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Hardy spaces and divergence operators on strongly Lipschitz domains of $\mathbb{R}^n$

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Abstract

Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Consider an elliptic second-order divergence operator L (including a boundary condition on $\partial\Omega$) and define a Hardy space by imposing the non-tangential maximal function of the extension of a function f via the Poisson semigroup for L to be in L^1. Under suitable assumptions on L, we identify this maximal Hardy space with $H^1(\mathbb{R}^n)$ if $\Omega = \mathbb{R}^n$, with $H^1_r(\Omega)$ under the Dirichlet boundary condition, and with $H^1_z(\Omega)$ under the Neumann boundary condition.

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1. Introduction

Hardy spaces on $\mathbb{R}^n$, and especially $H^1(\mathbb{R}^n)$, were studied in great detail in the 1960s and 1970s. A nice review of this is in [22].

Originally defined by means of Riesz transforms (see the seminal paper of Stein and Weiss [26]), the usefulness of this space in analysis as a substitute for $L^1(\mathbb{R}^n)$

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comes from its many characterizations, beginning with the work of Fefferman–Stein [13]. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. For all $t>0$, define $\phi_t(x) = t^{-n}\phi(x/t)$. A locally integrable function f on $\mathbb{R}^n$ is said to be in $H^1(\mathbb{R}^n)$ if the vertical maximal function

$$\mathscr{M}f(x) = \sup_{t>0} |\phi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^n)$. If it is the case, define

$$||f||_{H^1(\mathbb{R}^n)} = ||\mathscr{M}f||_1.$$

Recall that a function $f \in H^1(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} f(x)\,dx = 0$.

Another equivalent definition of $H^1(\mathbb{R}^n)$ involves the non-tangential maximal function associated with the Poisson semigroup (or the heat semigroup) generated by Δ, the Laplace operator on $\mathbb{R}^n$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the following are equivalent:

$$f \in H^1(\mathbb{R}^n),$$

$$\sup_{|y-x|\leqslant t} |e^{-t(-\Delta)^{1/2}} f(y)| \in L^1(\mathbb{R}^n), \tag{1}$$

see [13, Theorem 11, p. 183].

The atomic decomposition obtained by Coifman and Latter was a key step in the theory (see [10] when $n=1$, [18] when $n \geqslant 2$). A function a on $\mathbb{R}^n$ is an $H^1(\mathbb{R}^n)$-atom if it is supported in a cube Q, has mean-value zero and satisfies $||a||_\infty \leqslant |Q|^{-1}$. Then, $f \in H^1(\mathbb{R}^n)$ if and only if $f = \sum_Q \lambda_Q a_Q$, where the a_Q's are $H^1(\mathbb{R}^n)$-atoms and the sequence of complex numbers $(\lambda_Q)_Q$ is in l^1. The norm $||f||_{H^1(\mathbb{R}^n)}$ is comparable with the infimum of $\sum_Q |\lambda_Q|$ taken over all such decompositions.

In recent years, a quite complete theory of Hardy spaces on domains has been developed [8,9,17,20,27]. The Hardy spaces are either defined in terms of restrictions to Ω of $H_1(\mathbb{R}^n)$ functions (this is $H^1_r(\Omega)$) or in terms of those $H_1(\mathbb{R}^n)$ functions having support in $\overline{\Omega}$ (this is $H^1_z(\Omega)$). There are also possible definitions using some "grand" maximal functions. For these spaces, atomic decompositions have been obtained, in particular, on special Lipschitz domains and bounded Lipschitz domains of $\mathbb{R}^n$. However, *is there a maximal characterization using the Poisson semigroup?* More precisely, replace in (1), $\mathbb{R}^n$ by Ω and take for Δ the Laplacian with Dirichlet or Neumann boundary condition. This defines two maximal Hardy spaces on Ω. The aim of the present paper is to identify each one with one of the "geometrical" Hardy spaces mentioned above.

It turns out that the choice of boundary condition is meaningful in the answer. Indeed, the heuristic is the following: if $p_t(x,y)$ denotes the Poisson kernel on Ω and f is in some Hardy space on Ω, one can use the theory on $\mathbb{R}^n$ as soon as $e^{-t(-\Delta)^{1/2}} f(x)$

can be written as

$$\int_{\mathbb{R}^n} \tilde{p}_t(x,y)\tilde{f}(y)\,dy,$$

where $\tilde{f}$ is an extension of f in $H^1(\mathbb{R}^n)$ and $\tilde{p}_t(x,\cdot)$ is a smooth extension of $p_t(x,\cdot)$. If $f \in H^1_r(\Omega)$, such an extension exists, but as it is not explicit, our only choice is that $\tilde{p}_t(x,\cdot)$ is the zero extension of $p_t(x,\cdot)$. This forces us to impose a Dirichlet boundary condition on the Laplacian ($p_t(x,\cdot)$ vanishes on $\partial\Omega$). If $f \in H^1_z(\Omega)$, $\tilde{f}$ is its zero extension, so that "any" smooth extension $\tilde{p}_t(x,\cdot)$ of $p_t(x,\cdot)$ is acceptable. This leads to a Neumann boundary condition on the Laplacian.

Another question we ask here is: *does the Laplacian play a specific role*? In other words, can it be replaced by another second-order elliptic operator? In [2], it was shown that $H^1(\mathbb{R})$ has a maximal characterization using the Poisson semigroup of elliptic operators. We give here an affirmative answer in higher dimensions and on domains, provided the elliptic operator satisfies a technical condition (for example any real elliptic operator will do). This also emphasizes the prominent role of the boundary condition in these questions.

