Completeness and Riesz Bases of Reproducing Kernels in Model Subspaces

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Motivation

Notation

- $\mathbb{C}^+ =$ Upper Half Plane
- $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+ \cup \mathbb{R}$ and $E_\Lambda = \{e^{2\pi i \lambda_n t}\}$
- $H^2 := \{f \in \text{hol}(\mathbb{C}^+) : \|f\|_2^2 := \sup_{y>0} \int |f(x+iy)|^2 dx < \infty\}.$
- $H^\infty := \{f \in \text{hol}(\mathbb{C}^+) : \|f\|_\infty := \sup_{z \in \mathbb{C}^+} |f(z)| < \infty\}.$

When is E_Λ complete in $L^2([0,a])$?

Example 1. (Payley Wiener, 1935)

If $\Lambda \subseteq \mathbb{R}$,then E_Λ is complete in $L^2([0,a])$ if

$$\limsup_{x\to\infty} \frac{\#\left(\Lambda \cap (0,x)\right)}{x} > a$$

Example 2. (Beurling Malliavin, 1961)

Set $R(\Lambda) = \sup\{a : E_\Lambda \text{ is complete in } L^2([0,a])\}$. Then

$$R(\Lambda) = \text{ Exterior BM Density of } \Lambda$$

Rephrase the Setup

Definition 1. The **Payley Wiener Space** PW_a is the space of entire functions belonging to $L^2(\mathbb{R})$ of exponential type at most $2\pi a$, i.e.

$$|f(z)| \leq C \exp(2\pi a|z|)$$

Then $PW_a = \mathcal{F}\left(L^2([-a,a])\right)$.

Definition 2. Consider $\Phi := \exp(2\pi i a z)$. The **Model Subspace associated to** Φ is

$$K_\Phi := H^2 \ominus \Phi H^2$$

Then $K_\Phi = \exp(i\pi a z) PW_a$ and K_Φ is a reproducing kernel Hilbert space, so for every $z \in \mathbb{C}^+ \cup \mathbb{R}$, there is a function $k_z \in K_\Phi$ such that

$$\langle f, k_z \rangle_{K_\Phi} = f(z) \qquad \forall\ f \in K_\Phi.$$

Combining these Facts:

E_Λ is not complete in $L^2([-a,a])$ iff $\exists\ f \in L^2([-a,a])$ such that $f \perp e^{2\pi i \lambda_n t}\ \forall n$.

iff $\exists\ f \in PW_a$ such that $f(\lambda_n) = 0$ for all n

iff $\exists\ f \in K_\Phi$ such that $f(\lambda_n) = 0$ for all n

iff $\{k_{\lambda_n}\}$ is not complete in K_Φ.

Review of Model Subspaces

Definition 1. Φ is called **inner** if $\Phi \in H^\infty$ and

$$\lim_{y \searrow 0} |\Phi(x+iy)| = 1 \quad a.e. \text{ on } \mathbb{R}.$$

Definition 2. The **Model Subspace associated to** Φ is

$$K_\Phi := H^2 \ominus \Phi H^2 = H^2 \cap \Phi \overline{H^2}.$$

Then K_Φ is a reproducing kernel Hilbert space with reproducing kernels given by

$$k_w(z) := \frac{i}{2\pi} \frac{1 - \overline{\Phi(w)}\Phi(z)}{z - \bar{w}} \qquad \forall w \in \mathbb{C}^+.$$

Definition 3. Φ is **meromorphic inner** if Φ extends merimorphically to $\mathbb{C}$. Then

$$\Phi(z) = \exp(iaz)\, B(z)$$

where $a \in \mathbb{R}$ and $B(z)$ is a Blaschke product whose zeros don't accumulate on $\mathbb{R}$.

Question: For Φ inner and $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+ \cup \mathbb{R}$, when is $\mathcal{K}(\Lambda) := \{k_{\lambda_n}\}$ complete in K_Φ?

