Completeness and Riesz bases of reproducing kernels in model subspaces

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Abstract. We use the recent approach of N. Makarov and A. Poltoratski to give a criterion of completeness of systems of reproducing kernels in the model subspaces $K_\Theta = H^2 \ominus \Theta H^2$ of the Hardy class H^2. As an application we prove new results on stability of completeness with respect to small perturbations and obtain criteria of completeness in terms of certain densities. We also obtain a description of systems of reproducing kernels corresponding to real points which form a Riesz basis in a given model subspace.

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Introduction

Let Θ be an inner function in the upper half-plane $\mathbb{C}^+$, that is, a bounded analytic function such that $\lim\limits_{y\to 0+} |\Theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to the Lebesgue measure. With an inner function Θ we associate the model subspace

$$K_\Theta = H^2 \ominus \Theta H^2$$

of the Hardy class H^2 in the upper half-plane. These subspaces (and their analogs for the unit disc) play an outstanding role both in function and operator theory (see [10, 25, 26]), in particular, in the Sz.-Nagy–Foias model for contractions in a Hilbert space. Recall that any subspace of H^2 coinvariant with respect to the semigroup of shifts $(U_t)_{t\ge 0}$, $U_t f(x) = e^{itx} f(x)$, is of the form K_Θ for a certain inner function Θ.

We mention two important particular cases of the model subspaces. If $\Theta(z) = \exp(iaz)$, $a > 0$, then $K_\Theta = \exp(iaz/2) PW_{a/2}$, where PW_a is the Paley–Wiener space of entire functions of exponential type at most a, whose restrictions to the real axis $\mathbb{R}$ belong to $L^2(\mathbb{R})$. It is well known that the space PW_a coincides with the Fourier image of the space of square summable functions supported in the interval $(-a, a)$. On the other hand, if B is a Blaschke product with zeros z_n of multiplicities m_n, that is,

$$B(z) = \prod_n e^{i\alpha_n} \left(\frac{z - z_n}{z - \overline{z}_n} \right)^{m_n}$$

(here $\alpha_n \in \mathbb{R}$ and the factors $e^{i\alpha_n}$ ensure the convergence of the product), then the subspace K_B admits a simple geometrical description: it coincides with the closed linear span in $L^2(\mathbb{R})$ of the fractions $(z - \overline{z}_n)^{-k}$, $1 \le k \le m_n$.

We say that Θ is a meromorphic inner function if Θ is meromorphic in the whole complex plane $\mathbb{C}$. In this case Θ is of the form

$$\Theta(z) = \exp(iaz)B(z), \tag{1}$$

where $a \ge 0$ and B is a Blaschke product with zeros tending to infinity. If the inner function Θ is meromorphic, then each element of the model space K_Θ is also meromorphic and, in particular, admits an analytic continuation across the real axis. With each meromorphic inner function Θ we may associate an increasing branch of its argument on the real axis: there exists an increasing C^∞ function φ such that $\Theta(t) = \exp(i\varphi(t))$, $t \in \mathbb{R}$. Note also that

$$\varphi'(t) = |\Theta'(t)| = a + 2\sum_n \frac{m_n \mathrm{Im}\, z_n}{|t - z_n|^2}. \tag{2}$$

Recall that the function

$$\mathcal{K}_z(\zeta) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(z)}\Theta(\zeta)}{\zeta - \overline{z}}$$

is the reproducing kernel of the space K_Θ corresponding to the point $z \in \mathbb{C}^+$, that is,

$$f(z) = \langle f, \mathcal{K}_z \rangle_{L^2(\mathbb{R})}, \qquad f \in K_\Theta.$$

The last equality may be in some cases extended to the real values of z. For example, if Θ is a meromorphic inner function, then $\mathcal{K}_x \in K_\Theta$ for each $x \in \mathbb{R}$. For a general inner function Θ the criterion of the inclusion $\mathcal{K}_x \in K_\Theta$ is that $\lim_{y \to 0+} |\Theta(x+iy)| = 1$ and $|\Theta'(x)| < \infty$ (see [1]); here $|\Theta'(x)|$ stands for the modulus of the angular derivative at the point x.

We consider the following problem: given an inner function Θ, to describe the sets $\Lambda = \{\lambda_n\} \subset \mathbb{C}^+$ such that the system of reproducing kernels $\mathcal{K}(\Lambda) = \{\mathcal{K}_{\lambda_n}\}$ is complete in K_Θ. We will use repeatedly the following obvious but very important observation:

the system $\mathcal{K}(\Lambda)$ is complete in K_Θ if and only if Λ is a uniqueness set for K_Θ, that is, if the function f is in K_Θ and $f(\lambda_n) = 0$ for each n, then $f \equiv 0$.

In particular, if $\lambda_n \in \mathbb{C}^+$ and the system $\mathcal{K}(\Lambda)$ is not complete, then $\{\lambda_n\}$ is a Blaschke sequence.

We mention one important example motivating the interest to this problem. Consider the inner function $\Theta(z) = \exp(2\pi i z)$. Then, by the Paley-Wiener theorem, the model subspace K_Θ coincides with the Fourier image of the space $L^2(0, 2\pi)$. Moreover, a system of reproducing kernels $\mathcal{K}_{\lambda_n}$ in K_Θ corresponds to a system of complex exponentials $e^{i\lambda_n t}$ in $L^2(0, 2\pi)$. Completeness of systems of exponentials is a classical problem having a very long history. A detailed review of related results may be found in [22, 28]. One of the most deep results concerning this problem is the theorem of A. Beurling and P. Malliavin on the radius of completeness [8].

An important progress in the completeness problem is due to N. Makarov and A. Poltoratski [24] who recently obtained a criterion of completeness of systems of reproducing kernels for the model subspaces generated by meromorphic inner functions. Their criterion expresses the completeness in terms of an increasing branch of the argument of a meromorphic function Θ. Using this approach the authors obtain in [24] a new and essentially simpler proof of the Beurling–Malliavin theorem. They also relate their results on completeness to differential operator theory.

Making use of the similar ideas, we obtain here a slightly more general result on completeness of systems of reproducing kernels which is applicable in the case of an arbitrary, not necessarily meromorphic, inner function. As a corollary of this criterion we obtain a new result on stability of the completeness property under small perturbations of the set Λ and relate the completeness problem to certain densities.

Another long-standing problem concerning the geometric properties of systems of reproducing kernels is to describe the sets Λ such that the family $\mathcal{K}(\Lambda)$ is a Riesz basis in the given model subspace. Recall that a system of vectors $\{h_n\}$ in a Hilbert space H is said to be a Riesz basis if $\{h_n\}$ is an image of an orthogonal basis under a bounded and invertible linear operator in H. An equivalent definition is that each $h \in H$ may be represented as an unconditionally convergent series $h = \sum_n c_n h_n$ and there exist positive constants A and B such that

$$A \sum_n |c_n|^2 \|h_n\|_H^2 \le \left\| \sum_n c_n h_n \right\|_H^2 \le B \sum_n |c_n|^2 \|h_n\|_H^2.$$

In the case when $\Theta(z) = \exp(2\pi i z)$ the problem of description of Riesz bases of reproducing kernels is equivalent to the famous problem of non-harmonic Fourier series posed by Paley and Wiener. A solution of this problem was obtained by S.V. Hruscev, N.K. Nikolski and B.S. Pavlov in [21], where the case of general model subspaces is also considered and a Riesz bases' description is obtained under some additional restrictions (we state these results in Section 2).

In the present article, making use of the results of J. Ortega-Cerda and K. Seip [27], we describe Riesz bases of reproducing kernels corresponding to real points in the case of meromorphic inner functions.

1. Main results on completeness

Let Π denote the Poisson measure on $\mathbb{R}$, that is, $d\Pi(t) = \frac{dt}{t^2+1}$. Recall that the Hilbert transform of a function $g \in L^1(\Pi)$ is defined by

$$\tilde{g}(x) = v.p.\, \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) g(t)dt.$$

In what follows we denote by $\arg\Theta$ the main branch of the argument of Θ, that is, $\arg\Theta \in (-\pi, \pi]$.

The following theorem gives a criterion of completeness of $\mathcal{K}(\Lambda)$ for the case when $\Lambda \subset \mathbb{C}^+$.

Theorem 1.1. *Let $\Lambda = \{\lambda_n\} \subset \mathbb{C}^+$. Then the system $\mathcal{K}(\Lambda)$ is not complete in K_Θ if and only if there exist a nonnegative function $m \in L^2(\mathbb{R})$ such that $\log m \in L^1(\Pi)$, a measurable integer-valued function k and a real number γ such that*

$$\arg\Theta - \arg B_\Lambda = 2\widetilde{\log m} + 2\pi k + \gamma, \qquad a.e. \text{ on } \mathbb{R},$$

where B_Λ is the Blaschke product with the zeros $\{\lambda_n\}$.

An analogous description of non-complete systems is obtained in [24] for meromorphic inner functions (where an increasing branch of the argument is considered). The similar ideas were used in the papers of V.P. Havin and J. Mashreghi [19, 20] to parametrize the class of the so-called admissible majorants for model subspaces and to extend the Beurling–Malliavin multiplier theorem to the model subspaces. Moreover, in [20] a sufficient condition is obtained for a function f to be represented as $f = 2\widetilde{\log m} + 2\pi k$ for some m and k as above: such a representation takes place if the function f is *mainly increasing* (see the definition in Section 3).