Our main result is the following:

Theorem 1. *Let $\Omega = \mathbb{R}^n$ or Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and $L = -\operatorname{div} A\nabla$ be a second-order strongly elliptic operator with bounded measurable complex-valued coefficients. Assume that L satisfies the technical condition (G_∞).*

(a) *If $\Omega = \mathbb{R}^n$, one has $H^1_{\max,L}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.*
(b) *Under the Dirichlet boundary condition, if ${}^c\Omega$ is unbounded then one has $H^1_{\max,L}(\Omega) = H^1_r(\Omega)$.*
(c) *Under the Neumann boundary condition, one has $H^1_{\max,L}(\Omega) = H^1_z(\Omega)$.*

See below for precise definitions and notations. The technical condition may be valid or not under the Dirichlet or Neumann boundary condition. The statement is valid for general strongly Lipschitz domains (which include bounded and exterior domains).

The similar questions for local Hardy spaces have comparable answers. We shall obtain

Theorem 2. *Let Ω and L be as in Theorem* 1.

(a) *One has $h^1_{\max,L}(\mathbb{R}^n) = h^1(\mathbb{R}^n)$.*
(b) *Under the Dirichlet boundary condition, one has $h^1_{\max,L}(\Omega) = h^1_r(\Omega)$.*
(c) *Under the Neumann boundary condition, one has $h^1_{\max,L}(\Omega) = h^1_z(\Omega)$.*

Furthermore, if Ω is bounded, (b) *and* (c) *hold when L satisfies only* (G_1).

The plan of this paper is the following. First, we treat the case of global Hardy spaces on strongly Lipschitz domains: we introduce our maximal Hardy spaces and also some square functions. We then state the chain of gauges implying Theorem 1. We first prove the embeddings of the Hardy spaces in maximal Hardy spaces, then that maximal Hardy spaces imbed into spaces defined by square functions in L^1, and finally that the L^1-norm of the square functions controls the Hardy space norm. This follows a somewhat classical procedure in the subject but we had to take into account the boundary and the wide class of operators. In a second part, we study the corresponding theory for local Hardy spaces. We also present different vertical or non-tangential maximal functions characterizing our maximal Hardy space using the heat semigroup. We conclude with two appendices, one about kernel estimates and the other about the elementary geometry of Lipschitz domains.

Let us mention at this point that, using the recent work of Dafni et al. [8] in which they clarify the appropriate Hardy spaces on domains for $p<1$, one can certainly extend our results to a range of p's smaller than 1. We have not done so to keep the length of the paper reasonable.

2. Notation

In what follows, it is understood without mention that Ω belongs to the class of strongly Lipschitz domains of $\mathbb{R}^n$, that is Ω is a proper open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of these parts possibly unbounded.

This class includes special Lipschitz domains, bounded Lipschitz domains and exterior domains. Some facts about such domains are presented in Appendix B.

Some statements may be valid for a restricted class and we shall indicate this when it is the case.

We now describe the second-order operators considered in this work, the most typical example being the Laplacian with appropriate boundary condition. If $\Omega=\mathbb{R}^n$ or if Ω is a strongly Lipschitz domain of $\mathbb{R}^n$, we denote by $W^{1,2}(\Omega)$ the usual Sobolev space on Ω equipped with the norm $(||f||_2^2+||\nabla f||_2^2)^{1/2}$, whereas $W_0^{1,2}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

If $A:\mathbb{R}^n\to M_n(\mathbb{C})$ is a measurable function, define

$$||A||_\infty=\sup_{x\in\mathbb{R}^n,|\xi|=|\eta|=1}|A(x)\xi\cdot\eta|.$$

Here and subsequently in the paper, the notation sup is used for esssup and $\xi\cdot\eta$ denotes the hermitian product in $\mathbb{C}^n$. For all $\delta>0$, denote by $\mathscr{A}(\delta)$ the class of all measurable functions $A:\mathbb{R}^n\to M_n(\mathbb{C})$ satisfying, for all $x\in\mathbb{R}^n,\xi\in\mathbb{C}^n$:

$$||A||_\infty\leqslant\delta^{-1}\ \text{and}\ \operatorname{Re}A(x)\xi\cdot\xi\geqslant\delta|\xi|^2.$$

Denote by $\mathscr{A}$ the union of all $\mathscr{A}(\delta)$ for $\delta>0$.

When $A \in \mathscr{A}$ and V is a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$, denote by L the maximal-accretive operator on $L^2(\Omega)$ with largest domain $\mathscr{D}(L) \subset V$ such that

$$\langle Lf, g \rangle = \int_\Omega A\nabla f \cdot \nabla g, \ \forall f \in \mathscr{D}(L), \quad \forall g \in V. \tag{2}$$

We use the notation $L = (A, \Omega, V)$ to denote any operator defined as above. We say that L satisfies the Dirichlet boundary condition (DBC) when $V = W_0^{1,2}(\Omega)$, the Neumann boundary condition (NBC) when $V = W^{1,2}(\Omega)$.

Let $L = (A, \Omega, V)$ be as above. First, such an operator generates a semigroup $(e^{-tL})_{t \geqslant 0}$ of operators that is analytic (it has an extension to a complex half cone $|\arg z| < \mu$ for some $\mu \in (0, \pi/2)$) and contracting on $L^2(\Omega)$. Also, L has a unique maximal accretive square root $L^{1/2}$ so that $-L^{1/2}$ is the generator of the $L^2(\Omega)$-contracting semigroup $(P_t)_{t \geqslant 0}$ with $P_t = e^{-tL^{1/2}}$, that is the Poisson semigroup for L. We will need that P_t also acts on $L^1(\Omega)$. Let us then introduce a technical condition on L.

