambda$.

We define an entire function B_1 by

$$B_1(z) = \lim_{R\to\infty} \prod_{M<|\lambda|\leqslant R,\, \lambda\in\Lambda(B)} \Big(1-\frac{z}{\zeta_\lambda}\Big);$$

the limit on the right hand side exists because of (19) and (20). In a similar way, applying the maximum principle in every D_λ, we conclude that B_1 is of exponential type. Furthermore,

$$|B_1(z)| \geqslant c|B(z)|(1+|z|)^{-N}, \qquad z \in \partial D_\lambda,\ \lambda \in \Lambda(B) \setminus [-M, M],$$

where $N = \operatorname{card}(\Lambda(B) \cap [-M, M])$, and, hence,

$$|B_1'(\zeta_\lambda)| \geqslant c(1+|\zeta_\lambda|)^{-N}|B'(\lambda)|, \qquad \lambda \in \Lambda(B) \setminus [-M, M],$$

which implies that $B_1 \in \mathcal{K}(a, \widetilde{W}_{p+\ell})$ with $\ell = N + 2$. □

Now we are ready to pass to

Proof of Theorem 1.3: Let $\mathcal{E}(a)$ be stably dense in $L^2(\widetilde{\mu})$. By Bakan's Theorem 2.8, there exists a weight $\widetilde{W} \in L^2(\widetilde{\mu})$ such that $\mathcal{E}(a)$ is stably dense in $C_0(\widetilde{W})$. Then, by de Branges' Theorem 2.10, for each $t < \infty$, we have $\mathcal{K}(a, \widetilde{W}_t) = \emptyset$. We take p big enough. Then the function W defined in (16) is a weight function and belongs to $L^2(\mu)$ by Lemma 3.1. Hence, $\mathcal{K}(a, W) = \emptyset$ (otherwise, by Lemma 3.3, $\mathcal{K}(a, \widetilde{W}_{p+\ell}) \neq \emptyset$ for large p, which is impossible). Now, using again de Branges' Theorem, we obtain that $\mathcal{E}(a)$ is dense in $C_0(W)$. Applying again Bakan's Theorem, we see that $\mathcal{E}(a)$ is dense in $L^2(\mu)$, proving the theorem. □

3.1. **Remark on the polynomial approximation in $L^2(\mu)$.** Theorem 1.3 has a polynomial counterpart, which can be proved using the same lines of reasoning. *Let $\widetilde{\mu}$ be a non-negative measure with finite moments,*

$$\int_{\mathbb{R}} |x|^n \,\mathrm{d}x < \infty, \qquad n \geqslant 0,$$

and let $\mu \preccurlyeq \widetilde{\mu}$. If the set of the polynomials $\mathcal{P}$ is stably dense[6] *in $L^2(\widetilde{\mu})$, then $\mathcal{P}$ is dense in $L^2(\mu)$.* In other words, *if the measure $\widetilde{\mu}$ has infinite index of determinacy for the Hamburger moment problem, and $\mu \preccurlyeq \widetilde{\mu}$, then the measure μ is determinate.*

[6]That is, $\mathcal{P}$ is dense in $L^2(\widetilde{\mu}_t)$ for each $t < \infty$. Equivalently, one can say that the measure $\widetilde{\mu}$ has infinite index of determinacy for the Hamburger moment problem on the real axis, see Berg and Duran [5]

This extends a result of Yuditskii [37]. Answering a question posed by Berg in the 1990-s, Yuditskii obtained the same conclusion with a much stronger assumption that $\widetilde{\mu} = \mu + \nu$ where ν is a non-negative measure on $\mathbb{R}$ with a finite exponential moment: for some $\delta > 0$,

$$\int_{\mathbb{R}} e^{\delta|\lambda|}\,\mathrm{d}\nu(\lambda) > 0\,.$$

It is not clear whether an elegant operator-theoretical approach developed by Yuditskii in [37] (see also Section 5.2 below) can be used to obtain the aforementioned extension.

4. Sharpness. Proof of Theorem 1.6

4.1. **Proof of Theorem 1.6 (i).** We construct here a convex function f on $[0,\infty)$ such that

$$\int^{\infty} f(t)e^{-t}\,\mathrm{d}t = \infty, \tag{21}$$

$$\int^{\infty} \min\bigl(f(t), \varepsilon(e^t)e^t\bigr)e^{-t}\,\mathrm{d}t < \infty, \tag{22}$$

and set $\varphi(x) = e^{-f(\log^+|x|)}$, $x \in \mathbb{R}$.

Then

$$\int_{\mathbb{R}} \frac{\log(1/\varphi(x))}{x^2+1}\,\mathrm{d}x = \infty,$$

and by the result mentioned in Appendix A.1, we have $T(\varphi(x)\,\mathrm{d}x) = 0$. (In fact, a classical theorem of Izumi–Kawata (see [15, VID, p.170]) shows that already the polynomials are dense in $L^2(\varphi(x)\,dx)$.)

On the other hand, if $\psi(x) = \varphi(x) + e^{-\varepsilon(|x|)|x|}$, then

$$\int_{\mathbb{R}} \frac{\log(1/\psi(x))}{x^2+1}\,\mathrm{d}x < \infty,$$

and by the Krein theorem mentioned in Appendix A.2, $T(\psi(x)\,dx) = \infty$.

The function f will be built as the sum of the functions $f_n = \max(l_n, 0)$ for some linear functions l_n chosen in an inductive process. On step $n \geqslant 1$ we fix a sufficiently large a such that

$$\gamma = \sup_{t\geqslant a} \varepsilon(e^t) < 4^{-n},$$

and $b > a+1$ such that

$$(b-a)\gamma < 2^{-n}, \tag{23}$$

$$\frac{e^{b-a}-1}{b-a}\gamma \geqslant 10. \tag{24}$$

The linear function l_n is determined by the conditions

$$l_n(a) = \gamma e^a, \qquad l_n(b) = \gamma e^b.$$

To prove (21)–(22) we need only to verify that

$$\int_0^a f_n(x)e^{-x}\,\mathrm{d}x + \int_b^\infty f_n(x)e^{-x}\,\mathrm{d}x \leqslant C\gamma,$$

$$\int_a^b f_n(x)e^{-x}\,\mathrm{d}x \geqslant 1,$$

$$\int_a^b \gamma e^x \cdot e^{-x}\,\mathrm{d}x < 2^{-n}. \tag{25}$$

By (24), we have

$$\frac{e^b - e^a}{b-a} \geqslant \frac{10e^a}{\gamma} \geqslant e^a.$$

Therefore, $f_n = 0$ on $[0, a-1]$, and

$$\int_0^a f_n(x)e^{-x}\,\mathrm{d}x \leqslant \gamma \int_{a-1}^a e^{a-x}\,\mathrm{d}x = (e-1)\gamma.$$

Furthermore,

$$\int_b^\infty f_n(x)e^{-x}\,\mathrm{d}x = \gamma \int_0^\infty \Bigl(e^b + \frac{e^b-e^a}{b-a}s\Bigr)e^{-b-s}\,\mathrm{d}s$$
$$= \gamma \int_0^\infty \Bigl(e^{-s} + \frac{1-e^{a-b}}{b-a}se^{-s}\Bigr)\,\mathrm{d}s \leqslant 2\gamma,$$

and by (24),

$$\int_a^b f_n(x)e^{-x}\,\mathrm{d}x = \gamma \int_0^{b-a} \Bigl(e^a + \frac{e^b-e^a}{b-a}s\Bigr)e^{-a-s}\,\mathrm{d}s$$
$$= \gamma \int_0^{b-a} \Bigl(e^{-s} + \frac{e^{b-a}-1}{b-a}se^{-s}\Bigr)\,\mathrm{d}s \geqslant \gamma\frac{e^{b-a}-1}{b-a}\int_0^1 se^{-s}\,\mathrm{d}s \geqslant 1.$$

Finally, (25) follows from (23). □

4.2. **Proof of Theorem 1.6 (ii).** Here, the construction is more involved. Without loss of generality, we can assume that ε does not increase.