Now we consider an analogous statement for the case when $\Lambda \subset \mathbb{C}^+ \cup \mathbb{R}$. Namely, let $\Lambda = \{\lambda_n\} \subset \mathbb{C}^+$ and let $T = \{t_n\} \subset \mathbb{R}$. As above, we denote by B_Λ the Blaschke product with the zeros λ_n. We also introduce an inner function J such that

$$\{t \in \mathbb{R} : J(t) = 1\} = T.$$

Here $J(t)$ is interpreted in the sense of nontangential boundary values. Such a function J may be constructed in the following way. Let us take a measure $\nu = \sum_n \nu_n \delta_{t_n}$, where δ_x denotes the Dirac measure at the point x and $\nu_n > 0$. We assume also that $\sum_n \nu_n (t_n^2+1)^{-1} < \infty$. Consider the function

$$G(z) = \sum_n \nu_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right). \tag{3}$$

Clearly, G is analytic in $\mathbb{C}^+$ and $\operatorname{Im} G(z) > 0$, $z \in \mathbb{C}^+$. Therefore, the function

$$J = \frac{G-i}{G+i}$$

is an inner function in the upper half-plane and it is well-known that $\{J = 1\} = \{t_n\}$. Moreover, in this case ν is a so-called Clark measure corresponding to the function J (see [11]). Note also that $|J'(t_n)| = 2\nu_n^{-1}$

Theorem 1.2. *Let $\Lambda = \{\lambda_n\}$ and $T = \{t_n\}$, where $\lambda_n \in \mathbb{C}^+$ and $t_n \in \mathbb{R}$, and let B_Λ denote the Blaschke product with the zeros $\{\lambda_n\}$. Assume that for each of the points t_n the function Θ is analytic in some neighborhood of t_n. Then the system $\mathcal{K}(\Lambda) \cup \mathcal{K}(T)$ is not complete in K_Θ if and only if there exists an inner function J with $\{J = 1\} = T$ such that*

$$\arg \Theta - \arg B_\Lambda - \arg J = 2\widetilde{\log m} + 2\pi k + \gamma, \qquad \text{a.e. on } \mathbb{R},$$

for some function $m \ge 0$ with $m \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$, for a measurable integer-valued function k and for a real number γ.

We consider a few applications of these general criteria. Our first application is connected with the stability of the completeness property. It turns out that in many important particular cases the system $\{\mathcal{K}_{\lambda_n}\}$ under consideration is a small perturbation of some other system $\{\mathcal{K}_{\mu_n}\}$ of reproducing kernels (that is, $\lambda_n \approx \mu_n$ in a certain sense), which is already known to be complete in K_Θ. For the case of systems of exponentials certain results of this type were obtained by N. Levinson and R. Redheffer. General model subspaces were considered by E. Fricain [17]. In particular he proved the following theorem: *if the system $\{\mathcal{K}_{\mu_n}\}$ is complete in K_Θ and*

$$\sum_n \left| \frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu}_n} \right| < \infty, \tag{4}$$

then the system $\{\mathcal{K}_{\lambda_n}\}$ is also complete in K_Θ. We obtain the following generalization of this result.

Theorem 1.3. *Let $\Lambda = \{\lambda_n\}$, $M = \{\mu_n\}$, where $\lambda_n, \mu_n \in \mathbb{C}^+$, and let the system*

$\mathcal{K}(M)$ be complete in K_Θ. If for some choice of the arguments φ_Λ and φ_M of the Blaschke products B_Λ and B_M we have $\varphi_\Lambda - \varphi_M \in L^1(\Pi)$ and

$$(\varphi_\Lambda - \varphi_M)^\sim \in L^\infty(\mathbb{R}), \tag{5}$$

then the system $\mathcal{K}(\Lambda)$ is also complete in K_Θ. Condition (5) is satisfied if

$$\mathcal{R} \in L^\infty(\mathbb{R}), \quad \textit{where} \quad \mathcal{R}(t) = \sum_n \left| \frac{\lambda_n - \mu_n}{t - \mu_n} \right|. \tag{6}$$

Remarks. 1. Clearly, (4) implies that

$$\sum_n \frac{|\lambda_n - \mu_n|}{\operatorname{Im} \mu_n} < \infty,$$

and so the boundedness of the function (6) follows immediately from (4). On the other hand, condition (6) is more subtle since it takes into account the properties of the set $\{\mu_n\}$.

2. It should be emphasized that all the above results have their analogues for the L^p generalizations of the model subspaces, that is, for the subspaces $K^p_\Theta = H^p \cap \Theta \overline{H^p}$ of the Hardy class H^p, $1 < p < \infty$. Note that if $p > 1$, then $\mathcal{K}_z \in K^p_\Theta$ for any $z \in \mathbb{C}^+$ and also for $z = x \in \mathbb{R}$ if Θ is analytic in a neighborhood of x. Thus, one may consider the problem of completeness of a system $\mathcal{K}(\Lambda)$ in K^p_Θ which is equivalent to the problem of uniqueness for the space K^q_Θ with $1/p + 1/q = 1$ due to the well-known duality between K^p_Θ and K^q_Θ.

To obtain the corresponding results for K^p_Θ one should just replace "$m \in L^2(\mathbb{R})$" in the statements of Theorems 1 and 2 by "$m \in L^q(\mathbb{R})$". The statement of Theorem 1.3 requires no changes at all. The proofs of the L^p versions are analogous.

Our next result on stability of completeness shows that in the case when two meromorphic inner functions Θ and Θ° are in a sense sufficiently close to each other, the spaces K_Θ and K_{Θ° have the same sets of uniqueness.

Corollary 1.4. *Let Θ, Θ° be inner functions such that for a certain choice of their arguments φ and φ° we have $\varphi - \varphi^\circ \in L^1(\Pi)$ and*

$$(\varphi - \varphi^\circ)^\sim \in L^\infty(\mathbb{R}).$$

Let $\Lambda \subset \mathbb{C}^+$ and let $\mathcal{K}(\Lambda)$ and $\mathcal{K}^\circ(\Lambda)$ be the corresponding systems of reproducing kernels of the model subspaces K_Θ and K_{Θ°. Then the system $\mathcal{K}(\Lambda)$ is complete in K_Θ if and only if $\mathcal{K}^\circ(\Lambda)$ is complete in K_{Θ°.

Now we apply Theorem 1.2 to obtain necessary conditions and sufficient conditions of completeness in terms of certain densities. We will need the important class of orthogonal bases of reproducing kernels in the model subspaces. Such bases were studied by L. de Branges [9] for meromorphic inner functions and by D.N. Clark [11] in the general case. We restrict ourselves by the case of meromorphic inner functions.

Let Θ be a meromorphic inner function and let φ be an increasing branch of its argument. For $\alpha \in [0, 2\pi)$ we consider the set of points $s_n \in \mathbb{R}$ such that

$$\varphi(s_n) = \alpha + 2\pi n, \qquad n \in \mathbb{Z}. \tag{7}$$

It should be noted that the points s_n may exist not for all $n \in \mathbb{Z}$ (for example, the sequence $\{s_n\}$ may be one-side, that is, s_n may exist only for $n \geq n_0$).

If the points s_n are defined by (7), then the system of reproducing kernels $\{\mathcal{K}_{s_n}\}$ is an orthogonal basis for K_Θ for each $\alpha \in [0, 2\pi)$ except, may be, one (α is an exceptional value if and only if $\Theta - e^{i\alpha} \in L^2(\mathbb{R})$; a criterion of existence of such an α in terms of factorization parameters of Θ may be found in [30] or [3]). If α is an exceptional value, then the orthogonal complement of the span of $\{\mathcal{K}_{s_n}\}$ is the one-dimensional space generated by the function $\Theta - e^{i\alpha}$.

In what follows we assume without loss of generality that the system $\{\mathcal{K}_{s_n}\}$ corresponding to $\alpha = 0$ is a basis. Therefore each $f \in K_\Theta$ admits the expansion

$$f(z) = \sum_n \frac{f(s_n)}{\|\mathcal{K}_{s_n}\|_2^2} \mathcal{K}_{s_n}(z), \qquad z \in \overline{\mathbb{C}^+},$$

and the series converges uniformly on compact subsets of $\overline{\mathbb{C}^+}$. Recall also that $2\pi\|\mathcal{K}_x\|_2^2 = |\Theta'(x)| = \varphi'(x)$, $x \in \mathbb{R}$, and, therefore,

$$\|f\|_2^2 = 2\pi \sum_n \frac{|f(s_n)|^2}{\varphi'(s_n)}, \qquad f \in K_\Theta.$$

Clearly, $\{s_n\}$ is a uniqueness set for K_Θ and, at the same time, it is an interpolating sequence in the following sense: for each sequence $\{c_n\}$ such that $\sum_n |c_n|^2/|\Theta'(s_n)| < \infty$ there exists a function $f \in K_\Theta$ such that $f(s_n) = c_n$.

To determine whether a real sequence $T = \{t_m\}$ is a uniqueness set for K_Θ we introduce the following densities. For $r \in \mathbb{N}$ we put

$$D_+(T, r) = \sup_n \#\{m : t_m \in [s_n, s_{n+r})\}, \qquad D_-(T, r) = \inf_n \#\{m : t_m \in [s_n, s_{n+r})\}$$

and we put

$$D_+(T) = \lim_{r\to\infty} \frac{D_+(T, r)}{r}, \qquad D_-(T) = \lim_{r\to\infty} \frac{D_-(T, r)}{r};$$

both limits exist due to the superadditivity of $D_-(T,r)$ and the subadditivity of $D_+(T,r)$.

Assume that $D_+(T) < 1$, which means that the sequence $\{t_m\}$ is essentially more sparse than $\{s_n\}$. Since $\{s_n\}$ is both complete and interpolating sequence for K_Θ, it seems to be a natural conjecture that T is not a uniqueness set for K_Θ and so the system of reproducing kernels $\mathcal{K}(T)$ is not complete. On the other hand, if $D_-(T) > 1$, that is, T is more dense than a uniqueness set $\{s_n\}$, one can expect that T is a uniqueness set.

For the case of systems of exponentials (equivalently, for the case $\Theta(z) = \exp(iaz)$, where $s_n = 2\pi n/a$, $n \in \mathbb{Z}$) these statements are classical (see, for example, [22, 31]). We prove analogous results in a more general situation. It is well known that in the case when a meromorphic function Θ satisfies the condition

$$\Theta' \in L^\infty(\mathbb{R}), \tag{8}$$

the model subspace K_Θ shares many properties of the Paley–Wiener spaces (see [14, 15, 16]). Note that (8) implies that $\inf_n(s_{n+1} - s_n) > 0$ since $\int_{s_n}^{s_{n+1}} |\Theta'| = 2\pi$.