Definition 3. For $0 < \tau \leqslant +\infty$, we all (G_τ) the conjunction of (3) and (4) below: The kernel of e^{-tL}, denoted by $K_t(x, y)$, is a measurable function on $\Omega \times \Omega$ and there exist $C, \alpha > 0$ such that, for all $0 < t < \tau$ and almost every $x, y \in \Omega$,

$$|K_t(x, y)| \leqslant \frac{C}{t^{n/2}} e^{-\alpha \frac{|x-y|^2}{t}}. \tag{3}$$

For all $x \in \Omega$ and all $0 < t < \tau$, the functions $y \mapsto K_t(x, y)$ and $y \mapsto K_t(y, x)$ are Hölder continuous in Ω and there exist $C, \mu \in]0, 1]$ such that, for all $0 < t < \tau$ and all $x, y, y' \in \Omega$,

$$|K_t(x, y) - K_t(x, y')| + |K_t(y, x) - K_t(y', x)| \leqslant \frac{C}{t^{n/2}} \frac{|y - y'|^\mu}{t^{\mu/2}}. \tag{4}$$

When τ is finite, we set $\tau = 1$ without loss of generality.

For those readers only interested in the Laplacian or real symmetric operators (under DBC or NBC), this condition is always satisfied on $\mathbb{R}^n$ or on Lipschitz domains with $\tau = \infty$ except under NBC with Ω bounded for which we have τ finite. From the subordination formula, one obtains (see Appendix A)

Lemma 4. *When (G_∞) holds, the Poisson kernel of L, i.e. the kernel $p_t(x, y)$ of P_t satisfies*

$$|p_t(x, y)| \leqslant \frac{Ct}{(t + |x - y|)^{n+1}} \tag{5}$$

and

$$|p_t(x,y) - p_t(x,y')| + |p_t(y,x) - p_t(y',x)| \leqslant \frac{C}{t^n}\frac{|y-y'|^\mu}{t^\mu} \tag{6}$$

for all $t \in]0,\infty[$, *for some* $C>0$ *and* $\mu \in]0,1[$. *Furthermore, if* (G_1) *holds and* Ω *is bounded then* (5) *and* (6) *hold for* $0<t\leqslant 1$ *and for all* $t>1$ *one has*

$$|p_t(x,y)| \leqslant \frac{Ct}{(t+|x-y|)}. \tag{7}$$

Finally, an expression of the form $A \lesssim B$ between non-negative quantities A and B means that there is a constant C such that $A \leqslant CB$.

3. The global case

3.1. Strategy

Let us begin by recalling the definitions of various Hardy spaces on Ω.

A function f on Ω is said to be in $H^1_r(\Omega)$ if it is the restriction to Ω of a function $F \in H^1(\mathbb{R}^n)$. If $f \in H^1_r(\Omega)$, define $||f||_{H^1_r(\Omega)}$ by

$$||f||_{H^1_r(\Omega)} = \inf ||F||_{H^1(\mathbb{R}^n)},$$

the infimum being taken over all the functions $F \in H^1(\mathbb{R}^n)$ such that $F|_\Omega = f$.

A function f on Ω belongs to $H^1_z(\Omega)$ if the function F defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega \end{cases}$$

belongs to $H^1(\mathbb{R}^n)$. When $f \in H^1_z(\Omega)$, its norm $||f||_{H^1_z(\Omega)}$ is $||F||_{H^1(\mathbb{R}^n)}$. Note that it is a strict subspace of $H^1_r(\Omega)$ (in particular, a function f in $H^1_z(\Omega)$ satisfies $\int_\Omega f(x)\,dx = 0$, whereas this may not happen for $f \in H^1_r(\Omega)$). This space is nothing but the subspace of $H^1(\mathbb{R}^n)$ of all functions supported in $\overline{\Omega}$.

Since Ω is strongly Lipschitz, it is a space of homogeneous type and one also considers on Ω the Hardy space of Coifman and Weiss as defined in [12], which will be denoted in the sequel by $H^1_{\mathrm{CW}}(\Omega)$. An $H^1_{\mathrm{CW}}(\Omega)$-atom is a function a supported in $Q \cap \overline{\Omega}$, where Ω is a cube of $\mathbb{R}^n$ centered in Ω (but not necessarily included in Ω) with sidelength $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$ (this restriction is not really one as the traces of all such cubes give us the collection of all cubes of Ω) and satisfying

$$\int a(x)\,dx = 0 \text{ and } ||a||_\infty \leqslant |Q \cap \Omega|^{-1}.$$

If Ω has finite measure, the constant function $\frac{1}{|\Omega|}$ is not an atom with our definition in opposition with that of [12].

A function f is in $H^1_{\mathrm{CW}}(\Omega)$ if it can be written as

$$f = \sum_Q \lambda_Q a_Q,$$

where the a_Q's are $H^1_{\mathrm{CW}}(\Omega)$-atoms and $\sum_Q |\lambda_Q| < \infty$. The norm is defined as the infimum of such sums taken over all possible decompositions. Of course, $H^1_{\mathrm{CW}}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

We next introduce the maximal Hardy space on Ω associated with L. If (G_∞) holds and $f \in L^1_{\mathrm{loc}}(\Omega)$ with $y \mapsto |y|^{-n-1} f(y) \in L^1(\Omega)$, define, for all $x \in \Omega$,

$$f_L^*(x) = \sup_{y \in \Omega, t > 0,\ |y-x| < t} |P_t f(y)|.$$

Say that $f \in H^1_{\max,L}(\Omega)$ if $f_L^* \in L^1(\Omega)$ and define

$$||f||_{H^1_{\max,L}(\Omega)} = ||f_L^*||_{L^1(\Omega)}.$$

Note that $H^1_{\max,L}(\Omega)$ depends, in particular, on the boundary condition. Thanks to Lemma 4, P_t tends to the identity strongly in $L^1(\Omega)$ and, therefore, $H^1_{\max,L}(\Omega) \subset L^1(\Omega)$.