Lemma 4.1. *Let $\varepsilon(r) \searrow 0$, $r \to \infty$. There exists a system of disjoint intervals $I_k = [y_k, 2y_k]$, $k \geqslant 1$, and a convex function φ on $[1,\infty)$ such that*

$$\varepsilon(e^x)e^x = o(\varphi(x)), \qquad e^x \in \cup_{k\geqslant 1} I_k,\ x \to +\infty, \tag{26}$$

and

$$\int_0^\infty \varphi(t)e^{-t}\,dt < \infty. \tag{27}$$

Proof: We construct the function φ as the sum of functions φ_k,

$$\varphi_k(x) = \max\bigl(\gamma_k y_k(x + 1 - \log y_k), 0\bigr),$$

with

$$\gamma_k = k \sup_{t \geqslant y_k} \varepsilon(t).$$

Then (26) follows immediately. Since

$$\int_0^\infty \varphi_k(t) e^{-t}\, dt = \gamma_k y_k \int_{(\log y_k)-1}^\infty (t + 1 - \log y_k) e^{-t}\, dt$$

$$= e\gamma_k \int_0^\infty t e^{-t}\, dt = e\gamma_k,$$

we can find a sequence $\{y_k\}$ such that I_k are disjoint, $\sum_{k\geqslant 1} \gamma_k < \infty$, and hence, (27) holds. □

By Lemma 4.1, we obtain φ and $\{I_k\}$, and introduce an even weight

$$W(x) = \exp \varphi(\max(\log|2x|, 1)). \tag{28}$$

Definition 4.2. *We denote by $\mathcal{H}(W)$ the Hamburger class of transcendental entire functions f of zero exponential type a with simple real zeros $\Lambda(f)$ such that $f(\mathbb{R}) \subset \mathbb{R}$, and*

$$\sum_{\lambda\in\Lambda(f)} \frac{W(\lambda)}{|f'(\lambda)|} < \infty\,.$$

Lemma 4.3. *There exists $F \in \mathcal{H}(W)$ such that for some $c > 0$ and $E \subset \mathbb{R}$ of finite length, symmetric with respect to 0, we have*

$$|F(x)| \geqslant cW(x/2)^c, \qquad x \in \mathbb{R}\setminus E, \tag{29}$$

$$\operatorname{dist}(\lambda, \mathbb{R}\setminus E) \geqslant \frac{c}{1+|\lambda|^2}, \qquad \lambda \in \Lambda(F). \tag{30}$$

Proof:

A. First, we check that there is an entire function F in $\mathcal{H}(W)$ with the zero set Λ_F symmetric with respect to the origin. This will readily follow from a version of de Branges' theorem dealing with weighted polynomial approximation. We have

$$\lim_{x\to\infty} \frac{\log W(x)}{\log x} = \infty,$$

and the polynomials belong to $C_0(W)$. Furthermore, by (27),

$$\int_{-\infty}^\infty \frac{\log W(x)}{1+x^2}\, dx < \infty,$$

and by the Hall theorem [15, VI D], the polynomials are not dense in $C_0(W)$.

Let $\mu \in (C_0(W))^*$, $\mu \neq 0$, vanish on the polynomials. Consider the functional $\tilde{\mu} \in (C_0(W))^*$ defined by

$$\langle \tilde{\mu}, f \rangle = \langle \mu, x \mapsto f(-x) \rangle,$$

and define $\mu_{\text{even}} = (\mu + \tilde{\mu})/2$, $\mu_{\text{odd}} = (\mu - \tilde{\mu})/2$. Suppose that $\mu_{\text{even}} \neq 0$ (the case $\mu_{\text{odd}} \neq 0$ is dealt with analogously). Set $W_0(x) = W(\sqrt{x})$ if $x \geqslant 0$, $W_0(x) = \infty$ otherwise,

$$\langle \mu_{\text{right}}, f \rangle = \langle \mu_{\text{even}}, x \mapsto f(x^2) \rangle.$$

Then $\mu_{\text{right}} \in (C_0(W_0))^*$, $\mu_{\text{right}} \neq 0$, and μ_{right} vanishes on the polynomials. By the de Branges theorem [15, VI F2, VI F1], there exists a transcendental entire function F_0 of at most minimal type of order 1/2,

$$\limsup_{|z| \to \infty} \frac{\log |F_0(z)|}{|z|^{1/2}} = 0,$$

real on the real line, with zeros $\{x_k^2\}_{k \geqslant 1}$, $x_k \geqslant 0$, $k = o(x_k)$, $k \to \infty$, such that

$$\sum_{k \geqslant 1} \frac{W_0(x_k^2)}{|F_0'(x_k^2)|} < \infty.$$

Let

$$F(z) = F_0(z^2) = F(0) \prod_{k \geqslant 1} \Bigl(1 - \frac{z^2}{x_k^2}\Bigr) \tag{31}$$

(with an obvious modification if $x_1 = 0$). Then

$$\sum_{k \geqslant 1} \frac{x_k W(x_k)}{|F'(x_k)|} < \infty,$$

and $F \in \mathcal{H}(W)$.

B. Now, we prove estimates (29) and (30). Suppose that $0 < x_k < x_{k+1}$ are two consecutive zeros of F. Suppose that $\Delta = x_{k+1} - x_k > k^{-2}$ (otherwise, just add the closure of the interval $J_k = (x_k, x_{k+1})$ to E). By the Laguerre theorem, the zeros of F and F' interlace. Denote by λ the zero of F' on J_k, and set $G(z) = F'(z)/(z - \lambda)$. Then G has no zeros in the strip $J_k + i\mathbb{R}$. Since $|G(x + iy)|$ increases in y for positive y, we have

$$|G(x_k + \mathrm{i}y)| \geqslant |G(x_k)| \geqslant cW(x_k)/\Delta,$$
$$|G(x_{k+1} + \mathrm{i}y)| \geqslant |G(x_{k+1})| \geqslant cW(x_{k+1})/\Delta$$

for $y \in \mathbb{R}$. Hence, by the three lines theorem applied in the strip $J_k + i\mathbb{R}$ to the harmonic function $-\log |G(z)|$, we obtain

$$\log |G(x)| \geqslant \frac{x - x_k}{\Delta} \log W(x_{k+1}) + \frac{x_{k+1} - x}{\Delta} \log W(x_k) + \log \frac{c}{\Delta}, \ x \in J_k.$$

If $x_{k+1} < 2x_k$, then we obtain

$$\log|G(x)| \geqslant c\log W(x/2), \qquad x \in J_k,$$
$$|F'(x)| \geqslant cW(x/2)^c, \qquad x \in J_k,\ |x-\lambda| > k^{-2},$$
$$|F(x)| > c_1 W(x/2)^{c_1}, \qquad x_k + k^{-2} < x < x_{k+1} - k^{-2},\ |x-\lambda| > k^{-2}.$$

Otherwise, if $x_{k+1} \geqslant 2x_k$, then, in the same way, for some x'_k, x'_{k+1} with $x_k < x'_k < x_k + k^{-2}$, $x_{k+1} - k^{-2} < x'_{k+1} < x_{k+1}$, we obtain:

$$|F(x'_k)| \geqslant cW(x_k/2)^c,$$
$$|F(x'_{k+1})| \geqslant cW(x_{k+1}/2)^c.$$

Since the function $|F|$ is log-concave on J_k (this follows immediately from the representation (31)), and W is log-convex, we conclude that

$$|F(x)| \geqslant cW(x/2)^c, \qquad x'_k \leqslant x \leqslant x'_{k+1},$$

proving the lemma. □

Next we use the following simple lemma on perturbations of the sine function.

Lemma 4.4. *Let* $\Sigma = \bigcup_{k\geqslant 0}\{a_k, b_k, c_k, d_k\}$, *where for each* $k \geqslant 0$,

$$2k + \frac{6}{5} < a_k < b_k < 2k + \frac{7}{5}, \quad 2k + \frac{8}{5} < c_k < d_k < 2k + \frac{9}{5},$$
$$\eta_k = d_k - c_k = b_k - a_k, \quad a_k + b_k + c_k + d_k = 8k + 6\,.$$

Put

$$G(z) = z\prod_{\lambda\in\Sigma}\Bigl(1 - \frac{z^2}{\lambda^2}\Bigr).$$

Then G *is an entire function of exponential type* 2π, *and*

$$|G'(\lambda)| \geqslant c\eta_k, \qquad \lambda \in \Sigma \cap (2k+1, 2k+2). \tag{32}$$

Proof: (compare to that of Lemma 3.3) Let D_k be the disc centered at $2k + \frac{3}{2}$ of radius $2/3$,

$$g_k(z) = \frac{(1 - z/a_k)(1 - z/b_k)(1 - z/c_k)(1 - z/d_k)}{(1 - z/(2k+1))^2(1 - z/(2k+2))^2},$$
$$b_k = \frac{a_k b_k c_k d_k}{(2k+1)^2(2k+2)^2}.$$