Theorem 1.5. *Let Θ be a meromorphic inner function such that $\Theta' \in L^\infty(\mathbb{R})$ and let $\{s_n\}$, $T = \{t_m\}$, $D_+(T)$ and $D_-(T)$ be as above. We assume also that*

$$\sup_n \left| \sum_{k \neq n} \left(\frac{1}{s_n - s_k} + \frac{s_k}{s_k^2 + 1} \right) \right| < \infty. \tag{9}$$

Then

1. *if $D_+(T) < 1$, then T is not a uniqueness set for K_Θ;*
2. *if $D_-(T) > 1$, then T is a uniqueness set for K_Θ.*

Remarks. 1. Condition (9) means that the points s_n are sufficiently sparse or symmetric. For example, (9) is fulfilled if $\sup_n \sum_{k \neq n} |s_n - s_k|^{-1} < \infty$. On the other hand, if $\Theta(z) = \exp(iaz)$, then $s_n = 2\pi n/a$, $n \in \mathbb{Z}$, and (9) still holds due to the symmetry of the points s_n.

2. In the case of the Paley–Wiener space PW_a much stronger results are known (see [31], Theorems 2.1 and 2.2). For example, if $D_-(T) > 1$ and $D_+(T) < \infty$, then T is a sampling set for PW_a, that is,

$$\sum_m |f(t_m)|^2 \asymp \|f\|_2^2, \qquad f \in PW_a$$

(we write $g \asymp h$ if $C_1 g \le h \le C_2 g$ for some positive constants C_1 and C_2 and for all admissible values of the parameters). One may expect that in the case when $D_-(T) > 1$ and $D_+(T) < \infty$ in the conditions of Theorem 1.5 we have

$$\sum_m \frac{|f(t_m)|^2}{\varphi'(t_m)} \asymp \|f\|_2^2, \qquad f \in K_\Theta. \tag{10}$$

However, it is not true even for the class of inner functions with bounded derivative satisfying (9): the estimate of the sum in the left-hand side of (10) from above does not necessarily hold. Corresponding counterexamples were constructed in [4] and [6], where stability of Riesz bases of reproducing kernels under small perturbations was studied.

2. Description of Riesz bases of reproducing kernels

Description of Riesz bases of reproducing kernels in the model subspaces is closely connected with interpolation problems, namely, with the so-called *free interpolation* phenomenon. It was for the first time mentioned in [25] that, by duality arguments, the condition "$\mathcal{K}(\Lambda)$ is a Riesz basis in K_Θ" is equivalent to the following property: *for each* $\{c_n\} \in \ell^2$ *there exists a unique function* $f \in K_\Theta$ *such that*

$$f(\lambda_n) = c_n \|\mathcal{K}_{\lambda_n}\|_2,$$

and, moreover, $\|f\|_2 \asymp \|\{c_n\}\|_{\ell^2}$*, where the constants do not depend on* $\{c_n\}$. In this case we say that Λ is a *complete interpolating sequence* for the space K_Θ.

We will also consider a weaker *sampling* property: $\Lambda = \{\lambda_n\}$ is said to be a *sampling set* for K_Θ if

$$A\|f\|_2^2 \le \sum_n |f(\lambda_n)|^2/\|\mathcal{K}_{\lambda_n}\|_2^2 \le B\|f\|_2^2, \qquad f \in K_\Theta,$$

for some positive constants A and B. In terms of systems of reproducing kernels, the sampling property means that the normalized kernels $\{\mathcal{K}_{\lambda_n}/\|\mathcal{K}_{\lambda_n}\|_2\}$ form a frame in K_Θ. Recall that a system $\{h_n\}$ in a Hilbert space H is said to be a frame if there are positive constants A and B such that

$$A\|f\|_H^2 \le \sum_n |\langle f, h_n\rangle_H|^2 \le B\|f\|_H^2, \qquad f \in H.$$

Clearly, if the system $\{h_n\}$ is a Riesz basis, then $\{h_n/\|h_n\|_H\}$ is a frame in H.

A description of Riesz bases of exponentials was obtained by S.V. Hruscev, N.K. Nikolski and B.S. Pavlov [21] in terms the Helson–Szegö condition. In [21] also the case of general inner functions is treated and a necessary and sufficient condition is obtained under the additional restriction

$$\sup_n |\Theta(\lambda_n)| < 1. \tag{11}$$

In this case the system $\mathcal{K}(\lambda)$ is a basis if and only if the sequence Λ satisfies the Carleson interpolation condition and the Toeplitz operator $T_{\Theta\overline{B_\Lambda}}$ is invertible. The invertibility of $T_{\Theta\overline{B_\Lambda}}$ is, in its turn, equivalent to the representation $\Theta\overline{B_\Lambda} = \alpha h/\overline{h}$, where $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and $h \in H^2$ is an outer function such that $|h|^2$ satisfies the Helson–Szegö condition. Recall that a nonnegative function w is said to satisfy the Helson–Szegö condition if there are functions $u, v \in L^\infty(\mathbb{R})$ such that $\|v\|_\infty < \pi/2$ and $w = \exp(u + \tilde{v})$ (where $\tilde{v}$ is the Hilbert transform of v).

However, condition (11) seems to be too restrictive. In many cases there exist bases of reproducing kernels such that (11) does not hold. In particular, for the orthogonal de Branges–Clark bases $\{\mathcal{K}_{s_n}\}$, where s_n are defined by (7), we have $s_n \in \mathbb{R}$ and, thus, $|\Theta(s_n)| \equiv 1$.

Here we obtain a description of Riesz bases of the form $\mathcal{K}(T)$, $T = \{t_n\} \subset \mathbb{R}$, for a given meromorphic inner function Θ. To state this criterion we will need the relationship between the model subspaces generated by meromorphic inner functions and the de Branges spaces of entire functions.

Let E be an entire function such that

$$|E(z)| > |E(\overline{z})|, \qquad z \in \mathbb{C}^+.$$

In this case we say that E belongs to the Hermite–Biehler class HB. With the function $E \in HB$ we associate the de Branges space $\mathcal{H}(E)$ which consists of all entire functions F such that the functions F/E and F^*/E, where $F^*(z) = \overline{F(\overline{z})}$, belong to the Hardy class H^2. The norm in $\mathcal{H}(E)$ is defined by $\|F\|_E = \|F/E\|_{L^2(\mathbb{R})}$. The spaces $\mathcal{H}(E)$ introduced by L. de Branges have important applications in mathematical physics (see [9, 29]).

If $E \in HB$, then $\Theta = E^*/E$ is a meromorphic inner function. Conversely, each meromorphic inner function Θ admits the representation $\Theta = E^*/E$ for some entire function $E \in HB$ [19, Lemma 2.1]. Clearly, such a function E is unique up to a factor S, where S is an entire function without zeros in $\mathbb{C}^+$ and $\mathbb{C}^-$ which is real on the real axis.

It is easy to see that if $\Theta = E^*/E$, then the mapping $F \mapsto F/E$ is a unitary operator from $\mathcal{H}(E)$ onto K_Θ, that is, $K_\Theta = \mathcal{H}(E)/E$ (see, for example, [2] or [19, Theorem 2.10]).

Now we state the main results of this section. We will use essentially the properties of the de Branges spaces associated with meromorphic inner functions and the results of J. Ortega-Cerda and K. Seip [27]. We start with the following criterion.

Theorem 2.1. *Let $\Theta = E^*/E$, where $E \in HB$, be a meromorphic inner function, and let $T = \{t_n\} \subset \mathbb{R}$. Then the system $\mathcal{K}(T)$ is a Riesz basis for K_Θ if and only if there exists a meromorphic inner function $\Theta_1 = E_1^*/E_1$, $E_1 \in HB$, such that*

1. *$\mathcal{H}(E) = \mathcal{H}(E_1)$ as sets with equivalence of norms;*
2. *$T = \{\Theta_1 = 1\}$ and $\Theta_1 - 1 \notin L^2(\mathbb{R})$.*

For the case $\mathcal{H}(E) = PW_a$ this theorem is contained in the paper [27], where frames of exponentials are described. Only minor changes are required to adapt the proof to the more general situation. However, to apply this criterion one need to have a description of those functions $E_1 \in HB$ for which $\mathcal{H}(E) = \mathcal{H}(E_1)$. An obvious sufficient condition is that $|E(z)| \asymp |E_1(z)|$, $z \in \mathbb{C}^+ \cup \mathbb{R}$. This condition may be expressed in terms of the arguments of the corresponding inner functions.

Theorem 2.2. *Let Θ, Θ_1 be meromorphic inner functions with increasing branches of the arguments φ and φ_1. Assume that $\varphi - \varphi_1 \in L^1(\Pi)$ and*

$$(\varphi - \varphi_1)^{\sim} \in L^\infty(\mathbb{R}). \tag{12}$$

Then there exist entire functions $E, E_1 \in HB$ such that $\Theta = E^*/E$, $\Theta_1 = E_1^*/E_1$ and $|E(z)| \asymp |E_1(z)|$, $z \in \mathbb{C}^+ \cup \mathbb{R}$. Conversely, if $|E(z)| \asymp |E_1(z)|$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, then the arguments φ and φ_1 satisfy (12).

As a corollary we have the following sufficient condition.

Corollary 2.3. *Let Θ be a meromorphic inner function with an increasing branch of the argument φ, and let $T = \{t_n\} \subset \mathbb{R}$. Assume that there exists a meromorphic inner function Θ_1 with an increasing branch of the argument φ_1 such that*

1. *the function $\varphi - \varphi_1$ satisfies (12);*
2. *$T = \{\Theta_1 = 1\}$ and $\Theta_1 - 1 \notin L^2(\mathbb{R})$.*

Then the system $\mathcal{K}(T)$ is a Riesz basis for K_Θ.

Example. For an inner function Θ and for $\zeta \in \mathbb{C}$, $|\zeta| < 1$, one may consider the Frostman shift

$$\Theta_1 = \frac{\Theta - \zeta}{1 - \overline{\zeta}\Theta}.$$

If $\Theta = E^*/E$, where $E \in HB$, then $\Theta_1 = E_1^*/E_1$, where $E_1 = E - \overline{\zeta}E^*$. Clearly, $E_1 \in HB$ and $|E(z)| \asymp |E_1(z)|$, $z \in \overline{\mathbb{C}^+}$.

We state one more corollary of Theorem 2.2, which shows that if two inner functions are sufficiently close to each other, then the corresponding model subspaces have the same complete interpolating sequences.