Now, we introduce some square functions. The first one is an area functional using all partial derivatives of $P_t f(x)$: for $x \in \Omega$, set

$$Sf(x) = \left(\int_{\Gamma(x)} t^{1-n} |\overline{\nabla} P_t f(y)|^2 \, dy \, dt \right)^{1/2},$$

with $\overline{\nabla} u = (\nabla u, \partial_t u)$, $|\overline{\nabla} u|^2 = |\nabla u|^2 + |\partial_t u|^2$, and where $\Gamma(x)$ is the cone defined by $\Gamma(x) = \{(y,t) \in \Omega \times]0, +\infty[; |y - x| < t\}$.

The second one is also an area functional restricted to time derivatives

$$sf(x) = \left(\int_{\Gamma(x)} t^{1-n} |\partial_t P_t f(y)|^2 \, dy \, dt \right)^{1/2}.$$

It is clear that $sf \leqslant Sf$ pointwise.

The following proposition gives the strategy of proof of Theorem 1.

Proposition 5. *Assume that L satisfies (G_∞). Let $f \in L^1(\Omega)$.*

(a) *If $\Omega = \mathbb{R}^n$, one has*

$$||f||_{H^1(\mathbb{R}^n)} \lesssim ||sf||_1 \leqslant ||Sf||_1 \lesssim ||f||_{H^1_{\max,L}(\mathbb{R}^n)} \lesssim ||f||_{H^1(\mathbb{R}^n)}.$$

(b) *under DBC with* ${}^c\Omega$ *unbounded, one has*

$$||f||_{H^1_r(\Omega)} \lesssim ||sf||_1 \leqslant ||Sf||_1 \lesssim ||f||_{H^1_{\max,L}(\Omega)} \lesssim ||f||_{H^1_r(\Omega)}.$$

(c) *Under NBC one has*

$$||f||_{H^1_z(\Omega)} \lesssim ||f||_{H^1_{\mathrm{CW}}(\Omega)} \lesssim ||sf||_1 \leqslant ||Sf||_1 \lesssim ||f||_{H^1_{\max,L}(\Omega)} \lesssim ||f||_{H^1_z(\Omega)}.$$

The implicit constants depend neither on f, *nor its* L^1 *norm.*

Remark 6. (1) The restriction on Ω in (b) appears in the very left-hand inequality. We think this inequality is false for ${}^c\Omega$ bounded but we have not found an argument.

(2) The assumption (G_∞) in (c) forces Ω to be unbounded under NBC, as (G_∞) is never valid on bounded domains. The case of bounded domains will be discussed with local Hardy spaces.

(3) The relevance of the technical assumption on f will be addressed at the end of Section 3.4.

(4) A byproduct of (c) is the fact that $H^1_{\mathrm{CW}}(\Omega) = H^1_z(\Omega)$ (while the inclusion $\subset$ is, as we shall see, not too hard the converse is a deeper fact). In the case of special Lipschitz domain, or bounded domains and local Hardy spaces, this can also be seen as a consequence of the atomic decomposition for $H^1_z(\Omega)$ in [9] (but it was not mentioned there).

In the next sections we shall prove the three chains of inequalities in parallel, going from right to left, introducing when needed further ingredients.

3.2. *From Hardy spaces to maximal Hardy spaces*

Recall that $p_t(x,y)$ is the Poisson kernel for L and that $P_t = e^{-tL^{1/2}}$.

Proof of $||f||_{H^1_{\max,L}(\mathbb{R}^n)} \lesssim ||f||_{H^1(\mathbb{R}^n)}$: From the atomic decomposition, it suffices to let f be an $H^1(\mathbb{R}^n)$-atom. Indeed, for such an atom a it is plain to see, following classical arguments and using (G_∞), that $||a_L^*||_1 \leqslant C$ (see, for instance, [14, Chapter III, Theorem 3.4]). Briefly, write

$$P_t a(x) = \int_Q p_t(x,y) a(y)\, dy,$$

where Q is the support of a. When $x \in 2Q$ use

$$|P_t a(x)| \leqslant ||a||_\infty \int |p_t(x,y)|\, dy \lesssim |Q|^{-1}.$$

When $x \notin 2Q$ then use the moment condition on a to write

$$P_t a(x) = \int_Q (p_t(x,y) - p_t(x,y'))a(y)\, dy,$$

where y' is any point in Q, for example the center of Q. Then use the regularity of $p_t(x,y)$ in Lemma 4 to get $|P_t a(x)| \lesssim \frac{\ell(\Omega)^\mu}{|x-y'|^{n+\mu}}$, with $\ell(Q)$ the sidelength of Q.

Proof of $||f||_{H^1_{\max,L}(\Omega)} \lesssim ||f||_{H^1_r(\Omega)}$ *under DBC*: Since $f \in H^1_r(\Omega)$ it has an extension to $H^1(\mathbb{R}^n)$ with comparable norm. This extension can be decomposed into $H^1(\mathbb{R}^n)$-atoms. It suffices to look at them. Let a be an $H^1(\mathbb{R}^n)$-atom supported in a cube Q. In such a case, one has

$$P_t a(x) = \int_{Q\cap\Omega} p_t(x,y)a(y)\,dy$$

If $Q \subset \Omega$ then the argument above can be repeated. If $Q \cap \Omega = \emptyset$ there is nothing to do. It remains to see the case where Q intersects the boundary of Ω. Follow the argument above and choose for y' a point in Q on the boundary of Ω. This time, instead of using the moment condition on a, we use the Dirichlet boundary condition in the form of $p_t(x,y') = 0$ since $y' \in \partial\Omega$.