Then

$$b_k = 1 + O(1/k^2), \qquad k \to \infty,$$
$$|b_k g_k(z) - 1| \leqslant \frac{M}{1 + |z - 2k|^2}, \qquad z \in \mathbb{C} \setminus D_k,$$

with M independent of k. Therefore,

$$G_0(z) = \prod_{k\geqslant 0}[g_k(z)g_k(-z)]$$

is bounded outside $\bigcup_{k\geqslant 0}(D_k \cup \widetilde{D_k})$, where $\widetilde{D_k} = \{w : -w \in D_k\}$, and

$$|G_0(z)| \asymp |g_k(z)|, \qquad z \in D_k,$$
$$|G_0(z)| \asymp |g_k(-z)|, \qquad z \in \widetilde{D_k}.$$

Using the maximum principle in the discs D_k and $\widetilde{D_k}$, we conclude that

$$G(z) = G_0(z) \cdot \sin^2 \pi z$$

is an entire function of exponential type 2π; estimate (32) follows immediately. □

Now we return to the proof of Theorem 1.6 (ii). Let the function F and the set E be as in Lemma 4.3. Put

$$A =\{k \geqslant 0 : (2k + 1, 2k + 2) \not\subset \cup_{n\geqslant 1} I_n\},$$
$$B =\{k \geqslant 0 : (2k + 1, 2k + 2) \subset \cup_{n\geqslant 1} I_n\}.$$

We can choose $a_k, b_k, c_k, d_k,\ k \geqslant 0$, satisfying the conditions of Lemma 4.4 in such a way that

$$\eta_k = \frac{1}{10}, \qquad k \in A,$$
$$\eta_k = e^{-\varepsilon(2k+2)(2k+2)}, \qquad k \in B,$$

$\Sigma \cap E$ is bounded, and $\Sigma \cap \Lambda(F) = \emptyset$. By Lemma 4.4, we obtain an entire function G of exponential type 2π such that

$$|G'(\lambda)| \geqslant c > 0, \qquad \lambda \in \Sigma \cap (2k + 1, 2k + 2),\ k \in A,$$
$$|G'(\lambda)| \geqslant c\, e^{-\varepsilon(2k+2)(2k+2)}, \qquad \lambda \in \Sigma \cap (2k + 1, 2k + 2),\ k \in B,$$
$$|G(\lambda)| \geqslant c(1 + |\lambda|^2)^{-2}, \qquad \lambda \in \Lambda(F). \tag{33}$$

Let $H = FG$. Then H is of exponential type 2π, and by (28), (29), and (33),

$$|H'(\lambda)| \geqslant c(1 + |\lambda|), \qquad \lambda \in \Lambda(H).$$

The function

$$\Phi : \lambda \in \Lambda = \Lambda(H) \mapsto \frac{1}{H'(\lambda)}$$

belongs to $L^2(\mu)$, where

$$\mu = \sum_{\lambda\in\Lambda} \delta_\lambda.$$

Repeating the argument used in the proof of the simple direction of de Branges' Theorem 2.10, we see that Φ annihilates $\mathcal{E}(2\pi)$. Therefore, $T(\mu) \geqslant 2\pi$.

We have

$$\Lambda = \Lambda(F) \cup \bigcup_{k\geqslant 0}\{\pm a_k, \pm b_k, \pm c_k, \pm d_k\}.$$

Put

$$\Lambda^* = \Lambda(F) \cup \bigcup_{k\in A}\{\pm a_k, \pm b_k, \pm c_k, \pm d_k\} \cup \bigcup_{k\in B}\{\pm a_k, \pm c_k\},$$

$$\nu = \sum_{\lambda\in\Lambda^*} \delta_\lambda,$$

$$V(\lambda) = \begin{cases} (1+|\lambda|)^{-1}, & \lambda\in\Lambda^*, \\ +\infty, & \lambda\notin\Lambda^*. \end{cases}$$

Then $V\in L^2(\nu)$. Let $\varepsilon>0$. By de Branges' Theorem 2.10, if $\mathcal{E}(\pi+\varepsilon)$ is not dense in $C_0(V)$, then there exists a non-zero function $U\in\mathcal{K}(\pi+\varepsilon, V)$ such that $\Lambda(U)\subset\Lambda^*$, which contradicts to the Levinson theorem on the existence of the density of zeros for entire functions of Cartwright class (see [21, Lecture 17] or [15, III H2]). Therefore, by Bakan's theorem, $\mathcal{E}(\pi+\varepsilon)$ is dense in $L^2(\nu)$, $\varepsilon>0$. Thus, $T(\nu)\leqslant\pi$.

In a similar way, we verify that $T(\nu)\geqslant\pi$, $T(\mu)\leqslant 2\pi$.

Finally, we set $\{x_k\}=\Lambda^*$, and choose y_k such that

$$\{x_k\}\cup\{y_k\}=\Lambda, \quad \text{and} \quad |y_k-x_k|\leqslant e^{-\varepsilon(|x_k|)|x_k|}.$$

This completes the proof of Theorem 1.6. $\square$

5. Non-classical orthogonal spectral functions. Proof of Theorem 1.9

We prove Theorem 1.9 in two steps: first we perturb the stably orthogonal measure μ_0 on $\mathbb{R}$, and then we add a symmetric measure $\mu_{\mathrm{i}\mathbb{R}}$ supported by $\mathrm{i}\mathbb{R}$ with fast decaying tails.

5.1. **Perturbation on the real axis.** Let $\mu_{\mathbb{R}}$ be a symmetric measure supported by $\mathbb{R}$ such that the integral

$$\int_\lambda^\infty \mathrm{d}(\mu_{\mathbb{R}}-\mu_0)$$

(conditionally) converges and for some $\delta>0$,

$$\int_0^\infty e^{\delta\lambda}\left|\int_\lambda^\infty \mathrm{d}(\mu_{\mathbb{R}}-\mu_0)\right| < \infty\,. \tag{34}$$

Then $\widehat{\mu_{\mathbb{R}}} = \widehat{\mu_{\mathbb{R}}-\mu_0} + \widehat{\mu_0}$, and by condition (34), the function $\widehat{\mu_{\mathbb{R}}-\mu_0}$ has analytic continuation into the strip $\big\{|\mathrm{Im} z|<\delta\big\}$.

By the Gelfand–Levitan theorem 1.7, the function $\Phi[\mu_0]$ satisfies the Gelfand–Levitan condition (GL–$\mathbf{i}$), and, hence, the function $\Phi[\mu_{\mathbb{R}}]$ satisfies (GL–$\mathbf{i}$). Again by the Gelfand–Levitan theorem, $\mu_{\mathbb{R}}$ is a spectral measure of a Sturm–Liouville problem (3)–(4) on the interval $[0,a)$, with the potential q of the same class of smoothness on $[0,a)$ as the potential q_0 that corresponds to μ_0.

Next, we check that the spectral measure $\mu_{\mathbb{R}}$ is orthogonal. This follows from our Theorem 1.3 combined with the following claim:

Claim 5.1. *There exist positive constants $\delta' > 0$ and $C > 0$ such that, for all $x \in \mathbb{R}$,*

$$\mu_{\mathbb{R}}(I_x) \leqslant C\big(\mu_0(2I_x) + e^{-2\delta'|x|}\big),$$

where $I_x = [x - e^{-\delta'|x|}, x + e^{-\delta'|x|}]$ and $2I_x$ is the concentric interval of twice bigger length.

Proof: Suppose that the claim does not hold; i.e., for each $n \geqslant 3$ and each $\delta' > 0$, there exists λ_n such that

$$\mu_{\mathbb{R}}(I_{\lambda_n}) \geqslant n\big(\mu_0(2I_{\lambda_n}) + e^{-2\delta'|\lambda_n|}\big).$$

Since the measures μ_0 and $\mu_{\mathbb{R}}$ are locally finite, $|\lambda_n| \to \infty$ when $n \to \infty$. Without loss of generality, we assume that $\lambda_n \to +\infty$. Let $\psi(\lambda) = \int_\lambda^\infty \mathrm{d}(\mu_{\mathbb{R}} - \mu_0)$. Then

$$\begin{aligned}\psi\big(\lambda_n - e^{-\delta'\lambda_n}\big) - \psi\big(\lambda_n + e^{-\delta'\lambda_n}\big) &= \mu_{\mathbb{R}}(I_{\lambda_n}) - \mu_0(I_{\lambda_n}) \\ &\geqslant (n-1)\mu_0(2I_{\lambda_n}) + ne^{-2\delta'\lambda_n}\,.\end{aligned}$$