Corollary 2.4. *Let Θ, Θ° be meromorphic inner functions with increasing branches of the arguments φ and φ°. Denote by $\mathcal{K}_z$ and $\mathcal{K}_z^\circ$ the reproducing kernels of the spaces K_Θ and K_{Θ° respectively. Assume that $\varphi - \varphi^\circ \in L^1(\Pi)$ and*

$$(\varphi - \varphi^\circ)^{\sim} \in L^\infty(\mathbb{R}).$$

Let $\Lambda \subset \mathbb{C}^+ \cup \mathbb{R}$. Then $\mathcal{K}(\Lambda)$ is a Riesz basis in K_Θ if and only if $\mathcal{K}^\circ(\Lambda)$ is a Riesz basis in K_{Θ°.

However, the condition $|E(z)| \asymp |E_1(z)|$, $z \in \overline{\mathbb{C}^+}$, is not necessary for the equality $\mathcal{H}(E) = \mathcal{H}(E_1)$. Yu. Lyubarskii and K. Seip [23] have obtained a description of those entire functions E for which $\mathcal{H}(E) = PW_a$. Also in [23] an explicit example may be found of a de Branges space $\mathcal{H}(E)$ such that $\mathcal{H}(E) = PW_a$ and E is unbounded on $\mathbb{R}$.

Now we state a condition which is necessary and sufficient for $\mathcal{H}(E) = \mathcal{H}(E_1)$. Recall that a function f analytic in $\mathbb{C}^+$ is said to belong to the *Smirnov class* $\mathcal{N}_+$ if f may be represented as g/h, where $g, h \in H^\infty$ and h is an outer function.

Theorem 2.5. *Let $E, E_1 \in HB$ and let φ, φ_1 be increasing branches of the arguments of the inner functions $\Theta = E^*/E$ and $\Theta_1 = E_1^*/E_1$ respectively. Then $\mathcal{H}(E) = \mathcal{H}(E_1)$ if and only if the following two conditions hold:*

1. $E/E_1, E_1/E \in \mathcal{N}_+ \cap L^2(\Pi)$;

2. for each meromorphic inner function I with an increasing continuous branch of the argument ψ the inclusions

$$\exp((\varphi - \psi)^{\sim}) \in L^1(\mathbb{R})$$

and

$$\exp((\varphi_1 - \psi)^{\sim}) \in L^1(\mathbb{R})$$

are equivalent.

Remarks. 1. By analogy with the criterion of Hruscev, Nikolski and Pavlov, it is a natural question, whether equality $\mathcal{H}(E) = \mathcal{H}(E_1)$ implies that the Toeplitz operator $T_{\Theta\overline{\Theta}_1}$ is invertible, that is, $\Theta\overline{\Theta}_1 = \alpha h/\overline{h}$, where $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and $h \in H^2$ is an outer function such that $|h|^2$ satisfies the Helson–Szegö condition. However, it is not the case (see Section 6).

2. Making use of Theorems 2.2 and 2.5 one can obtain new results concerning stability of bases of reproducing kernels $\mathcal{K}(\Lambda)$ with respect to small perturbations of the "frequencies" λ_n in the spirit of the results of Paley, Wiener and M. Kadets (see [25]). Previously, certain results on stability of Riesz bases in the model subspaces were obtained in [6, 12, 18].

§3. General criteria of completeness

In this section we prove the general criteria of completeness (Theorems 1.1 and 1.2).

We recall the following equivalent definition of the model subspace K_Θ. It is well known (and easy to see) that a function $f \in L^2(\mathbb{R})$ is in K_Θ if and only if $f \in H^2$ and $\Theta\overline{f} \in H^2$ (here we identify the functions in H^2 with their boundary values on $\mathbb{R}$; with this identification H^2 becomes a closed subspace of $L^2(\mathbb{R})$).

Though Theorem 1.1 is a particular case of Theorem 1.2 we prefer to start with its proof to make the principal ideas more transparent.

Proof of Theorem 1.1. A system $\mathcal{K}(\Lambda)$ is not complete in K_Θ if and only if there is a nonzero function $f \in K_\Theta$ such that $f(\lambda_n) = 0$ for each n. Then $f = B_\Lambda g$ for some function $g \in H^2$. Moreover, it is easy to see that $g \in K_\Theta$ since $\Theta\overline{g} = B_\Lambda \Theta \overline{f}$.

We also may assume without loss of generality that g is an outer function (otherwise we divide it by the inner factor and the fraction will be still in K_Θ). Recall that each outer function $h \in H^2$ is of the form $h = O_m = \exp(\log m + i\widetilde{\log m})$, where $m \geq 0$, $m \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$ (recall that $\widetilde{\log m}$ stands for the Hilbert transform of $\log m$). Thus, $g = O_{|f|}$.

Since $B_\Lambda g \in K_\Theta$, we have $\Theta \overline{B}_\Lambda \overline{g} \in H^2$. Therefore

$$\Theta \overline{B}_\Lambda \overline{g} = I O_{|f|}$$

for some inner function I, and so

$$\Theta \overline{B}_\Lambda \overline{O_{|f|}} = I O_{|f|}.$$

Taking the arguments, we obtain

$$\arg \Theta - \arg B_\Lambda - \widetilde{\log|f|} = \arg I + \widetilde{\log|f|} + 2\pi k_0, \tag{13}$$

where k_0 is an integer-valued measurable function. Now we make use of the following statement: if I is an arbitrary inner function, then its argument may be represented as

$$\arg I = 2\widetilde{\log m_1} + 2\pi k_1 + \gamma_1, \tag{14}$$

where $m_1 \geq 0$, $m_1 \in L^\infty(\mathbb{R})$, $\log m_1 \in L^1(\Pi)$, $\gamma_1 \in \mathbb{R}$ and k_1 is an integer-valued measurable function. Indeed,

$$I = -\frac{1 - I}{1 - \overline{I}}$$

whence

$$\arg I = 2\widetilde{\log|1 - I|} + \pi. \tag{15}$$

(note that $1 - I$ is an outer function in H^∞ since $\mathrm{Re}\,(1 - I) > 0$ in $\mathbb{C}^+$). Moreover, it is shown in [7] that m_1 may be taken in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Combining equations (13) and (14) we obtain the representation

$$\arg \Theta - \arg B_\Lambda = 2\widetilde{\log m} + 2\pi k + \gamma, \tag{16}$$

where $m = m_1|f|$, $k = k_0 + k_1$ and $\gamma = \gamma_1$.

Now assume that we have the representation (16). Then we have

$$\Theta \overline{B}_\Lambda \overline{O_m} = e^{i\gamma} O_m$$

and consequently the function $f = B_\Lambda O_m$ is in K_Θ and vanishes at Λ. Hence, the system $\mathcal{K}(\Lambda)$ is not complete in K_Θ. ◯

Proof of Theorem 1.2. The proof of this theorem is analogous to the proof above. Assume that the system $\mathcal{K}(\Lambda) \cup \mathcal{K}(T)$ is not complete in K_Θ. Then there exists a function $f \in K_\Theta$ of the form $f = B_\Lambda g$, where g is an outer function in K_Θ. Since Θ is analytic near the points t_n, it follows that both g and B_Λ are analytic in a neighborhood of t_n for each n and $g(t_n) = 0$. By Lemma 2.6 of [24], choosing in the definition of the function (3) a sequence ν_n with sufficiently fast decay we may construct an inner function J with $\{J = 1\} = \{t_n\}$ such that $g/(1-J) \in L^2(\mathbb{R})$ (though in [24] the case of meromorphic functions is considered the proof uses only the fact that there exist disjoint neighborhoods of the points t_n, where all the elements of K_Θ are analytic).

Note also that $1-J$ is an outer function in H^∞ since $\mathrm{Re}\,(1-J) > 0$ in $\mathbb{C}^+$. Therefore, by the classical Smirnov theorem, $g/(1-J) \in H^2$.

Thus, there exists an outer function $h \in H^2$ such that $f = B_\Lambda(1-J)h \in K_\Theta$. Hence, we have $\Theta \overline{B}_\Lambda (1-\overline{J})\overline{h} \in H^2$. It follows that

$$\Theta \overline{B}_\Lambda \overline{O_{|1-J|}}\,\overline{O_{|h|}} = O_{|1-J|} O_{|h|} I$$

for some inner function I and, consequently,

$$\arg\Theta - \arg B_\Lambda = 2\widetilde{\log|h|} + 2\widetilde{\log|1-J|} + \arg I.$$

By (15), $2\widetilde{\log|1-J|} = \pi + \arg J + 2\pi k_0$ for some integer-valued function k_0. Taking (14) into account we get

$$\arg\Theta - \arg B_\Lambda - \arg J = 2\widetilde{\log m} + 2\pi k + \gamma, \tag{17}$$

where $m = m_1|h|$, $k = k_0 + k_1$ and $\gamma = \gamma_1 + \pi$.

Now assume that there exists an inner function J with $\{J = 1\} = \{t_n\}$ such that (17) holds. Then the function $f = B_\Lambda(1-J)O_m$ is in K_Θ and also $B_\Lambda J O_m \in K_\Theta$. Clearly, $f(\lambda_n) = 0$ for each n. Finally, note that since f is analytic in a neighborhood of t_n, both B_Λ and J are analytic in a neighborhood of t_n and, thus, the function O_m is meromorphic near t_n. Since $O_m \in H^2$, we conclude that O_m is analytic near t_n and so $f(t_n) = 0$. ◯

To apply Theorems 1.1 and 1.2 one should have a description of the functions representable as $\widetilde{\log m}$ for a nonnegative function $m \in L^2(\mathbb{R})$ up to a summand of the form $2\pi k$, where k is an integer-valued function. A condition sufficient for such a representation was proposed by V.P. Havin and J. Mashreghi [20]. To state it we need the notion of a *mainly increasing function*.