Proof of $||f||_{H^1_{\max,L}(\Omega)} \lesssim ||f||_{H^1_z(\Omega)}$ *under NBC*: For all $t>0$ and $x \in \Omega$, define

$$F_{x,t}(y) = t^n\left(1+\frac{|x-y|}{t}\right)^{n+1} p_t(x,y), \quad y \in \Omega.$$

An easy consequence of Lemma 4 is that for some $v \in (0,1]$

$$|F_{x,t}(y)| \leqslant C$$

and

$$|F_{x,t}(y) - F_{x,t}(y')| \leqslant C\left(\frac{|y-y'|}{t}\right)^v$$

for all $y, y' \in \Omega$. Thus, the function $F_{x,t}$ may be extended to a bounded Hölder continuous function on $\overline{\Omega}$, then on $\mathbb{R}^n$ (see [23, Chapter 6, Theorem 3, p. 174]). If this extension is denoted by $\tilde{F}_{x,t}$, one has

$$|\tilde{F}_{x,t}(y)| \leqslant C_0 C$$

and

$$|\tilde{F}_{x,t}(y) - \tilde{F}_{x,t}(y')| \leqslant C_0 C\left(\frac{|y-y'|}{t}\right)^v,$$

for all $y, y' \in \mathbb{R}^n$, where C_0 only depends on Ω. Define now

$$\tilde{p}_t(x,y) = t^{-n}\left(1 + \frac{|x-y|}{t}\right)^{-n-1} \tilde{F}_{x,t}(y).$$

Then, one has

$$|\tilde{p}_t(x,y)| \leqslant Ct^{-n}\left(1 + \frac{|x-y|}{t}\right)^{-n-1} \tag{8}$$

and

$$|\tilde{p}_t(x,y) - \tilde{p}_t(x,y')| \leqslant Ct^{-n}\left(\frac{|y-y'|}{t}\right)^{\nu} \tag{9}$$

for all $x \in \Omega, y, y' \in \mathbb{R}^n$ and all $t > 0$. Moreover, for all $t > 0$ and all $x, y \in \Omega, p_t(x,y) = \tilde{p}_t(x,y)$.

Consider now a function $f \in H^1_z(\Omega)$, extended by 0 outside Ω, so that $f \in H^1(\mathbb{R}^n)$ and $||f||_{H^1_z(\Omega)} = ||f||_{H^1(\mathbb{R}^n)}$. For all $x \in \Omega$, one has

$$\int_\Omega p_t(x,y) f(y)\, dy = \int_{\mathbb{R}^n} \tilde{p}_t(x,y) f(y)\, dy. \tag{10}$$

Using the atomic decomposition of f into $H^1(\mathbb{R}^n)$-atoms and estimates (8) and (9) for $\tilde{p}_t$, one deduces as before from (10) that

$$||f_L^*||_{L^1(\Omega)} \leqslant C||f||_{H^1(\mathbb{R}^n)}.$$

3.3. *From maximal functions to square functions*

This section is the most technical part of the paper. Here, we prove that maximal norms dominate the square functions namely

Proposition 7. *Assume that* (G_∞) *holds. For all* $f \in H^1_{\max,L}(\Omega)$, $||Sf||_1 \lesssim ||f||_{H^1_{\max,L}(\Omega)}$.

We have stated one inequality as the argument will work in the three different situations considered in Theorem 1.

The proof follows ideas from [13, Theorem 8, p. 161] for the Laplace operator on $\mathbb{R}^n$ and [11, Section 6]. See also [2, Lemme II.10] for the case of elliptic operators in one dimension. It relies on a "good λ" inequality. Here, no further condition on Ω is needed.

In order to proceed, we need square functions with different apertures. Define for $x \in \Omega$,

$$S_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} t^{1-n} |\overline{\nabla} P_t f(y)|^2\, dy\, dt\right)^{1/2}$$

and

$$S_\alpha^{\varepsilon,R} f(x) = \left(\int_{\Gamma_\alpha^{\varepsilon,R}(x)} t^{1-n} |\overline{\nabla} P_t f(y)|^2 \, dy \, dt \right)^{1/2},$$

with $\overline{\nabla} u = (\nabla u, \partial_t u)$, $|\overline{\nabla} u|^2 = |\nabla u|^2 + |\partial_t u|^2$, $P_t = e^{-tL^{1/2}}$ and where $\Gamma_\alpha(x)$ and $\Gamma_\alpha^{\varepsilon,R}(x)$ are, respectively, the cones and the truncated cones defined by

$$\Gamma_\alpha(x) = \{(y,t) \in \Omega \times]0, +\infty[; |y-x| < \alpha t\}$$

and

$$\Gamma_\alpha^{\varepsilon,R}(x) = \{(y,t) \in \Omega \times]\varepsilon, R[; |y-x| < \alpha t\},$$

for $\alpha > 0$, $0 < \varepsilon < R < +\infty$. We can also write $S_\alpha = S_\alpha^{0,\infty}$. Here, it is convenient to choose for the norm on $\mathbb{R}^n$ the supremum norm (for which balls are cubes with sides parallel to the axes).

The first lemma is of technical nature.

Lemma 8. *Assume $\alpha < 1$. Then, one has for $f \in L^2(\Omega)$,*

$$S_\alpha^{\varepsilon,R} f(x) \leqslant C(1 + |\ln(R/\varepsilon)|) f_L^*(x)$$

for some constant depending on α.

Proof. The truncated square function is well defined for $f \in L^2(\Omega)$ since $\overline{\nabla} P_t$ is bounded on $L^2(\Omega)$. Let us also recall that $u_t(y) = P_t f(y)$ satisfies the elliptic equation $\overline{\nabla} \cdot B \overline{\nabla} u_t(y) = 0$ (in the weak sense on $\Omega \times]0, \infty[$) where B is the $(n+1) \times (n+1)$ block diagonal matrix with components A and 1. Moreover, we have prescribed Dirichlet or Neumann data on the lateral boundary $\partial\Omega \times]0, \infty[$. Hence, we have interior and boundary Caccioppoli inequalities (see [5]): for some $\rho > 0$ and $C \geqslant 0$ depending on Ω and ellipticity,

$$\int_E |\overline{\nabla} u_t(y)|^2 \, dy \, dt \leqslant C r^{-2} \int_{\tilde{E}} |u_t(y)|^2 \, dy \, dt$$

for all sets $E = B((z,\tau), r) \cap (\Omega \times]0, \infty[)$ with $\tilde{E} = B((z,\tau), 2r) \cap (\Omega \times]0, \infty[)$ provided $x \in \Omega$, $\tau > 0$ and $r \leqslant \inf(\rho, \tau)/4$. Here $B((z,\tau), r)$ is the open ball defined by $\sup(|z-y|, |\tau - t) < r$.