Therefore, at least one of the following two conditions must hold: either

$$\psi\big(\lambda_n + e^{-\delta'\lambda_n}\big) \leqslant -\frac{1}{2}(n-1)\mu_0(2I_{\lambda_n}) - \frac{1}{2}ne^{-2\delta'\lambda_n}\,,$$

or

$$\psi\big(\lambda_n - e^{-\delta'\lambda_n}\big) \geqslant \frac{1}{2}(n-1)\mu_0(2I_{\lambda_n}) + \frac{1}{2}ne^{-2\delta'\lambda_n}\,.$$

We assume, for instance, that the first case occurs, the second case is quite similar. Then, for $\lambda \in [\lambda_n + e^{-\delta'\lambda_n}, \lambda_n + 2e^{-\delta'\lambda_n}]$ we have

$$\psi(\lambda) \leqslant \psi\big(\lambda_n + e^{-\delta'\lambda_n}\big) + \mu_0\big(2I_{\lambda_n}\big) \leqslant -e^{-2\delta'\lambda_n}\,,$$

whence

$$\int_{\lambda_n+e^{-\delta'\lambda_n}}^{\lambda_n+2e^{-\delta'\lambda_n}} |\psi(\lambda)|e^{\delta\lambda}\,\mathrm{d}\lambda \geqslant \int_{\lambda_n+e^{-\delta'\lambda_n}}^{\lambda_n+2e^{-\delta'\lambda_n}} e^{-2\delta'\lambda_n+\delta\lambda}\,\mathrm{d}\lambda \geqslant 1$$

provided that $\delta' \leqslant \delta/3$. Clearly, this contradicts to (34). Hence, the claim. $\square$

5.2. **Perturbation on the imaginary axis.** Here, we show that *if $\mathcal{E}(a)$ is dense in $L^2(\mu_{\mathbb{R}})$, where $\mu_{\mathbb{R}}$ is a stably orthogonal spectral measure supported by $\mathbb{R}$, then it is also dense in $L^2(\mu)$.* In the case $a=\infty$, a similar question was studied by Levitan and Meiman [23], and then by Vul [35]. Later, the same completeness problem appeared again in Gurarii's work [14, Theorem 5] on harmonic analysis in weighted Banach algebras of functions on $\mathbb{R}$ with asymmetric weights. We cannot use their results since we deal with the case of finite a. Instead, we use an idea from Yuditskii's work [37] pertaining to the density of polynomials. The following theorem completes the proof of Theorem 1.9 (condition (GL–**ii**) holds for all measures $\mu_{\mathrm{i}\mathbb{R}}$ we consider here).

Theorem 5.2. *Let $\mu_{\mathbb{R}}$ be a non-negative measure on $\mathbb{R}$ satisfying estimate* (6), *and let $\mathcal{E}(a)$ be stably dense in $L^2(\mu_{\mathbb{R}})$ for some $a>0$. Let $\mu_{\mathrm{i}\mathbb{R}}$ be a measure on $\mathrm{i}\mathbb{R}$ such that for some $\delta>0$,*

$$\int_{\mathbb{R}} e^{\delta\lambda^2}\,\mathrm{d}\mu_{\mathrm{i}\mathbb{R}}(\mathrm{i}\lambda)<\infty, \tag{35}$$

and let $\mu=\mu_{\mathbb{R}}+\mu_{\mathrm{i}\mathbb{R}}$. Then $\mathcal{E}(a)$ is dense in $L^2(\mu)$.

Proof: Suppose that $\mathcal{E}(a)$ is not dense in $L^2(\mu)$, and denote $X=\mathrm{clos}_{L^2(\mu)}\,\mathcal{E}(a)$. We will need a lemma which is a version of a classical result of M. Riesz and Mergelyan pertaining to the weighted polynomial approximation.

Lemma 5.3. *The elements $f\in X$ extend analytically to $\mathbb{C}$ with the following estimate: for each $\varepsilon>0$, there exists a constant C_ε (independent of f and z) such that*

$$|f(z)|\leqslant C_\varepsilon e^{\varepsilon|z|^2}\|f\|_{L^2(\mu)},\qquad z\in\mathbb{C}\,. \tag{36}$$

Proof: Let $h\in L^2(\mu)\ominus X$, $\|h\|_{L^2(\mu)}=1$, and let

$$H(z)=\int\frac{\overline{h(\lambda)}}{\lambda-z}\,\mathrm{d}\mu(\lambda),\qquad z\in\mathbb{C}\setminus(\mathbb{R}\cup\mathrm{i}\mathbb{R}).$$

Then for every $f\in\mathcal{E}(a)$ we have

$$\int\frac{f(\lambda)-f(z)}{\lambda-z}\overline{h(\lambda)}\,\mathrm{d}\mu(\lambda)=0,\qquad z\in\mathbb{C},$$

and, hence,

$$f(z)=\frac{1}{H(z)}\int\frac{f(\lambda)\overline{h(\lambda)}}{\lambda-z}\,\mathrm{d}\mu(\lambda),\qquad \lambda\in\mathbb{C}\setminus(\mathbb{R}\cup\mathrm{i}\mathbb{R}). \tag{37}$$

Since the function $z\mapsto\int\frac{f(\lambda)\overline{h(\lambda)}}{\lambda-z}\,\mathrm{d}\mu(\lambda)$ is bounded and the function H is of at most linear growth in $\overline{\Omega}$, where $\Omega=\{z\in\mathbb{C}:\mathrm{dist}(z,\mathbb{R}\cup\mathrm{i}\mathbb{R})>1\}$, we obtain that H and, hence, f are in the Nevanlinna class (see [12, Section II.5]) in Ω. The function f is of exponential type in the plane, and we conclude that

$$\log|f(z)| \leqslant \int_{\partial\Omega} \log|f(\lambda)|\,\omega(z, \mathrm{d}\lambda, \Omega),$$

where $\omega(z, E, \Omega)$ is the harmonic measure of $E \subset \partial\Omega$ in Ω with respect to $z \in \Omega$. By (37), $|f| \leqslant \|f\|_{L^2(\mu)}/|H|$ on $\partial\Omega$. Since $\log|H| \in L^1(\omega(z, \mathrm{d}\lambda, \Omega))$, $z \in \Omega$, for every $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$|f(z)| \leqslant c_\varepsilon \|f\|_{L^2(\mu)} e^{\varepsilon|z|^2}, \qquad z \in L \cap \Omega_1,$$

where $L = \{re^{i\theta} : 0 < r < \infty,\ |\sin 2\theta| > 1/2\}$, $\Omega_1 = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{R} \cup \mathrm{i}\mathbb{R}) \geqslant 2\}$.

Once again, since $f \in \mathcal{E}(a)$, the Phragmén-Lindelöf principle applied to f in each of the four sectors of $\mathbb{C} \setminus L$ gives

$$|f(z)| \leqslant c_\varepsilon \|f\|_{L^2(\mu)} e^{\varepsilon|z|^2}, \qquad z \in \mathbb{C}.$$

Passing to the limit, we obtain the same inequality for all $f \in X$. □

We resume the proof of Theorem 5.2, and define a linear operator K on X by the relation

$$\langle Kf, g\rangle_X = \int_{\mathrm{i}\mathbb{R}} f\bar{g}\,\mathrm{d}\mu_{\mathrm{i}\mathbb{R}}.$$

Since the form on the right-hand side is bounded, we see that the operator K is bounded. Furthermore, since

$$\langle (I-K)f, f\rangle_X = \int_{\mathbb{R}} |f|^2\,\mathrm{d}\mu_{\mathbb{R}}\,,$$

we see that $0 \leqslant K \leqslant I$ in the operator sense.