Summary of Results

(1) **When is $\mathcal{K}(\Lambda)$ complete in K_Φ?**

- Argument Criterion for Completeness
- Application: Stability of Completeness
  - Perturbing Λ
  - Perturbing Φ
- Density Criterion for Completeness (Meromorphic only)

(2) **When is $\mathcal{K}(\Lambda)$ a Riesz basis for K_Φ?**

- Definition: $\{h_n\}$ is a **Riesz basis** for H if $H = \overline{\text{Span}_n h_n}$ and there are $A, B > 0$ such that for every finite sum:

$$A \sum_n |c_n|^2 \leq \left\| \sum_n c_n h_n \right\|_H^2 \leq B \sum_n |c_n|^2$$

- Uses connections between entire functions and meromorphic inner functions

Argument Criterion for Completeness 1

- $\arg F$ denotes the main branch of the argument of F
- Π = Poisson measure on $\mathbb{R}$. Specifically

$$g \in L^1(\Pi) \quad \text{if} \quad \int \frac{|g(t)|}{1+t^2} dt < \infty.$$

- For $g \in L^1(\Pi)$ the **Hilbert Transform** is given by:

$$\tilde{g}(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-t|>\epsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) g(t) dt$$

Theorem 1: Points in $\mathbb{C}^+$

Let $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$. Then $\mathcal{K}(\Lambda)$ is *not complete* in K_Φ if and only if $\exists$

- a nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$
- a measurable $\mathbb{Z}$-valued function k and a $\gamma \in \mathbb{R}$

such that

$$\arg \Phi - \arg B_\Lambda = 2\widetilde{\log m} + 2\pi k + \gamma \quad a.e. \text{ on } \mathbb{R},$$

where B_Λ is the Blaschke product with zeros $\{\lambda_n\}$.

Argument Functions & Hilbert Transforms

- If O is outer in H^2, there is a nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$ such that
$$O(t) = \exp\left(\log m(t) + i\widetilde{\log m(t)}\right)$$
Actually m can be chosen to be $|O|$. Then, there is a $\mathbb{Z}$-valued k such that
$$\arg O = \widetilde{\log|O|} + 2\pi k \quad a.e. \text{ on } \mathbb{R}.$$
- If I is inner, there is a nonnegative $m_1 \in L^\infty(\mathbb{R})$ with $\log m_1 \in L^1(\Pi)$ and a $\mathbb{Z}$-valued k such that
$$\arg I = \widetilde{\log m_1} + 2\pi k + \pi \quad a.e. \text{ on } \mathbb{R}$$
and m_1 can be taken to be $|1 - I|$.

$\therefore$ If $f \in H^2$ then $f(z) = O(z)I(z)$ and so
$$\arg f = \widetilde{\log m} + 2\pi k + \widetilde{\log m_1} + \pi = \widetilde{\log m_1 m} + 2\pi k + \pi,$$
where $m_1 m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$ and k is measurable and $\mathbb{Z}$-valued.

Mainly Increasing Functions

A C^1 function f on $\mathbb{R}$ is **mainly increasing** if there is an increasing sequence $\{d_n\} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} |d_n| = \infty$ and

- $f(d_{n+1}) - f(d_n) \approx 1$
- There is a constant C such that

$$\sup_{s,t\in(d_n,d_{n+1})} |f(s) - f(t)| \leq C \quad \forall n$$

$$\sup_{s,t\in(d_n,d_{n+1})} |f'(s) - f'(t)| \leq C \quad \forall\ n.$$

Every mainly increasing function is of the form $2\widetilde{\log m} + 2\pi k$ for some nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$ and measurable $\mathbb{Z}$-valued k.

Argument Criterion for Completeness 2: Preliminaries

Let $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$ and $T = \{t_n\} \subseteq \mathbb{R}$.

Goal: Study the Completeness of $\mathcal{K}(\Lambda)$ and $\mathcal{K}(T)$ in K_Φ.

- Assume Φ is analytic in a neighborhood of each t_n.
- Construct inner function J with $\{J = 1\} = T$.

Step 1. Pick ν a Poisson-finite, positive measure supported on T.

$$\nu = \sum_n \nu_n \delta_{t_n} \quad \text{where} \quad \nu_n > 0 \quad \text{and} \quad \sum_n \frac{\nu_n}{1+t_n^2} < \infty$$

Step 2. Construct a meromorphic Herglotz function using ν.

$$G(z) = \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\nu = \sum_n \nu_n \left(\frac{1}{t_n - z} - \frac{t_n}{1+t_n^2} \right)$$

Step 3. Construct meromorphic inner function J using G as follows

$$J(z) = \frac{G(z) - i}{G(z) + i}$$

Then $\{J = 1\} = T$ and ν is the Clark measure of J.