For $(z,\tau) \in \Gamma_\alpha^{\varepsilon,R}(x)$, let $E_{(z,\tau)} = B((z,\tau), r) \cap (\Omega \times]0, \infty[)$ with $r = \delta \inf(\tau, \rho)$. Here δ is some small number. By a Besicovitch covering argument, pick a subcollection $E_j = E_{(z_j,\tau_j)}$ covering $\Gamma_\alpha^{\varepsilon,R}(x)$ and having bounded overlap. Remark that $(y,t) \in E_j$ implies $t \sim d_j$, the distance from E_j to the bottom boundary $\Omega \times \{0\}$. Remark also that if δ is small enough, $(y,t) \in \tilde{E}_j$ implies $(y,t) \in \Gamma_1(x)$, hence $|u_t(y)| \leqslant f_L^*(x)$. Thus, we obtain from the bounded overlap and Caccioppoli's

inequality,

$$S_\alpha^{\varepsilon,R} f(x)^2 \leqslant C \sum_j d_j^{1-n} r_j^{-2} |\tilde{E}_j| f_L^*(x)^2.$$

Observe that $|\tilde{E}_j| \leqslant C|E_j|$ and so that the bounded overlap of the E_j's again easily yields by inspection,

$$\sum_j d_j^{1-n} r_j^{-2} |\tilde{E}_j| \leqslant C(1 + |\ln(R/\varepsilon)|). \qquad \square$$

Next, in order to compensate for the lack of pointwise regularity for $P_t f(x)$ we introduce variants of the truncated square functions by adapting the well-known sweeping technique attributed to Titchmarsh. Set

$$\tilde{S}_\alpha^{\varepsilon,R} f(x) = \left(\int_1^2 \int_{\Gamma_{\alpha/a}^{a\varepsilon,aR}(x)} t^{1-n} |\overline{\nabla} P_t f(y)|^2 \, dy \, dt \, da \right)^{1/2} \quad x \in \Omega.$$

Fairly elementary arguments show that

$$S_\alpha^{2\varepsilon,R} f \leqslant \tilde{S}_\alpha^{\varepsilon,R} f \leqslant S_{2\alpha}^{\varepsilon,2R} f.$$

Our good λ inequality is

Lemma 9. *There exists $c > 0$ such that, for all $0 < \gamma \leqslant 1$, all $\lambda > 0$, all $0 < \varepsilon < R < \infty$ and all $f \in H^1_{\max,L} \cap L^2(\Omega)$,*

$$|\{x \in \Omega; \tilde{S}_{1/20}^{\varepsilon,R} f(x) > 2\lambda, f_L^*(x) \leqslant \gamma\lambda\}| \leqslant c\gamma^2 |\{x \in \Omega; \tilde{S}_{1/2}^{\varepsilon,R} f(x) > \lambda\}|.$$

We will also use the comparability of the square functions. See [11, Proposition 4, p. 309].

Lemma 10. *For $\alpha, \beta > 0, 0 \leqslant \varepsilon < R \leqslant +\infty$, one has*

$$||S_\alpha^{\varepsilon,R} f||_1 \sim ||S_\beta^{\varepsilon,R} f||_1,$$

where the implicit constants do not depend on f, ε, R.

Let us deduce Proposition 7. Assume first that $f \in H^1_{\max,L} \cap L^2(\Omega)$. As a consequence of Lemma 9, by integrating both sides with respect to λ, one obtains

$$||\tilde{S}_{1/20}^{\varepsilon,R} f||_1 \leqslant \gamma^{-1} ||f_L^*||_1 + c\gamma^2 ||\tilde{S}_{1/2}^{\varepsilon,R} f||_1.$$

Thanks to Lemma 10 and the comparisons between the square functions and their variants, one has

$$||S_1^{\varepsilon,R}f||_1 \leqslant C||\tilde{S}_{1/20}^{\varepsilon/2,R}f||_1$$

and by Lemma 8

$$||\tilde{S}_{1/2}^{\varepsilon/2,R}f||_1 \leqslant ||S_1^{\varepsilon/2,2R}f||_1 \leqslant ||S_1^{\varepsilon/2,\varepsilon}f||_1 + ||S_1^{\varepsilon,R}f||_1 + ||S_1^{R,2R}f||_1 \leqslant ||S_1^{\varepsilon,R}f||_1 + C||f_L^*||_1.$$

Hence, by choosing γ appropriately and using the a priori knowledge that $||S_1^{\varepsilon,R}f||_1 < +\infty$ one obtains

$$||S_1^{\varepsilon,R}f||_1 \leqslant C||f_L^*||_1.$$

By letting $\varepsilon \downarrow 0$ and $R \uparrow +\infty$, the conclusion of Proposition 7 in the case $f \in L^2$ follows.