We denote by $\operatorname{Osc}(f, E)$ the oscillation of a function f on the set E, that is,

$$\operatorname{Osc}(f, E) = \sup_{s,t \in E} (f(s) - f(t)).$$

Let f be a C^1-function on $\mathbb{R}$ and let $\{d_n\}$ (where $n \in \mathbb{Z}$ or $n \in \mathbb{N}$; in the latter case we assume that $d_1 = -\infty$) be an increasing sequence of real numbers such that $\lim_{|n|\to\infty} |d_n| = \infty$ and

$$f(d_{n+1}) - f(d_n) \asymp 1, \qquad n \in \mathbb{Z} \ \ (n \ge 2).$$

Assume also that there is a constant $C > 0$ such that

$$\operatorname{Osc}(f, (d_n, d_{n+1})) \le C \qquad \text{and} \qquad \operatorname{Osc}(f', (d_n, d_{n+1})) \le C$$

for all $n \in \mathbb{Z}$ ($n \in \mathbb{N}$). Such functions f will be referred to as *mainly increasing functions.*

It was shown in [20] (see, also, [7]) that each mainly increasing function f admits the representation

$$f = 2\widetilde{\log m} + 2\pi k + \gamma,$$

where $m \ge 0$, $m \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, $\log m \in L^1(\Pi)$, $\gamma \in \mathbb{R}$ and k is a measurable integer-valued function. This theorem is proved in [20] under a small additional restriction on the distances $d_{n+1} - d_n$ and in [7] in the general case. It should be mentioned that the condition $\operatorname{Osc}(f', (d_n, d_{n+1})) \le C$ may be replaced by a weaker integral estimate.

We have an immediate corollary of Theorem 1.2.

Corollary 3.1. *Let Θ be a meromorphic inner function. If $\arg\Theta - \arg B_\Lambda - \arg J$ is a mainly increasing function, then $\Lambda \cup T$ is not a uniqueness set for K_Θ.*

Example. Let $\Lambda \subset \mathbb{C}^+$, let $|\lambda_n| \to \infty$, $n \to \infty$, and let

$$|B'_\Lambda(t)| = 2\sum_n \frac{\operatorname{Im}\lambda_n}{|t - \lambda_n|^2} \to 0, \qquad |t| \to \infty. \tag{18}$$

Then the system $\{e^{i\lambda_n t}\}$ is not complete in $L^2(0, \varepsilon)$ for any $\varepsilon > 0$. Indeed, it follows from (18) that $\varepsilon t - \varphi_\Lambda$ is a mainly increasing function for each $\varepsilon > 0$ (here we denote by φ_Λ an increasing continuous branch of the argument of B_Λ) and therefore the system of reproducing kernels corresponding to Λ is not complete in $K_{e^{i\varepsilon z}}$.

Condition (18) appears in [16] as a criterion of boundedness of the differentiation operator in K_{B_Λ}. In particular, (18) is satisfied if Λ is contained in a Stolz angle $\{z \in \mathbb{C}^+ : \operatorname{Im} z \ge \delta|\operatorname{Re} z|\}$, $\delta > 0$, and $|\lambda_n| \to \infty$ (in this case we have $\sum_n |\lambda_n|^{-1} < \infty$ and the incompleteness of $\{e^{i\lambda_n t}\}$ follows also from [28, Theorem 41]).

We will see in the proof of Theorem 1.5 that sometimes it is more convenient to represent T as a finite union $T = \bigcup_{l=1}^{L} T_l$ and consider the inner functions J_l such that $\{J_l = 1\} = T_l$. Repeating the arguments from the proof of Theorem 1.2 we obtain the following completeness criterion:

Theorem 3.2. *Let $T = \bigcup_{l=1}^{L} T_l \subset \mathbb{R}$ and let the function Θ be analytic in a neighborhood of t for each $t \in T$. Then the system $\mathcal{K}(T)$ is not complete in K_Θ if and only if there exist inner functions J_l with $\{J_l = 1\} = T_l$ such that*

$$\arg\Theta - \sum_{l=1}^{L} \arg J_l = 2\widetilde{\log m} + 2\pi k + \gamma, \qquad \text{a.e. on } \mathbb{R},$$

for some function $m \ge 0$ satisfying $m \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$, for a measurable integer-valued function k and for a real number γ.

In the proof of Theorem 1.5 we will use the following corollary of Theorem 3.2.

Corollary 3.3. *Let Θ be a meromorphic inner function. If for some inner functions J_l such that $\{J_l = 1\} = T_l$ the function*

$$\arg\Theta - \sum_{l=1}^{L} \arg J_l$$

is mainly increasing, then $T = \bigcup_{l=1}^{L} T_l$ is not a uniqueness set for K_Θ.

§4. Stability of completeness and density criteria

We start with the proof of Theorem 1.3 on stability of the completeness property under small perturbations of "frequencies" λ_n.

Proof of Theorem 1.3. Assume that Λ is not a uniqueness set for K_Θ. Then, by Theorem 1.1,

$$\arg\Theta - \varphi_\Lambda = 2\widetilde{\log m} + 2\pi k + \gamma.$$

The proof will be completed as soon as we show that

$$\varphi_\Lambda - \varphi_M = 2\widetilde{\log m_1}, \tag{19}$$

where $m_1 \geq 0$, $m_1 \in L^\infty(\mathbb{R})$ and $\log m_1 \in L^1(\Pi)$. Indeed, in this case

$$\arg\Theta - \arg B_M = 2\widetilde{\log(m_1 m)} + 2\pi k + \gamma,$$

and, thus, $M = \{\mu_n\}$ is not a uniqueness set for K_Θ which contradicts the hypothesis.

Since $(\varphi_\Lambda - \varphi_M)^\sim \in L^\infty(\mathbb{R})$, it follows that

$$\varphi_\Lambda - \varphi_M = 2\tilde{u} = 2\widetilde{\log m_1},$$

where $u \in L^\infty(\mathbb{R})$ and so $m_1 = e^u \asymp 1$.

Now we show that the boundedness of (6) implies (19) for certain choice of the arguments φ_Λ and φ_M. Put

$$h(z) = \prod_n \left(\frac{z - \overline{\lambda}_n}{z - \overline{\mu}_n}\right)^2, \qquad z \in \overline{\mathbb{C}^+}.$$

It follows from (6) that the product converges and both h and and h^{-1} are in H^∞. Thus, h is an outer function. Note that

$$h B_\Lambda \overline{B}_M = |h|, \qquad \text{a.e. on } \mathbb{R}.$$

Therefore, $B_\Lambda \overline{B}_M = \exp(i\widetilde{\log|h|})$ and we have

$$\varphi_\Lambda - \varphi_M = \widetilde{\log|h|} = 2\widetilde{\log m_1},$$

with $m_1 = \sqrt{|h|} \in L^\infty(\mathbb{R})$ for a certain choice of the arguments φ_Λ and φ_M. $\bigcirc$

Proof of Corollary 1.4. Since, $(\varphi - \varphi^\circ)^\sim \in L^\infty(\mathbb{R})$, we have

$$\varphi - \varphi^\circ = 2\widetilde{\log m_1},$$

where $m_1 \asymp 1$. The statement follows immediately from Theorem 1.1. $\bigcirc$

Now we turn to the proof of Theorem 1.5. The following lemma will play the key role in this proof.

Lemma 4.1. *Assume that the real sequence $\{t_n\}$ satisfies the following two conditions:*

$$\inf_n (t_{n+1} - t_n) = \delta > 0 \tag{20}$$

and

$$\sup_k \left| \sum_{n \neq k} \left(\frac{1}{t_n - t_k} + \frac{t_k}{t_k^2 + 1} \right) \right| < \infty. \tag{21}$$

Let G be the function of the form (3) with $\nu_n \equiv 1$, that is,

$$G(z) = \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right),$$

and let $J = (G - i)/(G + i)$. Then $J' \in L^\infty(\mathbb{R})$.

Proof. By the definition of J we have

$$|J'(t)| = \frac{2|G'(t)|}{|G(t) + i|^2} \le \frac{2}{|G(t) + i|^2} \sum_n \frac{1}{|t - t_n|^2}, \qquad t \in \mathbb{R}.$$

Note that $G(t) \in \mathbb{R}$, $t \in \mathbb{R}$. Therefore, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that $|J'(t)| \le C$ if $|t - t_n| \ge \varepsilon$ for each n. Now assume that $|t - t_n| < \varepsilon < \delta/2$ for some n. Then

$$|J'(t)| \le \sum_{k \ne n} \frac{2}{|t - t_k|^2} + 2 \left(1 + (t_n - t) \left[-\frac{t_n}{t_n^2 + 1} + \sum_{k \ne n} \left(\frac{1}{t_k - t} - \frac{t_k}{t_k^2 + 1} \right) \right] \right)^{-2}.$$

The conditions (20) and (21) imply that for sufficiently small ε

$$\left| (t - t_n) \sum_{k \ne n} \left(\frac{1}{t_k - t} - \frac{t_k}{t_k^2 + 1} \right) \right| \le \frac{1}{2}, \qquad |t - t_n| < \varepsilon.$$

Thus, $J' \in L^\infty(\mathbb{R})$. ◯

We will also need two more lemmas. The first one is a particular case of the results of [2] and [5, Theorem 1.5] on embeddings of model subspaces.

Lemma 4.2. *Let $\Theta' \in L^\infty(\mathbb{R})$ and let the sequence $\{t_n\}$ satisfy (20). Then there exists $C > 0$ such that*

$$\sum_n |f(t_n)|^2 \le C \|f\|_2^2, \qquad f \in K_\Theta.$$

Lemma 4.3. *Let $\Theta' \in L^\infty(\mathbb{R})$, let the sequence $\{t_n\}$ satisfy (20) and (21), and let J be the inner function constructed in Lemma 4.1. If $f \in K_\Theta$ and $f(t_n) = 0$ for each n, then $f/(1 - J) \in K_\Theta$.*

Proof. It is sufficient to show that $f/(1 - J) \in L^2(\mathbb{R})$. Indeed, then, by the Smirnov theorem, $f/(1 - J) \in H^2$ and, analogously, $\Theta\overline{f}/(1 - J) \in L^2(\mathbb{R})$. Hence,

$$\Theta \frac{\overline{f}}{1 - \overline{J}} = \frac{\Theta \overline{f}}{J - 1} J \in H^2$$

and, therefore, $f/(1 - J) \in K_\Theta$.