Argument Criterion for Completeness 2

Theorem 2: Points in $\mathbb{C}^+$ and $\mathbb{R}$

Let $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$ and $T = \{t_n\} \subseteq \mathbb{R}$. Let Φ be an inner function analytic in a neighborhood of each t_n. Then:

$\mathcal{K}(\Lambda) \cup \mathcal{K}(T)$ is *not complete* in K_Φ if and only if there exist

- an inner function J with $\{J = 1\} = T$
- a nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$
- a measurable $\mathbb{Z}$-valued function k and a $\gamma \in \mathbb{R}$

such that

$$\arg \Phi - \arg B_\Lambda - \arg J = 2\widetilde{\log m} + 2\pi k + \gamma \quad a.e. \text{ on } \mathbb{R},$$

where B_Λ is the Blaschke product with zeros $\{\lambda_n\}$.

Future Use: Let $T \subseteq \mathbb{R}$ and Φ be meromorphic inner. If we find an inner J s.t.

$$\{J = 1\} = T \quad \text{and} \quad \arg \Phi - \arg J \text{ mainly increasing}$$

then $\mathcal{K}(T)$ is not complete in K_Φ.

Application: Perturbing Λ

Corollary 1

Let $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$ and $M = \{\mu_n\} \subseteq \mathbb{C}^+$. Assume $\mathcal{K}(M)$ is complete in K_Φ.

If for some choice of arguments ψ_Λ of B_Λ and ψ_M of B_M, $(\psi_\Lambda - \psi_M) \in L^1(\Pi)$ and

$$(\psi_\Lambda - \psi_M)^\sim \in L^\infty(\mathbb{R})$$

then $\mathcal{K}(\Lambda)$ is also complete in K_Φ.

Proof: Assume $\mathcal{K}(\Lambda)$ is not complete. Then

$$\arg \Phi - \psi_\Lambda = 2\widetilde{\log m} + 2\pi k + \gamma.$$

The assumption $(\psi_\Lambda - \psi_M)^\sim = u \in L^\infty(\mathbb{R})$ implies

$$\psi_\Lambda - \psi_M = -\tilde{u} = 2\widetilde{\log m_1},$$

for $m_1 = e^{-u/2}$ and $u \in L^\infty \Rightarrow m_1 \in L^\infty$ and clearly $\log m_1 \in L^1(\Pi)$. So:

$$\arg \Phi - \psi_M = \arg \Phi - \psi_\Lambda + (\psi_\Lambda - \psi_M) = 2\widetilde{\log m_1 m} + 2\pi k + \gamma,$$

a contradiction. □

Application 2: Perturbing Φ

Alternate Assumption $\mathcal{K}(\Lambda)$ complete $\Rightarrow \mathcal{K}(M)$ complete if

$$R(t) := \sum_n \left| \frac{\lambda_n - \mu_n}{t - \mu_n} \right| \in L^\infty(\mathbb{R})$$

Now: Fix $\Lambda \subseteq \mathbb{C}^+$ and assume Φ and Φ^o are both inner functions. Then

$\mathcal{K}(\Lambda)$ = The set of reproducing kernels of K_Φ corresponding to Λ
$\mathcal{K}^o(\Lambda)$ = the set of reproducing kernels of K_{Φ^o} corresponding to Λ

Corollary 2

Let Φ and Φ^o be inner functions such that for a certain choice of their arguments ψ and ψ^o, $(\psi - \psi^o) \in L^1(\Pi)$ and

$$(\psi - \psi^o)^\sim \in L^\infty(\mathbb{R})$$

Then $\mathcal{K}(\Lambda)$ is complete in K_Φ if and only if $\mathcal{K}^o(\Lambda)$ is complete in K_{Φ^o}.

Proof: The assumption implies: $\psi - \psi^o = 2\widetilde{\log m}$ for a nonnegative $m \in L^2(\mathbb{R})$ with $\log m \in L^1(\Pi)$. Just apply Theorem 2.