Lemma 5.4. *The operator K is compact.*

Proof: Let $f_n \in X$ tend weakly to 0. By the Banach–Steinhaus uniform boundedness principle, $\sup_n \|f_n\|_{L^2(\mu)} < \infty$. Therefore, by (36), the family of entire functions $\{f_n\}$ is equicontinuous on every compact subset of $\mathbb{C}$. Again applying (36), we see that f_n tend to 0 pointwise on $\mathbb{C}$, and hence, uniformly on compact subsets of $\mathbb{C}$. Furthermore,

$$|\langle Kf_n, g\rangle_X|^2 \leqslant \|g\|^2_{L^2(\mu)} \cdot \int_{\mathbb{R}} |f_n(\mathrm{i}y)|^2\,\mathrm{d}\mu_{\mathrm{i}\mathbb{R}}(\mathrm{i}y)\,.$$

By the dominated convergence theorem (it can be applied due to estimates (36) and (35)), the integral in the right-hand side tends to 0 when $n \to \infty$, whence,

$$\lim_{n\to\infty} \sup_{\|g\|_{L^2(\mu)} \leqslant 1} |\langle Kf_n, g\rangle_X| = 0\,,$$

proving the compactness of the operator K. □

We proceed with the proof of Theorem 5.2, and denote by $\sigma(K)$ the spectrum of the operator K. First, suppose that $1\notin\sigma(K)$. Then the operator $I-K$ is invertible. Therefore, for each $g\in\mathcal{E}(a)\subset X$, we have

$$\begin{aligned}\|g\|^2_{L^2(\mu)} = \|g\|^2_X &= \langle (I-K)^{-1}(I-K)g, g\rangle_X \\ &\leqslant \|(I-K)^{-1}\|\cdot\langle (I-K)g,g\rangle_X = \|(I-K)^{-1}\|\cdot\|g\|^2_{L^2(\mu_{\mathbb{R}})}\,,\end{aligned}$$

and by (36),

$$|g(z)|\leqslant C_\varepsilon e^{\varepsilon|z|^2}\,\|(I-K)^{-1}\|^{1/2}\cdot\|g\|_{L^2(\mu_{\mathbb{R}})}\,,\qquad z\in\mathbb{C}\,.$$

In particular, $\kappa=\inf\bigl\{\|g\|_{L^2(\mu)}:|g(\mathrm{i})|=1\bigr\}>0$. However, this contradicts to the stable density of $\mathcal{E}(a)$ in $L^2(\mu_{\mathbb{R}})$. Indeed, for any function $f\in\mathcal{E}(a)$ with $f(\mathrm{i})=1$, and any $h\in\mathcal{E}(a)$, we have

$$\begin{aligned}\bigl\|h(x)-f(x)(x-\mathrm{i})^{-1}\bigr\|_{L^2(\mu_2)} &\asymp \|h(x)(x-\mathrm{i})-f(x)\|_{L^2(\mu)} \\ &\geqslant \inf\{\|g\|_{L^2(\mu)}:|g(\mathrm{i})|=1\}=\kappa>0\,,\end{aligned}$$

where, as above, $\mathrm{d}\mu_2(\lambda)=(1+|\lambda|)^2\,\mathrm{d}\mu(\lambda)$. Recalling Theorem 2.3, we conclude that $\mathcal{E}(a)$ is not dense in $L^2(\mu_2)$, and hence, is not stably dense in $L^2(\mu)$.

Now, we suppose that $1\in\sigma(K)$. In this case, the operator $I-K$ is not invertible, but its kernel is finite dimensional, and we can modify the previous argument.

Denote $X_0=\ker(I-K)$, $X_1=X\ominus X_0$, $N=\dim(X_0)$, and $K_1=K|\,X_1$. Note that $\|K_1\|<1$. Take a basis $g_1,\dots,g_N$ in X_0, and choose points $x_1,\dots,x_N$ on $\mathbb{R}$ such that the matrix $\bigl[g_j(x_k)\bigr]_{1\leqslant j,k\leqslant N}$ is non-degenerate (this choice is possible due to the linear independence of the functions $g_1,\dots,g_N$). Denote $\nu=\sum_k\delta_{x_k}$. Then $|g(i)|\leqslant c\|g\|_{L^2(\nu)}$, $g\in X_0$.

Next, set $\widetilde{\mu}_{\mathbb{R}}=\mu_{\mathbb{R}}+\nu$. Let $f=f_0+f_1\in X$, $f_0\in X_0$, $f_1\in X_1$, $\|f\|_{L^2(\widetilde{\mu}_{\mathbb{R}})}\leqslant 1$. Since $(I-K)(f_0+f_1)=(I-K_1)f_1$, and

$$\langle (I-K)(f_0+f_1),f_0+f_1\rangle_X=\langle (I-K_1)f_1,f_1\rangle_X,$$

we obtain, as above, that $\|f_1\|_{L^2(\mu)}\leqslant c$. By Lemma 5.3, $\|f_1\|_{L^2(\nu)}\leqslant c_1$ and $|f_1(i)|\leqslant c_2$. Therefore, $\|f_0\|_{L^2(\nu)}\leqslant 1+c_1$, $|f_0(i)|\leqslant c_3$, and $|f(i)|\leqslant c_2+c_3$.

As a result, we obtain that $|f(i)|\leqslant c\|f\|_{L^2(\widetilde{\mu}_{\mathbb{R}})}$ for each $f\in\mathcal{E}(a)$. By Lemma B.4 and Theorem 2.3, this contradicts to the stable density of $\mathcal{E}(a)$ in $L^2(\mu_{\mathbb{R}})$. $\square$

Appendix A. Cases when the type is explicitly computable

There are several cases when the type $T(\mu)$ can be explicitly computed, or at least estimated. Here, we briefly list some of these cases.

A.1. **Measures of zero type.** If the tails of the measure μ decay sufficiently rapidly or if there are large gaps in the support of μ, then the measure has zero type. A useful sufficient condition that deals with these two kinds of behavior is due to de Branges [8, Theorem 63]; an equivalent result was obtained by Beurling, see [15, Section VII.A.2]. *Let* $K\colon \mathbb{R}\to[1,+\infty]$, $\log K$ *be uniformly continuous, and let*

$$\int_{\mathbb{R}}\frac{\log K(t)}{t^2+1}\,\mathrm{d}t=\infty\,.$$

If $\int_{\mathbb{R}} K\,\mathrm{d}\mu<\infty$, *then* $T(\mu)=0$.

A.2. **Measures of infinite type.** A useful sufficient condition for $T(\mu)=\infty$ is due to Krein (and goes back to Szegö). *Suppose that* μ *has a bounded density* μ' *with respect to Lebesgue measure, and that*

$$\int_{\mathbb{R}}\frac{\log \mu'(t)}{t^2+1}\,\mathrm{d}t>-\infty\,.$$

Then $T(\mu)=\infty$.

Another useful result follows from a theorem of Duffin and Schaeffer [9]: if the measure μ is relatively dense with respect to Lebesgue measure, then it must have a positive type. More precisely, *suppose that for some* $L<\infty$ *and* $\delta>0$,

$$\mu[x-L,x+L]\geqslant\delta,\qquad x\in\mathbb{R}.$$

Then $T(\mu)\geqslant\frac{2\pi}{L}$. This can be regarded as a certain stability of the infinite type of Lebesgue measure.

A.3. **Measures of positive type supported by discrete separated sets.** If the measure μ is supported by the set of the integers $\mathbb{Z}$, then the Fourier transforms of the measures $g\,\mathrm{d}\mu$ are 2π-periodic functions, whence $T(\mu)\leqslant\pi$. A result of Koosis [16] yields that *if* $\mu=\sum_{\mathbb{Z}}\omega(n)\delta_n$, *where* δ_n *is the point mass at* n, *and* $\omega\colon\mathbb{Z}\to\mathbb{R}_+$ *is an arbitrary sequence such that* $\sum_{n\in\mathbb{Z}}\frac{\omega(n)}{1+n^2}<\infty$ *and* $\sum_{n\in\mathbb{Z}}\frac{\log\omega(n)}{1+n^2}>-\infty$, *then* $T(\mu)=\pi$. This is a deep result which readily yields one of the equivalent forms of the Beurling–Malliavin multiplier theorem. It is worth mentioning that this fact has a much simpler proof in the case when ω satisfies additionally some regularity assumptions, for instance, if it is an even non-decreasing sequence.

Another deep result is a recent theorem of Mitkovski and Poltoratski [28] which, in its turn, goes back to de Branges. It yields that if $\Lambda=\{\lambda\}\subset\mathbb{R}$ is a separated sequence of points and $\mu=\sum_{\Lambda}\delta_{\lambda_n}$, then $T(\mu)=\pi\mathcal{D}_*(\Lambda)$, where $\mathcal{D}_*(\Lambda)$ is the *lower Beurling–Malliavin density* of the sequence Λ.

Appendix B. Stable and unstable density

Here, we will discuss measures μ such that $\mathcal{E}(a)$ is dense but not stably dense in $L^2(\mu)$. Our discussion is close to the one in [6, Appendix 1] where we dealt with the weighted polynomial approximation.