Note that $|J'(t_n)| = 2\nu_n^{-1} = 2$. By Lemma 4.1, $J' \in L^\infty(\mathbb{R})$ and, in particular, $\inf_n \operatorname{Im} z_n > 0$, where z_n are zeros of Θ. Now it follows immediately from the formula (2) for the modulus of the derivative of an inner function that for each $\varepsilon > 0$ there exists positive constants C_1 and C_2 such that

$$C_1 \le |J'(s)|/|J'(t)| \le C_2, \qquad s, t \in \mathbb{R},\ |s-t| \le \varepsilon,$$

and, in particular, $|J'(t)| \asymp 1$, $|t - t_n| \le \varepsilon$. Let us fix $\varepsilon < \delta/2$ and denote by ψ an increasing branch of the argument of J. Since $|J'(t)| = \psi'(t)$, it follows that

$$0 < C_3 \le \psi(t_n + \varepsilon) - \psi(t_n) = \int\limits_{t_n}^{t_n+\varepsilon} \psi' \le C_4 < 2\pi$$

and

$$0 < C_3 \le \psi(t_n) - \psi(t_n - \varepsilon) \le C_4 < 2\pi$$

for some constants C_3 and C_4 independent of n. Recall that $\{t_n\} = \{J = 1\}$ and, therefore,

$$|J(t) - 1| = |e^{i\psi(t)} - 1| \ge C_5 > 0, \qquad |t - t_n| > \varepsilon,$$

for each n. Hence, the function $f/(1-J)$ is square summable on the set $\mathbb{R} \setminus E$, where

$$E = \bigcup_n [t_n - \varepsilon, t_n + \varepsilon].$$

It remains to estimate the function $f/(1-J)$ on the set E. Let $|t - t_n| \le \varepsilon$. Then

$$\left| \frac{f(t)}{1 - J(t)} \right| = |f(t)| \cdot \left| 2 \sin \frac{\psi(t) - \psi(t_n)}{2} \right|^{-1} \le C_6 \frac{|f(t)|}{|t - t_n|}$$

since $\psi'(t) \asymp 1$ and

$$\left| \sin \frac{\psi(t) - \psi(t_n)}{2} \right| \asymp |\psi(t) - \psi(t_n)| \asymp |t - t_n|.$$

Thus, we have

$$\left| \frac{f(t)}{1 - J(t)} \right| = \left| \frac{f(t) - f(t_n)}{1 - J(t)} \right| \le C_6 \max_{|s - t_n| \le \varepsilon} |f'(s)|, \qquad |t - t_n| \le \varepsilon.$$

It is easy to see that $f' \in K_{\Theta^2}$ as soon as $f \in K_\Theta$ (see [16]). Now it follows from Lemma 4.2 (applied to Θ^2 and f' instead of f and Θ) that

$$\sum_n \int\limits_{t_n - \varepsilon}^{t_n + \varepsilon} \left| \frac{f(t)}{1 - J(t)} \right|^2 dt \le C_7 \sum_n \max_{|s - t_n| \le \varepsilon} |f'(s)|^2 < \infty,$$

which completes the proof. $\bigcirc$

Proof of Theorem 1.5. We consider here the case when $T = \{t_n\}_{n\in\mathbb{Z}}$ and $\lim_{|n|\to\infty} |t_n| = \infty$; only minor changes are required in the case of one-side sequences. Let $D_+(T) < 1$. Then there exists $L \in \mathbb{N}$, $L > 2$, such that

$$\#\{m : t_m \in [s_n, s_{n+L})\} \le L - 2$$

for any n. We may add to T arbitrary points (if necessary) so that

$$\#\{m : t_m \in [s_{nL}, s_{(n+1)L})\} = L - 2.$$

Then we may represent T as the union $T = \bigcup_{l=1}^{L-2} T_l$ where $T_l = \{t_n^l\}$ and t_n^l is the element of the set $T \cap [s_{nL}, s_{(n+1)L})$ with the number l (if we number the points in the increasing order).

Clearly,

$$\inf_{n\neq k} |t_n^l - t_k^l| > 0.$$

Moreover it is easy to verify that (9) implies that the sequence $\{t_n^l\}$ satisfies condition (21) of Lemma 4.1. Let J_l be an inner function such that $\{J_l = 1\} = T_l$ and $J_l' \in L^\infty(\mathbb{R})$ constructed in Lemma 4.1. Denote by φ and ψ_l increasing branches of the arguments of Θ and J_l respectively. Note that $\varphi', \psi_l' \in L^\infty(\mathbb{R})$. We will show that

$$f = \varphi - \sum_{l=1}^{L-2} \psi_l$$

is a mainly increasing function. Then, by Corollary 3.3, T is not a uniqueness set for K_Θ.

Indeed, let r be a sufficiently large positive integer. Put $d_n = s_{nrL}$. Let us show that the function f and sequence $\{d_n\}$ satisfy the definition of a mainly increasing function. Clearly, $\mathrm{Osc}(\varphi, I_n) = 2\pi r L$. For a fixed n there are exactly r points $t_{m+1}^l, \dots t_{m+r}^l$ of the set T_l in the interval $I_n = [s_{nrL}, s_{(n+1)rL})$. Hence,

$$\int_{d_n}^{d_{n+1}} \psi_l' \le \int_{t_m^l}^{t_{m+r+1}^l} \psi_l' \le 2\pi(r+1).$$

We used the fact that

$$\int_{t_k^l}^{t_{k+1}^l} \psi_l' = 2\pi$$

for any k and l. Therefore, $\mathrm{Osc}(\sum_{l=1}^{L-2}\psi_l', I_n) \le 2\pi(r+1)(L-2)$ and, in particular,

$$f(d_{n+1}) - f(d_n) \ge 2\pi rL - 2\pi(r+1)(L-2) = 4\pi(r+1) - 2\pi L > 2\pi L$$

if $r > L-1$. Finally, since $\varphi', \psi_l' \in L^\infty(\mathbb{R})$, we have $\sup_n \mathrm{Osc}(f', I_n) < \infty$. Hence, the function f is mainly increasing which proves the first statement.

Now we prove the second statement of the theorem. Assume that $D_-(T) > 1$. Choosing a subsequence if necessary, we may also assume that $D_+(T) < \infty$. By the same arguments as above, in this case we may split the set T (after eliminating part of the points) into a finite union of disjoint sets T_l, $l = 1, \dots L$, and construct the inner functions J_l such that the function $\sum_{l=1}^{L}\psi_l - \varphi$ is mainly increasing. Therefore

$$\sum_{l=1}^{L}\psi_l - \varphi = 2\widetilde{\log m_1} + 2\pi k_1 + \gamma_1, \tag{22}$$

where $m_1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $m_1 \ge 0$, $\log m_1 \in L^1(\Pi)$, $\gamma_1 \in \mathbb{R}$ and k_1 is a measurable integer-valued function.

Assume that T is not a uniqueness set for K_Θ. Then, there exists a nonzero function $f \in K_\Theta$ such that $f(t) = 0$, $t \in T$. By Lemma 4.3, the function

$$\frac{f}{(1-J_1)(1-J_2)\dots(1-J_L)}$$

is in K_Θ, and therefore (as in the proof of Theorem 1.2)

$$\varphi - \sum_{l=1}^{L}\psi_l = 2\widetilde{\log m_2} + 2\pi k_2 + \gamma_2, \tag{23}$$

where $m_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $m_2 \ge 0$, $\log m_2 \in L^1(\Pi)$, $\gamma_2 \in \mathbb{R}$ and k_2 is a measurable integer-valued function. The inclusion $m_2 \in L^\infty(\mathbb{R})$ follows from the fact that in the case when $\Theta' \in L^\infty(\mathbb{R})$ we have $K_\Theta \subset L^\infty(\mathbb{R})$ (see [14, 15]).

Combining (22) and (23) we get

$$2\widetilde{\log m_3} + 2\pi k_3 + \gamma_3 = 0, \tag{24}$$

where $m_3 = m_1 m_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\gamma_3 \in \mathbb{R}$ and k_3 is an integer-valued function. To complete the proof we apply an argument from [7], Lemma 14, which shows that (24) may hold only if $m_3 \asymp 1$ on $\mathbb{R}$ (thus, we get a contradiction since $m_3 \in L^2(\mathbb{R})$).

We say that a nonnegative function w on the real line is an admissible majorant for some model subspace K_θ if there is a non-zero function $f \in K_\theta$ such that $|f| \le w$ a.e. on $\mathbb{R}$. In [19, 20] the following admissibility criterion is given: w is an admissible majorant for K_θ if and only if there exists a function $m \in L^\infty(\mathbb{R})$ with $m \geq 0$, $mw \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$ such that

$$\arg\theta = 2\widetilde{\log(mw)} + 2\pi k + \gamma, \qquad \text{a.e. on } \mathbb{R},$$

for a measurable integer-valued function k and for some $\gamma \in \mathbb{R}$.

Now it follows from (24) that for any θ and for any K_θ-admissible majorant w the majorant $m_3 w$ is also admissible for K_θ. For example, let $\theta(z) = \frac{z-i}{z+i}$. Then K_θ is a one-dimensional space generated by the function $f(z) = \frac{1}{z+i}$. Thus, the majorant $w(t) = \frac{1}{|t+i|}$ is admissible, and, clearly, it follows from admissibility of $m_3 w$ that $\operatorname{ess\,inf}_{\mathbb{R}} m_3 > 0$. ◯

§5. Proof of the Riesz bases' criterion

We start with the following theorem on sampling for the de Branges spaces, which is analogous to the theorem of Ortega-Cerda and Seip on sampling in the Paley–Wiener spaces. We say that the $T = \{t_n\} \subset \mathbb{R}$ is a sampling set for the space $\mathcal{H}(E)$ if

$$A\|F\|_E^2 \leq \sum_n \frac{|F(t_n)|^2}{|E(t_n)|^2 \varphi'(t_n)} \leq B\|F\|_E^2, \qquad F \in \mathcal{H}(E), \tag{25}$$

which is equivalent to say that T is sampling for K_Θ with $\Theta = E^*/E$.

Theorem 5.1. *Let $E \in HB$ and let $T = \{t_n\} \subset \mathbb{R}$. If T is a sampling set for K_Θ, then there exist entire functions E_1, E_2, where $E_1 \in HB$ and E_2 is either in HB or a constant function, such that*

1. *$\mathcal{H}(E) = \mathcal{H}(E_1)$;*
2. *T is the zero set for the function $E_1E_2 - E_1^*E_2^*$;*
3. *$1 - \Theta_1\Theta_2 \notin L^2(\mathbb{R})$, where $\Theta_1 = E_1^*/E_1$ and $\Theta_2 = E_2^*/E_2$.*

This theorem was proved in [27] for the case $\mathcal{H}(E) = PW_a$ (that is, $E(z) = \exp(-iaz)$). However, the proof works in the general case (the only difference is in the definition of the function G on page 795 of [27]: one should replace the canonical product of order one by an arbitrary Weierstrass product with simple zeros t_n). We omit the details.