Clark Points

For Φ meromorphic inner, there is an increasing, smooth function $\psi(t)$ such that

$$\Phi(t) = \exp(i\psi(t)) \quad \text{on} \quad \mathbb{R}.$$

Definition 1. Assume $\Phi - 1 \notin L^2(\mathbb{R})$ and define the set $S = \{s_n\} \subseteq \mathbb{R}$ by

$$\psi(s_n) = 2\pi n \qquad \forall\ n.$$

The set of reproducing kernels $\{k_{s_n}\}$ is an orthogonal basis for K_Φ and is called a **de Branges-Clark basis.**

The set $\{s_n\}$ is simultaneously "large" and "small:"

(1) $\{s_n\}$ is a set of uniqueness ($\{k_{s_n}\}$ is complete.)

$$f(z) = \sum_n \frac{f(s_n)}{\|k_{s_n}\|_2^2} k_{s_n}(z) = \sum_n \left\langle f, \frac{k_{s_n}}{\|k_{s_n}\|_2} \right\rangle \frac{k_{s_n}(z)}{\|k_{s_n}\|_2}.$$

(2) $\{s_n\}$ is a *complete interpolating set.* I.e, if $\{c_n\}$ satisfies

$$\sum_n \frac{|c_n|^2}{\|k_{s_n}\|_2^2} < \infty \ \Rightarrow \ \exists\ f \in K_\Phi \text{ with } f(s_n) = c_n.$$

Density Criterion: Use Clark Points

Main Idea

- If $T = \{t_n\}$ is *much denser* then $S \Rightarrow \mathcal{K}(T)$ is *complete* in K_Φ.
- If $T = \{t_n\}$ is *much sparser* then $S \Rightarrow \mathcal{K}(T)$ is *not complete* in K_Φ.

Definition 1. Upper and Lower Densities of T of Length r:

$$D_+(T,r) = \sup_n \#\{m : t_m \in [s_n, s_{n+r})\} \text{ and } D_-(T,r) = \inf_n \#\{m : t_m \in [s_n, s_{n+r})\}$$

Definition 2. Upper and Lower Densities of T:

$$D_+(T) = \lim_{r\to\infty} \frac{D_+(T,r)}{r} \quad \text{and} \quad D_-(T) = \lim_{r\to\infty} \frac{D_-(T,r)}{r}$$

Rigorous Idea

- $D_-(T) > 1 \Rightarrow \mathcal{K}(T)$ is *complete* in K_Φ.
- $D_+(T) < 1 \Rightarrow \mathcal{K}(T)$ is *not complete* in K_Φ.

Density Criterion for Completeness

Theorem 3: Density for Points in $\mathbb{R}$

Assume Φ is meromorphic, inner and $\Phi' \in L^\infty(\mathbb{R})$. Further, assume $\{s_n\}$ satisfy

$$\sup_n \left| \sum_{k \neq n} \left(\frac{1}{s_n - s_k} + \frac{s_k}{1 + s_k^2} \right) \right| < \infty$$

Taking D_+, D_-, and T as before:

- $D_-(T) > 1 \Rightarrow \mathcal{K}(T)$ is *complete* in K_Φ.
- $D_+(T) < 1 \Rightarrow \mathcal{K}(T)$ is *not complete* in K_Φ.

$\Phi' \in L^\infty(\mathbb{R})$ implies that $\inf_n (s_{n+1} - s_n) > 0$ since

$$\int_{s_n}^{s_{n+1}} |\Phi'(t)| dt = \int_{s_n}^{s_{n+1}} \psi'(t) dt = 2\pi.$$

The "S" Condition is satisfied if

- $\{s_n\}$ is sufficiently sparse, e.g. $\sup_n \sum_{n \neq k} |s_n - s_k|^{-1} < \infty$.
- $\{s_n\}$ is sufficiently symmetric, e.g. $s_n = 2\pi n / a$

Riesz Basis Criterion: Preliminaries

- $\Lambda = \{\lambda_n\}$ is a *sampling set* of K_Φ if there are $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum \frac{|f(\lambda_n)|^2}{\|k_{\lambda_n}\|_2^2} \leq B\|f\|_2^2 \qquad \forall f \in K_\Phi.$$