As above, given a real t, we define the measure μ_t and the weight W_t as follows:

$$\mathrm{d}\mu_t(\lambda) = (1+|\lambda|)^t\, \mathrm{d}\mu(\lambda)\,, \quad W_t(\lambda) = W(\lambda)(1+|\lambda|)^{-t}\,.$$

Lemma B.1. *Let $a > 0$, and let μ be a non-negative measure on $\mathbb{R}$ of at most polynomial growth such that* $\operatorname{supp}\mu$ *does not contain the zero set of any function from the Krein class $\mathcal{K}(a)$. Then $\mathcal{E}(a)$ is stably dense in $L^2(\mu)$.*

Proof: Suppose that for some $t \in \mathbb{R}$, $\mathcal{E}(a)$ is not dense in $L^2(\mu_t)$. For some $N < \infty$, the weight function W defined by

$$W(x) = \begin{cases} (1+|x|)^{-N}, & x \in \operatorname{supp}\mu, \\ +\infty, & x \notin \operatorname{supp}\mu, \end{cases}$$

belongs to $L^2(\mu_t)$. By Bakan's Theorem 2.8, $\mathcal{E}(a)$ is not dense in $C_0(W)$. By de Branges' Theorem 2.10, there exists $f \in \mathcal{K}(a)$ such that

$$\Lambda(f) \subset \{x : W(x) \neq +\infty\} = \operatorname{supp}\mu\,.$$

$\square$

Definition B.2. *Let $a > 0$. We say that the measure μ is a-singular, if there exist real t and s, $t < s$, such that $\mathcal{E}(a)$ is dense in $L^2(\mu_t)$ and is not dense in $L^2(\mu_s)$. Similarly, we say that the weight W is a-singular, if there exist real t and s, $t < s$, such that $\mathcal{E}(a)$ is dense in $C_0(W_t)$ and is not dense in $C_0(W_s)$.*

Lemma B.3.

(I) *Suppose W is an a-singular weight. Then there exists a function B of Krein's class $\mathcal{K}(a)$ such that $\{\lambda\colon W(\lambda) \neq \infty\} = \Lambda(B)$.*

(II) *Suppose μ is an a-singular measure. Then there exists a function B of Krein's class $\mathcal{K}(a)$ such that* $\operatorname{supp}(\mu) = \Lambda(B)$.

Proof of Lemma B.3: The proof will be similar to the previous one.

(I) Let W be an a-singular weight. Since the weights W and W_t are finite on the same set of points, we may assume that $\mathcal{E}(a)$ is dense in $C_0(W)$ and is not dense in $C_0(W_1)$. Then by de Branges' Theorem 2.10, there is a function B in $\mathcal{K}(a)$ such that

$$\sum_{\lambda\in\Lambda(B)} \frac{W(\lambda)}{(1+|\lambda|)|B'(\lambda)|} < \infty\,.$$

Clearly, $\{\lambda\colon W(\lambda) \neq \infty\} \supset \Lambda(B)$. Suppose that $\{\lambda\colon W(\lambda) \neq \infty\} \supsetneqq \Lambda(B)$, take a point $\lambda_0 \in \{\lambda\colon W(\lambda) \neq \infty\} \setminus \Lambda(B)$, and consider the entire function

$B_1(z) = (z-\lambda_0)B(z)$. Clearly, this is again a function of Krein's class $\mathcal{K}(a)$, $\Lambda(B_1) = \Lambda(B) \cup \{\lambda_0\}$, and for $\lambda \in \Lambda(B)$, we have $|B_1'(\lambda)| = |\lambda - \lambda_0||B'(\lambda)|$. Therefore,

$$\sum_{\lambda\in\Lambda(B_1)} \frac{W(\lambda)}{|B_1'(\lambda)|} < \infty\,,$$

and by the other half of de Branges' Theorem, $\mathcal{E}(a)$ is dense in $C_0(W)$, which contradicts our assumption.

(II) Let μ be an a-singular measure. As above, we assume that $\mathcal{E}(a)$ is dense in $L^2(\mu)$ and is not dense in $L^2(\mu_1)$. By Bakan's theorem 2.8, there exists an a-singular weight $W \in L^2(\mu)$. Therefore, applying the first part of the lemma, we see that the support Λ of the measure μ is contained in the (discrete) set $\{\lambda\colon W(\lambda) \ne \infty\}$ which coincide with the zero set of a function of Krein's class. In particular,

$$\sum_{\lambda\in\Lambda} \frac{1}{(1+|\lambda|)^2} < \infty\,.$$

Introduce an auxiliary weight V,

$$V(\lambda) = \begin{cases} \mu\{\lambda\}^{-1/2}, & \lambda\in\Lambda, \\ \infty, & \text{otherwise}\,. \end{cases}$$

Then for every function φ with a compact support,

$$\|\varphi\|^2_{C_0(V)} = \max_{\lambda\in\Lambda} |\varphi(\lambda)|^2 \mu\{\lambda\} \leqslant \sum_{\lambda\in\Lambda} |\varphi(\lambda)|^2\mu\{\lambda\} = \|\varphi\|^2_{L^2(\mu)},$$

$$\|\varphi\|^2_{L^2(\mu_1)} \leqslant \max_{\lambda\in\Lambda} |\varphi(\lambda)|^2(1+|\lambda|)^3\mu\{\lambda\}\cdot \sum_{\lambda\in\Lambda}\frac{1}{(1+|\lambda|)^2} \leqslant C(\Lambda)\|\varphi\|^2_{C_0(V_{3/2})}\,.$$

Thus, the weight V is a-singular, and the first part of the lemma completes the proof. □

We also use the following lemma:

Lemma B.4. *Suppose that $\mathcal{E}(a)$ is stably dense in $L^2(\mu)$, $\lambda_k\in\mathbb{R}$, $1\leqslant k\leqslant N$, and let $\widetilde{\mu} = \mu + \sum_{1\leqslant k\leqslant N} \delta_{\lambda_k}$. Then $\mathcal{E}(a)$ is dense in $L^2(\widetilde{\mu})$.*

Proof: By Bakan's Theorem, there exists a weight $W\in L^2(\mu)$ such that $\mathcal{E}(a)$ is stably dense in $C_0(W)$. Let $\widetilde{W}\leqslant W$ be a (lower semi-continuous) weight such that $W(\lambda_k) < +\infty$, $1\leqslant k\leqslant N$, $\widetilde{W} = W$ outside of a finite interval I.

We claim that $\mathcal{E}(a)$ is dense in $C_0(\widetilde{W})$. Otherwise, by de Branges' Theorem 2.10, we find a Krein class function $\widetilde{B}$ such that

$$\sum_{\lambda\in\Lambda(\widetilde{B})} \frac{\widetilde{W}(\lambda)}{|\widetilde{B}'(\lambda)|} < \infty .$$

Set $\Lambda = \Lambda(\widetilde{B}) \cap I$, and let

$$B(z) \stackrel{\rm def}{=} \frac{\widetilde{B}(z)}{\prod\limits_{\lambda\in\Lambda} (z-\lambda)} .$$

Clearly, B is a Krein class function, and for some $s < \infty$, we have $|B'(\lambda)| \asymp (1+|\lambda|)^{-s}|\widetilde{B}'(\lambda)|$ for $\lambda \in \Lambda(B)$. Thereby,

$$\sum_{\lambda\in\Lambda(B)} \frac{W(\lambda)}{(1+|\lambda|)^s|B'(\lambda)|} < \infty ,$$

which by de Branges' theorem contradicts to the stable density of $\mathcal{E}(a)$ in $C_0(W)$.

Thus, $\mathcal{E}(a)$ is dense in $C_0(\widetilde{W})$. Since $\widetilde{W} \in L^2(\widetilde{\mu})$, using again Bakan's Theorem, we conclude that $\mathcal{E}(a)$ is dense in $L^2(\widetilde{\mu})$, completing the proof. $\Box$

It is not difficult to prove the inverse statement: *if for any points* $\lambda_1, \dots, \lambda_N \in \mathbb{R}$, $\mathcal{E}(a)$ *is dense in* $L^2(\widetilde{\mu})$, *where* $\widetilde{\mu} = \mu + \sum_k \delta_{\lambda_k}$, *then* $\mathcal{E}(a)$ *is stably dense in* $L^2(\mu)$.

Appendix C. Nazarov's construction of spectral measures

We describe an elegant Nazarov's construction of a wide class of spectral measures supported by $\mathbb{R}$. This construction is based on a distorted Poisson formula.

Definition C.1. *Denote by* Γ *the set of* C^∞*-diffeomorphisms* X *of* $\mathbb{R}$ *satisfying the following three conditions:*

(I) $X'(t) \to 1$ *as* $|t| \to \infty$*;*

(II) *There exists a sequence of (strictly) positive numbers* s_k, $\lim\limits_{k\to\infty} s_k = \infty$, *such that for every* $k \geqslant 2$, $X^{(k)}(t) = O(|t|^{-s_k})$ *as* $|t| \to \infty$.