Remark. It is shown in [27] that in the case $\mathcal{H}(E) = PW_a$ the necessary condition of Theorem 5.1 is sufficient if we assume also that the sequence $\{t_n\}$ is uniformly separated. To have the converse statement in the general case one have to assume that the right-hand side estimate in (25) holds.

Proof of Theorem 2.1. Assume that there exists an inner function $\Theta_1 = E_1^*/E_1$

such that $\mathcal{H}(E) = \mathcal{H}(E_1)$, $T = \{\Theta_1 = 1\}$ and $1 - \Theta_1 \notin L^2(\mathbb{R})$. We denote by φ and φ_1 increasing branches of the arguments of the functions Θ and Θ_1.

Denote by $\mathcal{K}_z^{(1)}$ the reproducing kernel of the space K_{Θ_1} corresponding to the point z. It follows that the system $\mathcal{K}^{(1)}(T) = \{\mathcal{K}_{t_n}^{(1)}\}$ is a de Branges–Clark basis for K_{Θ_1}. Hence, T is a complete interpolation set for K_{Θ_1}, that is, for each sequence $\{c_n\}$ such that

$$\sum_n |c_n|^2/\varphi_1'(t_n) < \infty$$

there exists a unique function $f \in K_{\Theta_1}$ such that $f(t_n) = c_n$ and, moreover, $\|f\|_2^2 \asymp \sum\limits_n |c_n|^2/\varphi_1'(t_n)$. In view of the relationship between K_{Θ_1} and $\mathcal{H}(E_1)$ it is equivalent to say that for each sequence $\{d_n\}$ satisfying

$$\sum_n |d_n|^2(|E_1(t_n)|^2\varphi_1'(t_n))^{-1} < \infty$$

there exists a unique function $F \in \mathcal{H}(E_1)$ such that $F(t_n) = d_n$, and $\|F\|_E^2 \asymp \sum\limits_n |d_n|^2(|E_1(t_n)|^2\varphi_1'(t_n))^{-1}$.

Note that the function $K^{(1)}(z,\cdot) = E_1\mathcal{K}_z^{(1)}$ is the reproducing kernel of the space $\mathcal{H}(E_1)$ corresponding to the point z and, therefore,

$$\frac{1}{2\pi}|E_1(t)|^2\varphi_1'(t) = K^{(1)}(t,t) = \sup_{F\in\mathcal{H}(E_1),\, \|F\|_{E_1}\le 1} |F(t)|^2, \qquad t \in \mathbb{R}.$$

Since $\mathcal{H}(E) = \mathcal{H}(E_1)$ with equivalence of the norms, we have

$$|E_1(t)|^2\varphi_1'(t) \asymp |E(t)|^2\varphi'(t), \qquad t \in \mathbb{R}. \tag{26}$$

Hence, $|E_1(t_n)|^2\varphi_1'(t_n) \asymp |E(t_n)|^2\varphi'(t_n)$ and for each, sequence $\{d_n\}$ satisfying

$$\sum_n |d_n|^2(|E(t_n)|^2\varphi'(t_n))^{-1} < \infty$$

there exists a unique function $F \in \mathcal{H}(E)$ such that $F(t_n) = d_n$. Thus, T is a complete interpolating set for K_Θ and, consequently, $\mathcal{K}(T)$ is a Riesz basis in K_Θ.

Let us prove the converse statement. Assume that $\mathcal{K}(T)$ is a Riesz basis for K_Θ. Hence, T is a complete interpolating set for K_Θ and for $\mathcal{H}(E)$, that is, for each sequence $\{c_n\}$ such that

$$\sum_n \frac{|c_n|^2}{|E(t_n)|^2\varphi'(t_n)} < \infty$$

there exists a unique function $F \in \mathcal{H}(E)$ such that $F(t_n) = c_n$, and

$$\|F\|_E^2 \asymp \sum_n \frac{|c_n|^2}{|E(t_n)|^2 \varphi'(t_n)}.$$

In particular, T is a sampling set for $\mathcal{H}(E)$.

Let E_1, E_2 be entire functions from Theorem 5.1. To complete the proof it remains to show that E_2 is a constant function.

Assume that E_2 is a nontrivial HB-function. It follows from conditions 2 and 3 of Theorem 5.1 that T is a complete interpolating set for the space $\mathcal{H}(E_1E_2)$, that is, for each $\{d_n\}$ such that

$$\sum_n \frac{|d_n|^2}{|E_1(t_n)|^2 |E_2(t_n)|^2 (\varphi_1'(t_n) + \varphi_2'(t_n))} < \infty$$

there exists a unique function $G \in \mathcal{H}(E_1E_2)$ such that $G(t_n) = d_n$.

It is easy to see that the space $\mathcal{H}(E_1E_2)$ admits the orthogonal decomposition

$$\mathcal{H}(E_1E_2) = E_2\mathcal{H}(E_1) \oplus E_1^*\mathcal{H}(E_2).$$

By (26), we have $|E_1(t_n)|^2 \varphi_1'(t_n) \asymp |E(t_n)|^2 \varphi'(t_n)$. Therefore, for each sequence $\{d_n\}$ such that

$$\sum_n \frac{|d_n|^2}{|E_1(t_n)|^2 |E_2(t_n)|^2 \varphi_1'(t_n)} < \infty$$

there exists a unique function $F \in E_2\mathcal{H}(E)$ such that $F(t_n) = d_n$. Clearly, the functions $G \in \mathcal{H}(E_1E_2)$ such that

$$\sum_n \frac{|G(t_n)|^2}{|E_1(t_n)|^2 |E_2(t_n)|^2 \varphi_1'(t_n)} < \infty$$

are dense in $\mathcal{H}(E_1E_2)$. Hence, the closed proper subspace $E_2\mathcal{H}(E_1)$ is dense in $\mathcal{H}(E_1E_2)$ and we got a contradiction. ◯

Proof of Theorem 2.2. Let E, E_1 be HB entire functions without real zeros such that $\Theta = E^*/E$ and $\Theta_1 = E_1^*/E_1$. We show that one can choose another entire function $E_2 \in HB$ such that $\Theta = E_2^*/E_2$ and $|E_2(z)| \asymp |E_1(z)|$, $z \in \mathbb{C}^+ \cup \mathbb{R}$.

Put $f = (\varphi_1 - \varphi)/2$ and let

$$w(t) = \exp(\tilde{f}(t) + if(t)), \qquad t \in \mathbb{R}.$$

By the hypothesis, $\tilde{f} \in L^\infty(\mathbb{R})$, and, therefore, w (extended to the upper half-plane) is an outer function in H^∞ and $w^{-1} \in H^\infty$. Moreover, since f is in $C^\infty(\mathbb{R})$, it follows that the functions w and w^{-1} are continuous in $\overline{\mathbb{C}^+}$.

Note that $E(t)e^{i\varphi(t)/2} \in \mathbb{R}$, $t \in \mathbb{R}$. Put

$$S(z) = \frac{E_1(z)}{E(z)} w(z).$$

Clearly, S is analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}$. We have

$$S(t) = \frac{E_1(t)\exp(i\varphi_1(t)/2)}{E(t)\exp(i\varphi(t)/2)} \exp(\tilde{f}(t)) \in \mathbb{R}, \qquad t \in \mathbb{R}.$$

Hence, S may be extended to an entire function in the whole complex plane.

Now put $E_2 = SE$. Since S is non-vanishing and real on the real axis, it follows that $E_2 \in HB$. We have also $E_2/E_1 = w$ and, therefore, $|E_2(z)| \asymp |E_1(z)|$, $z \in \overline{\mathbb{C}^+}$.

The converse statement is immediate. Indeed, if $|E(z)| \asymp |E_1(z)|$, $z \in \overline{\mathbb{C}^+}$, then, up to a constant summand,

$$\varphi_1 - \varphi = 2\widetilde{\log|w|},$$

where $w = E/E_1 \in H^\infty$ and $w^{-1} \in H^\infty$. $\bigcirc$

Proof of Corollary 2.3. By Theorem 2.2, there exist entire functions $E, E_1 \in HB$ such that $\Theta = E^*/E$, $\Theta_1 = E_1^*/E_1$ and $\mathcal{H}(E) = \mathcal{H}(E_1)$. Now the statement follows from Theorem 2.1. $\bigcirc$

Proof of Corollary 2.4. By Theorem 2.2, condition $(\varphi - \varphi^\circ)^\sim \in L^\infty(\mathbb{R})$ implies that there exist entire functions $E, E^\circ \in HB$ such that $\Theta = E^*/E$, $\Theta^\circ = (E^\circ)^*/E^\circ$ and $\mathcal{H}(E) = \mathcal{H}(E^\circ)$. Hence, the classes of complete interpolating sequences for the spaces K_Θ and K_{Θ° coincide. $\bigcirc$

Example. Let $\Theta = B_\Lambda$ and $\Theta^\circ = B_M$, where B_Λ and B_M are meromorphic Blaschke products associated with the sequences $\Lambda = \{\lambda_n\}$ and $M = \{\mu_n\}$ such that $|\lambda_n| \to \infty$, $|\mu_n| \to \infty$, $n \to \infty$. Assume that Λ and M satisfy the conditions of Theorem 1.3, that is, the function (6) is bounded. We have shown in the proof of Theorem 1.3 that for certain choice of the arguments $(\arg B_\Lambda - \arg B_M)^\sim \in L^\infty(\mathbb{R})$. Hence, the spaces K_{B_Λ} and K_{B_M} have the same complete interpolating sequences.

In the proof of Theorem 2.5 we will use a version of the description of the moduli of elements of a given model subspace obtained by K.M. Dyakonov [13] (see, also, [19, Lemma 4.2]).