- $\mathcal{K}(\Lambda)$ is a *Riesz basis* is a stronger condition: For every sequence $\{c_n\}$ such that

$$\sum \frac{|c_n|^2}{\|k_{\lambda_n}\|_2^2} < \infty \ \Rightarrow \ \exists \text{ unique } f \in K_\Phi \ s.t. \ f(\lambda_n) = c_n$$

and $\|f\|_2^2 \approx \sum |c_n|^2 / \|k_{\lambda_n}\|_2^2$. Λ is called *a complete interpolating set.*

Theorem (Hruscev, Nikolski, & Pavlov, 1981)

Let Φ be an inner function and $\Lambda = \{\lambda_n\} \subseteq \mathbb{C}^+$ such that $\sup_n |\Phi(\lambda_n)| < 1$. Then $\mathcal{K}(\Lambda)$ is a Riesz basis for K_Φ if and only if Λ satisfies the Carleson Interpolation condition

$$\inf_{k \in \mathbb{Z}} \prod_{n \neq k} \left| \frac{\lambda_k - \lambda_n}{\lambda_k - \bar{\lambda}_n} \right| \geq \delta > 0$$

and $T_{\Phi \overline{B_\Lambda}}$ is invertible.

Hermite-Biehler Functions

Definition Let E be an entire function such that

$$|E(z)| > |E(\bar{z})| \qquad \forall\, z \in \mathbb{C}^+.$$

Then E is called a *Hermite-Biehler* function and Φ defined by

$$\Phi(z) := \frac{E^*(z)}{E(z)} = \frac{\overline{E(\bar{z})}}{E(z)} \text{ is meromorphic inner.}$$

If Φ is meromorphic inner, then $\Phi = E^*/E$.

- For $\Phi(z) = \exp(aiz)$, then $\Phi = E^*/E$, where $E(z) = \exp(-iaz/2)$.
- For B_Λ a meromorphic Blaschke product, write

$$B_\Lambda(z) = \prod_n \frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n} \qquad z \notin \Lambda.$$

For each λ_n define:

$$E_n(z) = \left(1 - \frac{z}{\bar{\lambda}_n}\right) \exp\left(\mathrm{Re}\left[\frac{1}{\bar{\lambda}_n}\right] z + \cdots + \mathrm{Re}\left[\frac{1}{\bar{\lambda}_n^n}\right]\frac{z^n}{n}\right)$$

Then $E(z) := \prod E_n(z)$ is entire and $B_\Lambda = E^*/E$.

Definition. Let E be an entire function such that

$$|E(z)| > |E(\bar{z})| \qquad \forall\, z \in \mathbb{C}^+.$$

The **de Branges Space associated to E**, $\mathcal{H}(E)$ is

$$\mathcal{H}(E) := \{\text{entire functions } F : F/E, F^*/E \in H^2\}.$$

Then $\mathcal{H}(E)$ is a (reproducing kernel) Hilbert space with norm

$$\|F\|_E = \|F/E\|_2$$

Connection Between Model Spaces & de Branges Spaces

Let $\Phi := E^*/E$. Then

$$F \mapsto F/E \text{ is a unitary operator from } \mathcal{H}(E) \text{ onto } K_\Phi.$$

Specifically

$F \in \mathcal{H}(E) \Rightarrow F/E, F^*/E \in H^2$ Show: $\Phi\bar{F}/\bar{E} \in H^2$

$F/E \in K_\Phi \Rightarrow F/E \in H^2, F^*/E \in L^2$ Show: $F^*/E \perp \overline{H^2}, F$ is entire

Riesz Basis Criterion

Theorem 3

Let $\Phi = E^*/E$, where $E \in$ HB be meromorphic inner and let $T = \{t_n\} \subseteq \mathbb{R}$.

Then $\mathcal{K}(T)$ is a Riesz Basis for K_Φ if and only if there is a meromorphic inner $\Phi_1 = E_1^*/E_1$ such that

(1) $\mathcal{H}(E) = \mathcal{H}(E_1)$ as sets with equivalent norms

(2) $T = \{\Phi_1 = 1\}$ and $\Phi_1 - 1 \notin L^2(\mathbb{R})$

Observation: (2) says $\{k^1_{t_n}\}$ is a de Branges-Clark basis for K_{Φ_1}.