Given a diffeomorphism $X \in \Gamma$ and given $c > 0$, we define the measure

$$\mu = \mu_X = \sum_{k\in\mathbb{Z}} X'(ck)\delta_{X(ck)} = \sum_{\lambda\colon Y(\lambda)\in c\mathbb{Z}} \frac{\delta_\lambda}{Y'(\lambda)} . \tag{38}$$

Here and below, $Y = X^{-1}$ is the inverse diffeomorphism. By $\widehat{\mu}$ we denote the distributional Fourier transform of μ, i.e., $\langle \widehat{\mu}, \varphi\rangle = \langle \mu, \widehat{\varphi}\rangle$; as above,

$$\widehat{\varphi}(\lambda) = \int_{\mathbb{R}} \varphi(x) e^{-\mathrm{i}\lambda x}\, \mathrm{d}\lambda\,.$$

Theorem C.2. *There exists a function $M \in C^\infty(-2\pi c^{-1}, 2\pi c^{-1})$ such that $\widehat{\mu} = \delta_0 + M$ on $(-2\pi c^{-1}, 2\pi c^{-1})$.*

We start with an obvious claim:

Claim C.3. *Let s_k, $k \geqslant k_0$, be any sequence of (strictly) positive numbers satisfying $\lim\limits_{k\to\infty} s_k = \infty$. Then the sequence*

$$S_m = \inf\Big\{ \sum_j s_{k_j} \colon\ \sum_j k_j \geqslant m \Big\}$$

also satisfies $\lim\limits_{m\to\infty} S_m = \infty$.

Next, we prove

Claim C.4. *If $X \in \Gamma$, then $Y = X^{-1} \in \Gamma$.*

Proof: Condition (I) implies that $X(t) \asymp t$ for large t and, thereby, $Y(t) \asymp t$ for large t as well. Then condition (I) for Y follows immediately from the identity $Y' = \frac{1}{X'\circ Y}$. Differentiating this identity $k-1$ times, we conclude that $Y^{(k)}$ is a finite linear combination of terms of the kind

$$(X'\circ Y)^{-m} \prod_j X^{(k_j)} \circ Y, \quad \text{with } m, k_j \geqslant 1, \quad \sum_j (k_j - 1) = k-1.$$

Now applying Claim C.3 to the sequence $\{s_{k+1}\}_{k\geqslant 1}$, we conclude that each such term is $O(|t|^{-S_k})$ where $\{S_k\}_{k\geqslant 2}$ is a sequence of positive numbers tending to infinity. □

Proof of Theorem C.2: It suffices to deal only with the case $c = 1$ (otherwise, just replace X by $t \mapsto c^{-1}X(c\cdot t)$). We fix any $0 < a < b < 2\pi$, and take a function φ in the Schwartz class $\mathcal{S}$ such that $\widehat{\varphi} = 1$ on $(-a, a)$ and $\widehat{\varphi} = 0$ outside $(-b, b)$. Clearly, $\widehat{\mu} = \widehat{\mu}\widehat{\varphi} = \widehat{\mu * \varphi}$ on $(-a, a)$. Taking into account that the point mass at the origin is the Fourier transform of the constant $\frac{1}{2\pi}$, it remains to prove that $(\mu * \varphi)(t) - \frac{1}{2\pi}$ decays faster than any power of $|t|$ as $|t| \to \infty$.

To this end, fix large $t \in \mathbb{R}$ and consider the function $\psi(s) = \varphi(t - X(s))X'(s)$. Since all the derivatives of X are bounded and $X(s) \asymp s$ for large s, the function ψ belongs to $\mathcal{S}$. Moreover, we have $\|\psi^{(k)}\|_{L^1} \leqslant C(k, X, \varphi)$ independently of t, which allows us to conclude that $|\widehat{\psi}(\ell)| \leqslant C(N, X, \varphi)(1 + |\ell|)^{-N}$ for any $N > 0$, independently of t and ℓ.

Next,

$$(\mu * \varphi)(t) = \sum_{k\in\mathbb{Z}} X'(k)\varphi(t - X(k)) = \sum_{k\in\mathbb{Z}} \psi(k) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \widehat{\psi}\,(2\pi k)\ .$$

Since

$$\widehat{\psi}(0) = \int_{\mathbb{R}} \psi = \int_{\mathbb{R}} \varphi(t - X(s))X'(s)\,\mathrm{d}s = \int_{\mathbb{R}} \varphi = \widehat{\varphi}(0) = 1\,,$$

we need only to prove that $\sum\limits_{\ell\in\mathbb{Z}\setminus\{0\}} |\widehat{\psi}(2\pi\ell)|$ decays faster than any power of $|t|$ as $|t| \to \infty$.

Note that if X is the identical map, this property holds because the support of $\widehat{\psi}$ is contained in $(-b, b)$ with $b < 2\pi$. We need to show that this property is preserved under distortion of φ by any diffeomorphism X which is asymptotically close to the identity.

We put $Y = X^{-1}$, $R(t) = Y(t) - t$, and write

$$\begin{aligned}
\widehat{\psi}(2\pi\ell) &= \int_{\mathbb{R}} \varphi(t - X(s))X'(s)e^{-2\pi\mathrm{i}\ell s}\,\mathrm{d}s \\
&= \int_{\mathbb{R}} \varphi(t - s)e^{-2\pi\mathrm{i}\ell Y(s)}\,\mathrm{d}s = \int_{\mathbb{R}} \varphi(s)e^{-2\pi\mathrm{i}\ell Y(t-s)}\,\mathrm{d}s \\
&= e^{-2\pi\mathrm{i}\ell R(t) - 2\pi\mathrm{i}\ell t}\int_{\mathbb{R}} \varphi(s)e^{2\pi\mathrm{i}\ell s + 2\pi\mathrm{i}\ell[R(t) - R(t-s)]}\,\mathrm{d}s\,.
\end{aligned}$$

Claim C.5. *There exists $0 < \delta < \frac{1}{4}$ such that for large N,*

$$\int_{\mathbb{R}} \varphi(s)e^{2\pi\mathrm{i}\ell s + 2\pi\mathrm{i}\ell[R(t) - R(t-s)]}\,\mathrm{d}s = O\big(|t|^{-N}\big), \qquad |t| \to \infty,$$

uniformly in $1 \leqslant \ell \leqslant |t|^{\delta}$.

Since $\sum\limits_{\ell\colon\, |\ell| > |t|^{\delta}} |\widehat{\psi}(2\pi\ell)|$ decays faster than any power of $|t|$ as $|t| \to \infty$, proving the claim, we prove the theorem.

Proof of Claim C.5: Fix $N \in (1, \infty)$ and write the Taylor formula:

$$\begin{aligned}
Q(t, s) &\overset{\text{def}}{=} R(t) - sR'(t) - R(t - s) \\
&= \sum_{k=2}^{p-1} a_k(t)s^k + O(a_p(t)s^p), \qquad |t| \to \infty,\ |s| < |t|/2.
\end{aligned}$$

By Claim C.4 and the definition of the class Γ, we can find $0 < \delta < \frac{1}{4}$, $p < \infty$, and $C < \infty$ such that for large $|t|$,

$$\begin{aligned}
|a_k(t)| &\leqslant C|t|^{-4\delta}, \qquad 2 \leqslant k \leqslant p - 1, \\
|a_p(t)| &\leqslant C|t|^{-2N}.
\end{aligned} \tag{39}$$

We fix $0<\varepsilon<\frac{\delta}{p}$, and set

$$A(t,s)=\sum_{k=2}^{p-1}a_k(t)s^k.$$

Then for $|s|<|t|^\varepsilon$, $|t|>1$, we have

$$|Q(t,s)-A(t,s)|\leqslant c|t|^{-2N+\delta}, \tag{40}$$

$$|A(t,s)|\leqslant c|t|^{-3\delta}. \tag{41}$$

Furthermore,

$$\int_{\mathbb{R}}\varphi(s)e^{-2\pi \mathrm{i} su}\,\mathrm{d}s=0, \qquad |u|\geqslant b\,,$$

and hence for large $|t|$ we have

$$\int_{\mathbb{R}}s^k\varphi(s)e^{2\pi\mathrm{i}s\ell(1+R'(t))}\,\mathrm{d}s=0, \qquad \ell\in\mathbb{Z}\setminus\{0\},\ k\geqslant 0\,.$$

Since $\varphi\in\mathcal{S}$, we have

$$|\varphi(t)|=O(t^{-1-3N/\varepsilon}), \qquad |t|\to\infty, \tag{42}$$

whence,

$$\int_{|s|<|t|^\varepsilon}s^k\varphi(s)e^{2\pi\mathrm{i}s\ell(1+R'(t))}\,\mathrm{d}s=O(t^{-2N}), \qquad |t|\to\infty\,, \tag{43}$$

uniformly in $0\leqslant k\leqslant pN/\delta$.