To complete the proof of Proposition 7, we have to relax the assumption $f \in L^2(\Omega)$. But, if $f_L^* \in L^1(\Omega)$, then $f \in L^1(\Omega)$ and together with the kernel estimates on the kernel of P_t, one has $f_\varepsilon \in L^2(\Omega)$ for all $\varepsilon > 0$, where $f_\varepsilon(x) = P_\varepsilon f(x)$. It follows that $||Sf_\varepsilon||_1 \lesssim ||(f_\varepsilon)_L^*||_1 \leqslant ||f_L^*||_1$. Letting $\varepsilon \downarrow 0$, one obtains $||Sf||_1 \leqslant C||f_L^*||_1$ by monotone convergence. $\square$

We turn to the proof of Lemma 9. In the next argument, ε, R, λ are fixed. Also $f \in H^1_{\max,L} \cap L^2(\Omega)$. Define $O = \{x \in \Omega; \tilde{S}_{1/2}^{\varepsilon,R}f(x) > \lambda\}$. We may assume that $O \neq \Omega$. Let $O = \bigcup_k Q_k$ be a Whitney decomposition of O (with respect to Ω) by dyadic cubes (of $\mathbb{R}^n$), so that, for all $k, 2Q_k \subset O \subset \Omega$, but $4Q_k$ intersects $\Omega \backslash O$. Since $\{\tilde{S}_{1/20}^{\varepsilon,R}f > 2\lambda\} \subset \{\tilde{S}_{1/2}^{\varepsilon,R}f > \lambda\}$, it is enough to show that

$$|\{x \in Q_k; \tilde{S}_{1/20}^{\varepsilon,R}f(x) > 2\lambda,\ f_L^*(x) \leqslant \gamma\lambda\}| \leqslant c\gamma^2|Q_k|.$$

From now on, fix k and denote by l the side length of Q_k.

If $x \in Q_k$,

$$\tilde{S}_{1/20}^{\sup(10l,\varepsilon),R}f(x) \leqslant \lambda.$$

Indeed, pick $x_k \in 4Q_k$ with $x_k \notin O$. If $|y - x| < \frac{t}{20}$ and with $t \geqslant \sup(10l, \varepsilon)$, then one has $|x_k - y| < \frac{t}{20} + 4l \leqslant \frac{t}{2}$. Hence $\tilde{S}_{1/20}^{\sup(10l,\varepsilon),R}f(x) \leqslant \tilde{S}_{1/2}^{\sup(10l,\varepsilon),R}f(x_k) \leqslant \lambda$.

If $\varepsilon \geqslant 10l$, we are done. Otherwise, using, $\tilde{S}_{1/20}^{\varepsilon,R}f(x) \leqslant \tilde{S}_{1/20}^{\varepsilon,10l}f(x) + \tilde{S}_{1/20}^{10l,R}f(x)$, it remains to show that

$$|\{x \in Q_k \cap F;\ g(x) > \lambda\}| \leqslant c\gamma^2|Q_k|,$$

where

$$g(x) = \tilde{S}_{1/20}^{\varepsilon,10l}f(x)$$

and

$$F = \{x \in \Omega; f_L^*(x) \leqslant \gamma\lambda\}.$$

By Tchebytchev's inequality, this follows from

$$\int_{Q_k \cap F} g^2 \leqslant c\gamma^2\lambda^2 |Q_k|.$$

We note that the condition (G_∞) combined with Lemma 4 and analyticity of the heat semigroup (see Appendix A) implies that F is a closed set of Ω as $(x,t) \mapsto u_t(x) = P_t f(x)$ is a continuous function.

If $5l \leqslant \varepsilon$, then the argument using Caccioppoli's inequality shows that

$$\int_{Q_k \cap F} g^2 \leqslant c \int_{Q_k \cap F} (f_L^*)^2 \leqslant c\gamma^2\lambda^2 |Q_k \cap F|.$$

Assume from now on that $\varepsilon < 5l$. By geometric considerations,

$$\int_{Q_k \cap F} g(x)^2\, dx \leqslant c \int_1^2 \int_{\mathscr{E}_a} t|\overline{\nabla} u_t(y)|^2\, dy\, dt\, da$$

where

$$\mathscr{E}_a = \{(y,t) \in \Omega \times]\varepsilon a, 10la[; a\psi(y) < t\}$$

with $\psi(y)$ the Lipschitz function equal to 20 dist $(y, Q_k \cap F)$. Recall also that $u_t(y) = P_t f(y)$.

Observe that $\mathscr{E}_a = \{(y, at); (y,t) \in \mathscr{E}_1\}$. Define $E = \{y; (y,t) \in \mathscr{E}_1\}$: this is an open set in Ω. For a connected component C of E, we let $\mathscr{C}_a = \{(y,t) \in \mathscr{E}_a; y \in C\}$. It suffices to show that

$$\int_1^2 \int_{\mathscr{C}_a} t|\overline{\nabla} u_t(y)|^2\, dy\, dt\, da \leqslant c\gamma^2\lambda^2 |C|.$$

Indeed, summing over all connected components of E, we get

$$\int_1^2 \int_{\mathscr{E}_a} t|\overline{\nabla} u_t(y)|^2\, dy\, dt\, da \leqslant c\gamma^2\lambda^2 |E|,$$

and it remains to observe that $E \subset 2Q_k$. Indeed, if $y \in E$, there is a point (y,t) above contained $\mathscr{E}_1$, hence there exists $x \in Q_k \cap F$ such that $|y - x| < \frac{t}{20}$. Since $t < 10l$, we have $|y - x| < \frac{l}{2}$ and the desired inclusion follows.

We next fix a connected component C of E. Consider $a \in]1,2[$ and note that $\mathscr{C}_a$ is connected and has Lipschitz boundary. The ellipticity condition for A

shows that

$$\int_{\mathscr{C}_a} t|\overline{\nabla} u_t(y)|^2\, dy\, dt \leqslant C \operatorname{Re} \int_{\mathscr{C}_a} tB\overline{\nabla} u_t(y) \cdot \overline{\nabla} u_t(y)\, dy\, dt = C \operatorname{Re}\, I_a,$$

where B is the $(n+1)\times(n+1)$ block diagonal matrix with components A and 1. The function $u_t(y)$ satisfies the equation $\overline{\nabla}\cdot B\overline{\nabla} u_t(y) = 0$ (in the weak sense on $\Omega\times]0,\infty[$) so that we wish to integrate by parts.