Idea of the Proof: $(\Leftarrow)$

- $\{k^1_{t_n}\}$ is a Riesz basis for K_{Φ_1}.
- T is a complete interpolation set for K_{Φ_1}
- Reinterpret the interpolation statement in terms of the norm of $\mathcal{H}(E_1)$.
- Use norm equivalency to obtain the result for $\mathcal{H}(E)$ (and K_Φ).

Riesz Basis Criterion II

($\Rightarrow$) Use the following result by Ortega-Cerda and Seip:

Theorem 4

Let $\Phi = E^*/E$ and let $T = \{t_n\} \subseteq \mathbb{R}$. If T is a sampling set for K_Φ, then there exist entire functions E_1 and E_2 where $E_1 \in HB$ and $E_2 \in HB$ or is constant s.t.:

(1) $\mathcal{H}(E) = \mathcal{H}(E_1)$ with norm equivalence.

(2) If $\Phi_1 = E_1^*/E_1$ and $\Phi_2 = E_2^*/E_2$, then $\{\Phi_1\Phi_2 = 1\} = T$.

(3) $(1 - \Phi_1\Phi_2) \notin L^2(\mathbb{R})$

Proof Strategy: Show E_2 is constant

- Decompose $\mathcal{H}(E_1E_2) = E_2\mathcal{H}(E_1) \oplus E_1^*\mathcal{H}(E_2)$
- Show T satisfies interpolation property for $E_2\mathcal{H}(E_1)$. (& by assumption for $\mathcal{H}(E_1E_2)$.)
- This will imply $\mathcal{H}(E_2) = \{0\}$.

$\mathcal{H}(E) = \mathcal{H}(E_1)$?

Observation: If $|E(z)| \approx |E_1(z)|$ for all $z \in \mathbb{C}^+ \cup \mathbb{R}$, then $\mathcal{H}(E) = \mathcal{H}(E_1)$ as sets with equivalent norms.

Example: The Frostman Shift Let Φ be inner and $|\zeta| < 1$. Define

$$\Phi_1 := \frac{\Phi - \zeta}{1 - \bar{\zeta}\Phi}.$$

IF $\Phi = E^*/E$, then $\Phi_1 = E_1^*/E_1$, where $E_1 = E - \bar{\zeta}E^*$.

Theorem 5

Let Φ, Φ_1 be meromorphic inner with increasing arguments ψ,ψ_1. Assume $(\psi - \psi_1) \in L^1(\Pi)$.

Then there are entire functions $E_1, E \in \mathrm{HB}$ such that $\Phi = E^*/E$ and $\Phi_1 = E_1^*/E_1$ and

$$|E(z)| \approx |E_1(z)| \qquad \forall\, z \in \mathbb{C}^+ \cup \mathbb{R}$$

if and only if

$$(\psi - \psi_1)^{\sim} \in L^\infty(\mathbb{R}).$$

Corollaries

Corollary 1

Let Φ be meromorphic inner with an increasing argument ψ and let $T = \{t_n\} \subseteq \mathbb{R}$.

Assume that there is an meromorphic inner Φ_1 with an increasing branch of its argument ψ_1 such that

(1) $(\psi - \psi_1) \in L^1(\Pi)$ and $(\psi - \psi_1)^{\sim} \in L^{\infty}(\mathbb{R})$

(2) $T = \{\Phi_1 = 1\}$ and $(\Phi_1 - 1) \notin L^2(\mathbb{R})$.

Then $\mathcal{K}(T)$ is a Riesz basis for K_Φ.

Corollary 2

Let Φ, Φ^0 be meromorphic inner functions with increasing branches of arguments ψ, ψ^0. Let $T = \{t_n\} \subseteq \mathbb{R}$. Assume $(\psi - \psi^0) \in L^1(\Pi)$ and

$$(\psi - \psi^0)^{\sim} \in L^{\infty}(\mathbb{R}).$$

Then $\mathcal{K}(T)$ is a Riesz basis for K_Φ iff $\mathcal{K}^0(T)$ is a Riesz basis for K_{Φ^0}.