Using (42) once more, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\varphi(s)e^{2\pi\mathrm{i}\ell s+2\pi\mathrm{i}\ell[R(t)-R(t-s)]}\,\mathrm{d}s\\
&\qquad=\int_{|s|<|t|^\varepsilon}\varphi(s)e^{2\pi\mathrm{i}\ell s+2\pi\mathrm{i}\ell[R(t)-R(t-s)]}\,\mathrm{d}s+O\big(|t|^{-N}\big)\\
&\qquad=\int_{|s|<|t|^\varepsilon}\varphi(s)e^{2\pi\mathrm{i}\ell s(1+R'(t))}e^{2\pi\mathrm{i}\ell Q(t,s)}\,\mathrm{d}s+O\big(|t|^{-N}\big), \qquad |t|\to\infty.
\end{aligned}$$

Next, choosing an integer q, $\frac{N+1}{2\delta}<q<\frac{N}{\delta}$, we expand

$$\begin{aligned}
e^{2\pi\mathrm{i}\ell Q}&=\sum_{j=0}^{q-1}\frac{(2\pi\mathrm{i}\ell Q)^j}{j!}+O\big(|\ell Q|^q\big)=\sum_{j=0}^{q-1}\frac{(2\pi\mathrm{i}\ell\,[A+(Q-A)])^j}{j!}+O\big(|\ell Q|^q\big)\\
&=\sum_{j=0}^{q-1}\frac{(2\pi\mathrm{i}\ell)^j}{j!}A^j+\sum_{j=1}^{q-1}O\Big(|\ell|^j\sum_{m=0}^{j-1}|A|^m|Q-A|^{j-m}\Big)+O\big(|\ell Q|^q\big)
\end{aligned}$$

for $|t| \to \infty$, uniformly in $|s| \leqslant |t|^\varepsilon$ and $|\ell| \leqslant |t|^\delta$. Using estimates (40)-(41), and recalling that $N > 4\delta$, and that $\frac{1}{2}(N+1) < \delta q < N$, we get

$$e^{2\pi \mathrm{i}\ell Q} = \sum_{j=0}^{q-1} \frac{(2\pi \mathrm{i}\ell)^j}{j!} A^j + O\bigl(|t|^{-N-2\delta}\bigr), \qquad |t| \to \infty\,,$$

again, uniformly in $|s| \leqslant |t|^\varepsilon$ and $|\ell| \leqslant |t|^\delta$. Then we substitute this expansion into the integral we are estimating:

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(s) e^{2\pi \mathrm{i}\ell s + 2\pi \mathrm{i}\ell[R(t)-R(t-s)]}\, \mathrm{d}s \\
&\qquad = \sum_{j=0}^{q-1} \frac{(2\pi \mathrm{i}\ell)^j}{j!} \int_{|s|<|t|^\varepsilon} \varphi(s) A(t,s)^j e^{2\pi \mathrm{i}\ell s(1+R'(t))}\, \mathrm{d}s + O\bigl(|t|^{-N}\bigr) \\
&\qquad\qquad \overset{(43)}{=} \sum_{j=1}^{q-1} \frac{(2\pi \mathrm{i}\ell)^j}{j!} \int_{|s|<|t|^\varepsilon} \varphi(s) A(t,s)^j e^{2\pi \mathrm{i}\ell s(1+R'(t))}\, \mathrm{d}s + O\bigl(|t|^{-N}\bigr)
\end{aligned}$$

for $|t| \to \infty$, uniformly in $1 \leqslant |\ell| \leqslant |t|^\delta$. At last, note that the functions $s \mapsto A(t,s)^j$ are polynomials of degree at most $(p-1)(q-1)$ which is less than pN/δ by the choice of q. Hence, integrating these polynomials we can apply estimates (43). By (39), the coefficients of these polynomials are bounded by $O(|t|^{-4\delta j})$. Hence,

$$\begin{aligned}
&\sum_{j=1}^{q-1} \frac{(2\pi \mathrm{i}\ell)^j}{j!} \int_{|s|<|t|^\varepsilon} \varphi(s) A(t,s)^j e^{2\pi \mathrm{i}\ell s(1+R'(t))}\, \mathrm{d}s \\
&\qquad = O\bigl(\sum_{j=1}^{q-1} |\ell|^j |t|^{-4\delta j}\, |t|^{-2N}\bigr) = O(|t|^{-N}), \qquad |t| \to \infty\,,
\end{aligned}$$

uniformly in $1 \leqslant |\ell| \leqslant |t|^\delta$. This proves the claim and finishes off the proof of Theorem C.2. ∎

Appendix D. Stably orthogonal spectral measures

In this appendix, we show that Nazarov's construction of spectral measures described in Appendix C gives a wide class of explicitly defined discrete stably orthogonal spectral measures. Let us take a diffeomorphism X satisfying Definition C.1. Suppose that X is an odd function. Then the measure μ defined in (38) is symmetric with respect to the origin, and hence, is a spectral measure of a Sturm–Liouville problem with a potential $q \in C^\infty[0, \pi c^{-1})$. Now, we show that if

$$\lim_{t\to+\infty} \bigl(X(t) - t\bigr) = +\infty\,, \tag{44}$$

then the spectral measure μ is stably orthogonal.

For a discrete set $\Lambda \subset \mathbb{R}$, we denote

$$n_\Lambda(t) = \mathrm{Card}\{\Lambda \cap [-t,t]\}, \qquad N_\Lambda(R) = \int_1^R \frac{n_\Lambda(t)}{t}\,\mathrm{d}t\,.$$

Let $\Lambda = X(c\mathbb{Z})$. Then by condition (44),

$$\lim_{t\to\infty}\bigl(n_\Lambda(t) - 2tc^{-1}\bigr) = -\infty\,. \tag{45}$$

By Lemma B.1, the following lemma does the job.

Lemma D.1. *Suppose that the set Λ satisfies condition* (45). *Then Λ cannot contain the zero set of any function $F \in \mathcal{K}(\pi c^{-1})$.*

Proof: Suppose that for some function $F \in \mathcal{K}(\pi c^{-1})$, we have $\Lambda(F) \subset \Lambda$, where $\Lambda(F)$ is the zero set of F. Take a polynomial P with real coefficients and with simple zeros in $\mathbb{R}\setminus\Lambda(F)$. Then the entire function $G = P\cdot F$ again belongs to the Krein class $\mathcal{K}(\pi c^{-1})$, and $|G'(\lambda)| \geqslant c(1+|\lambda|^2)$, provided that the degree of P is big enough.

Take any $b < \pi c^{-1}$, and write the Lagrange interpolation formula,

$$\frac{\sin(bz)}{G(z)} = \sum_{\Lambda(G)} \frac{\sin(b\lambda)}{G'(\lambda)(z-\lambda)}\,.$$

Then

$$|\sin(bz)| \leqslant \frac{|G(z)|}{|\operatorname{Im} z|}\sum_{\Lambda(G)}\frac{1}{1+|\lambda|^2} = C\frac{|G(z)|}{|\operatorname{Im} z|}\,.$$

Since the constant C does not depend on the choice of $b < \pi c^{-1}$, letting $b \to \pi c^{-1}$, we obtain $|\sin(\pi c^{-1}z)| \leqslant C|\operatorname{Im} z|^{-1}\,|G(z)|$, and then

$$\begin{aligned}\log|\sin(\pi c^{-1}z)| &\leqslant \log|G(z)| + \log\frac{1}{|\operatorname{Im} z|} + C' \\ &= \log|F(z)| + \log|P(z)| + \log\frac{1}{|\operatorname{Im} z|} + C'\,.\end{aligned}$$

Letting $z = Re^{\mathrm{i}\theta}$, integrating over θ, and using Jensen's formula, we get

$$\begin{aligned}2c^{-1}R &= \frac{1}{2\pi}\int_0^{2\pi}\log|\sin(\pi c^{-1}Re^{\mathrm{i}\theta})|\,\mathrm{d}\theta + O(\log R) \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|F(Re^{\mathrm{i}\theta})|\,\mathrm{d}\theta + O(\log R) \\ &= N_{\Lambda(F)}(R) + O(\log R) \leqslant N_\Lambda(R) + O(\log R)\,, \qquad R\to\infty\,.\end{aligned}$$

It remains to note that, by our assumption (45), for each $A < \infty$, we have

$$\lim_{R\to\infty}\bigl(N_\Lambda(R) - 2c^{-1}R + A\log R\bigr) = -\infty\,.$$

We arrive at a contradiction that proves the lemma. □

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