Lemma 5.2. *Let Θ be a meromorphic inner function and let $m \ge 0$ be a continuous on $\mathbb{R}$ function such that $m \in L^2(\mathbb{R})$ and $\log m \in L^1(\Pi)$. Then $m = |f|$ for some function $f \in K_\Theta$ which has no real zeros if and only if there exists a meromorphic inner function I with an increasing continuous branch of the argument ψ such that*

$$2\widetilde{\log m} = \varphi - \psi. \tag{27}$$

Proof. Recall that $f \in K_\Theta$ if and only if $f \in H^2$ and $\Theta\overline{f} \in H^2$. Hence, the function $f = O_m I_1$, where $m \geq 0$, $m \in L^2(\mathbb{R})$, $\log m \in L^1(\Pi)$ and I_1 is an inner function, is in K_Θ if and only if

$$\overline{O_m}\,\overline{I}_1\Theta = O_m I_2$$

for some inner function I_2. Since Θ is meromorphic, both I_1 and I_2 are meromorphic inner functions and so

$$2\widetilde{\log m} = \varphi - \psi_1 - \psi_2 + 2\pi k, \qquad \text{a.e. on } \mathbb{R},$$

where ψ_1 and ψ_2 are continuous increasing branches of the arguments of I_1 and I_2 respectively and k is an integer-valued measurable function. Since $m = |f| \neq 0$ on $\mathbb{R}$, the function $\widetilde{\log m}$ is continuous on $\mathbb{R}$ and, therefore, k is a constant function. Choosing another branch of the argument of I_1 we can make $k = 0$. To obtain (27) we put $I = I_1 I_2$. The converse is analogous. ◯

Proof of Theorem 2.5: necessity. Let $\mathcal{H}(E) = \mathcal{H}(E_1)$. Denote by $K(\zeta, \cdot)$ the reproducing kernel of the space $\mathcal{H}(E)$ corresponding to the point ζ. Recall that

$$K(\zeta, z) = \frac{i}{2\pi} \cdot \frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{z - \overline{\zeta}}.$$

Take $\zeta_0 \in \mathbb{C}^+$. Since $K(\zeta_0, \cdot) \in \mathcal{H}(E_1)$, we have $h = K(\zeta_0, \cdot)/E_1 \in H^2$. Hence,

$$\frac{E(z)}{E_1(z)} = \frac{2\pi}{i\overline{E(\zeta_0)}} \cdot h(z) \cdot \frac{z - \overline{\zeta}_0}{1 - \Theta(z)\overline{\Theta(\zeta_0)}}.$$

Now inclusions $h \in \mathcal{N}_+$ and $\frac{z - \overline{\zeta}_0}{1 - \Theta(z)\overline{\Theta(\zeta_0)}} \in \mathcal{N}_+$ imply that $E/E_1 \in \mathcal{N}_+$. Clearly, we have also $E/E_1 \in L^2(\Pi)$. Analogously, $E_1/E \in \mathcal{N}_+ \cap L^2(\Pi)$.

Put $w = E/E_1$. Since $w \in \mathcal{N}_+$, we have, in particular, $\log|w| \in L^1(\Pi)$. Note also that w is an outer function, that is $w = O_{|w|} = |w| \exp(i\widetilde{\log|w|})$ on $\mathbb{R}$. We have already mentioned that $E(t)\exp(i\varphi(t)/2) \in \mathbb{R}$ and so

$$w^2(t) = |w(t)|^2 \exp(i(\varphi_1(t) - \varphi(t))).$$

Note that the functions $\widetilde{\log w}$ and $\varphi_1 - \varphi$ are continuous. Hence,

$$2\widetilde{\log|w|} = \varphi_1 - \varphi \tag{28}$$

up to a constant summand of tne form $2\pi n$. Choosing, if necessary, another branch of the argument of Θ_1, we may assume $n = 0$.

Since $\mathcal{H}(E) = \mathcal{H}(E_1)$, we have $wf \in K_{\Theta_1}$ for each $f \in K_\Theta$. In particular, if $m = |f|$, then $wm \in L^2(\mathbb{R})$. Assume that for some meromorphic inner function I we have $\exp((\varphi - \psi)^{\sim}) \in L^1(\mathbb{R})$ or, equivalently,

$$\varphi - \psi = 2\widetilde{\log m}$$

for some $m \in L^2(\mathbb{R})$ (recall that, up to a constant, $\tilde{\tilde{g}} = g$ for any $g \in L^1(\Pi)$). Then, by Lemma 5.2, $m = |f|$ for some $f \in K_\Theta$, and, in particular, $wm \in L^2(\mathbb{R})$. By (28),

$$2\widetilde{\log |w|m} = \varphi_1 - \psi,$$

which is equivalent to $\exp((\varphi_1 - \psi)^{\sim}) \in L^1(\mathbb{R})$.

Proof of Theorem 2.5: sufficiency. Now assume that E_1 and E_2 satisfy conditions 1 and 2 of the theorem. We will show that $wK_\Theta \subset K_{\Theta_1}$. Since $w = E/E_1 \in \mathcal{N}_+$, we have (28) up to a constant summand.

Let $f \in K_\Theta$. To prove the inclusion $wf \in K_{\Theta_1}$ it suffices to show that $wf \in L^2(\mathbb{R})$. Indeed, $wf \in \mathcal{N}_+$ and we have $wf \in H^2$ by the Smirnov theorem. By the definition of w,

$$w(t)/\overline{w(t)} = \overline{\Theta(t)}\Theta_1(t), \qquad t \in \mathbb{R}.$$

Therefore, we have on $\mathbb{R}$

$$\overline{w}\overline{f}\Theta_1 = \overline{w}\overline{\Theta}\Theta_1\overline{f}\Theta = w\Theta\overline{f} \in H^2$$

since $\Theta\overline{f} \in H^2$. Thus, $wf \in K_{\Theta_1}$.

Note also that we can verify the inclusion $wf \in L^2(\mathbb{R})$ only for functions f without real zeros. If $f(t_n) = 0$, $t_n \in \mathbb{R}$, we can choose a sequence y_n tending to zero sufficiently rapidly such that the function

$$g(z) = f(z) \prod_n \frac{z - t_n - iy_n}{z - t_n}$$

is also in $L^2(\mathbb{R})$. Clearly, $g \in K_\Theta$, g has no real zeros and $|g(t)| \geq |f(t)|$, $t \in \mathbb{R}$. Thus, if $wg \in L^2(\mathbb{R})$, then $wf \in L^2(\mathbb{R})$.

Now let $f \in K_\Theta$ and $f \neq 0$ on $\mathbb{R}$. Then, by Lemma 5.2,

$$2\widetilde{\log |f|} = \varphi - \psi$$

for some meromorphic inner function I with an argument ψ. By (28),

$$2\widetilde{\log |wf|} = \varphi_1 - \psi$$

and the condition 2 of the theorem implies that $wf \in L^2(\mathbb{R})$. ◯

Remark. We show that, in contrast to the criterion of Hruscev, Nikolski and Pavlov, the invertibility of the Toeplitz operator $T_{\Theta\overline{\Theta}_1}$ is not necessary for the equality $\mathcal{H}(E) = \mathcal{H}(E_1)$. We will use an example from [23]. Let $0 < \delta < 1/4$ and let $\lambda_0 = i$,

$$\lambda_n = \begin{cases} n - \delta + in^{-4\delta}, & n > 0, \\ n + \delta + i|n|^{-4\delta}, & n < 0. \end{cases}$$

Let $E(z) = \exp(-\pi i z)$ and let

$$E_1(z) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{z}{\overline{\lambda}_n}\right). \tag{29}$$

Then $\mathcal{H}(E) = \mathcal{H}(E_1)$. However, the Toeplitz operator $T_{\Theta\overline{\Theta}_1}$, where $\Theta = \exp(2\pi i z) = E^*/E$ and $\Theta_1 = B_\Lambda = E_1^*/E_1$, is not invertible.

It is well known (see [21]) that $T_{\Theta\overline{\Theta}_1}$ is invertible if and only if $P_\Theta|_{K_{\Theta_1}}$ (where P_Θ denotes the orthogonal projector in H^2 onto K_Θ) is an isomorphism onto K_Θ. Clearly,

$$P_\Theta \mathcal{K}^{(1)}_{\lambda_n} = \frac{i}{2\pi} P_\Theta \left(\frac{1}{z - \overline{\lambda}_n}\right) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(\lambda_n)}\Theta(z)}{z - \overline{\lambda}_n} = \mathcal{K}_{\lambda_n}.$$

Since $\|(z-\lambda_n)^{-1}\|_2 \asymp |n|^{2\delta}$, $n \neq 0$, and $\|\mathcal{K}_{\lambda_n}\|_2 \asymp 1$, it follows that the operator $P_\Theta|_{K_{\Theta_1}}$ is not invertible.

On the other hand, invertibility of $T_{\Theta\overline{\Theta}_1}$, even combined with some growth conditions on E/E_1 (say, E/E_1 and E_1/E belong to $\mathcal{N}_+ \cap L^2(\Pi)$), is not sufficient for $\mathcal{H}(E) = \mathcal{H}(E_1)$. Let $E(z) = \exp(-\pi i z)$ and define E_1 by (29) but with another zero sequence

$$\lambda_n = \begin{cases} n + i, & n \le 0, \\ n + \delta + i, & n > 0, \end{cases}$$

where $0 < \delta < 1/4$. Then $\Theta = E^*/E = \exp(2\pi i z)$ and $\Theta_1 = E_1^*/E_1 = B_\Lambda$. By the Ingham–Kadets 1/4 theorem, the system of exponentials $\{e^{i\lambda_n t}\}$ is a Riesz basis in $L^2(0, 2\pi)$ and, therefore, the Toeplitz operator $T_{e^{2\pi i z}\overline{B_\Lambda}}$ is invertible. However, $\mathcal{H}(E) \neq \mathcal{H}(E_1)$. Indeed, if $\mathcal{H}(E) = \mathcal{H}(E_1)$, then

$$|E(t)|^2 \varphi'(t) \asymp |E_1(t)|^2 \varphi_1'(t), \qquad t \in \mathbb{R}.$$

In our case, $\varphi' \asymp \varphi_1' \asymp 1$, but it is easily seen that E/E_1 is unbounded on $\mathbb{R}$ ($|E(x)/E_1(x)| \asymp |x|^\delta$, $|x \to \infty|$).

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