To do so let us make some observations. We claim that for $(y,t)\in\overline{\mathscr{C}_a}$, then $y\in 2Q_k\subset\Omega$ and $(y,t)\in\mathscr{E}_1$. Indeed, since F is closed, there exists $x\in Q_k\cap F$ such that $|y-x|\leqslant\frac{t}{20a}$. Since $t\leqslant 10la$, we have $|y-x|\leqslant\frac{l}{2}$ and the first claim is true. Moreover, $|y-x|\leqslant\frac{t}{20a}<t$, hence the second claim.

It follows in particular that $\overline{\mathscr{C}_a}$ remains far from the boundary of $\Omega\times]0,\infty[$, so that we do not care about the boundary values of $u_t(y)$, and that $|u_t(y)|\leqslant\gamma\lambda$ on $\overline{\mathscr{C}_a}$.

The Green–Riemann formula shows that I_a is equal to

$$-\int_{\mathscr{C}_a} \partial_t u_t(y)\overline{u_t(y)}\, dy\, dt + \int_{\partial\mathscr{C}_a} tB\overline{\nabla} u_t(y)\cdot N_a(y,t)\overline{u_t(y)}\, d\sigma_a(y,t).$$

In this computation, $N_a(y,t)$ is the unit normal vector outward $\mathscr{C}_a$ whereas $d\sigma_a$ is the surface measure over $\partial\mathscr{C}_a$.

Moreover, the Green–Riemann formula again yields

$$2\operatorname{Re}\int_{\mathscr{C}_a} \partial_t u_t(y)\overline{u_t(y)}\, dy\, dt = \int_{\partial\mathscr{C}_a} |u_t(y)|^2 N_a(y,t)\cdot(0,\ldots,0,1)\, d\sigma_a(y,t).$$

Finally,

$$\begin{aligned}\int_{\mathscr{C}_a} t|\overline{\nabla} u_t(y)|^2\, dy\, dt \leqslant{}& C\int_{\partial\mathscr{C}_a} |u_t(y)|^2\, d\sigma_a(y,t)\\ &+ C\int_{\partial\mathscr{C}_a} t|u_t(y)||\overline{\nabla} u_t(y)|\, d\sigma_a(y,t).\end{aligned}$$

Since $|u_t(y)|\leqslant\gamma\lambda$ on $\partial\mathscr{C}_a$, we obtain that

$$\int_1^2\int_{\partial\mathscr{C}_a} |u_t(y)|^2\, d\sigma_a(y,t)\, da \leqslant \gamma^2\lambda^2\int_1^2\int_{\partial\mathscr{C}_a} d\sigma_a(y,t)\, da.$$

We claim that

$$\int_1^2\int_{\partial\mathscr{C}_a} d\sigma_a(y,t)\, da \leqslant c|C|.$$

Indeed, this integral is bounded by $c\int_{\mathscr{G}}\frac{dz\, ds}{s}$ where $\mathscr{G}$ is the union of the sets $\partial\mathscr{C}_a$ for $1<a<2$. This is the set of points (z,s) with $z\in C$ and $\varepsilon<s<2\varepsilon$ or $\psi(z)<s<2\psi(z)$ or $10l<s<20l$. The claim follows readily.

It remains to establish

$$\int_1^2 \int_{\partial \mathscr{C}_a} t|u_t(y)||\overline{\nabla} u_t(y)|\, d\sigma_a(y,t)\, da \leqslant c\gamma^2\lambda^2|C|.$$

Using the previous notation and a change of variables, this integral is bounded by

$$\gamma\lambda \int_{\mathscr{G}} |\overline{\nabla} u_t(y)|\, dy\, dt.$$

Pick a covering of $\mathscr{G}$ with bounded overlap by balls $B_j = B((x_j, t_j), \frac{\varepsilon t_j}{20})$. Remark that $(x,t) \in B_j$ implies $t \sim t_j \sim r(B_j)$, the radius of B_j. Then using Hölder's inequality and again Caccioppoli's inequality

$$\begin{aligned}
\int_{\mathscr{G}} |\overline{\nabla} u_t(y)|\, dy\, dt &\leqslant c \sum \int_{B_j} |\overline{\nabla} u_t(y)|\, dy\, dt \\
&\leqslant c \sum |B_j|^{1/2} r(B_j)^{-1} \left(\int_{2B_j} |u_t(y)|^2\, dy\, dt \right)^{1/2} \\
&\leqslant c\gamma\lambda \sum |B_j| r(B_j)^{-1} \\
&\leqslant c\gamma\lambda \int_{\tilde{\mathscr{G}}} \frac{dz\, ds}{s} \\
&\leqslant c\gamma\lambda |C|.
\end{aligned}$$

Here, $\tilde{\mathscr{G}}$ is a set like $\mathscr{G}$ but slightly enlarged: it is contained set of points (z,s) with $z \in C$ and $\varepsilon/2 < s < 4\varepsilon$ or $\psi(z)/2 < s < 4\psi(z)$ or $5l < s < 40l$. □

Remark 11. (1) The argument simplifies when L is the Laplacian on Ω. Indeed, then $(y,t) \mapsto u_t(y)$ is harmonic so that the Caccioppoli inequality can be replaced by the mean value property and one can proceed directly using the square functions (and not their averaged variants).

(2) When $\Omega \neq \mathbb{R}^n$, the Whitney cubes Q_k are designed to stay away from the boundary of Ω so that interior estimates suffice. In fact the boundary conditions appear in a subtle way in the first technical lemma. When $\Omega = \mathbb{R}^n$, one can proceed directly (without introducing the Q_k's) and prove the good λ inequality on $\mathbb{R}^n$. Details are left to the reader.

3.4. *From square functions to Hardy spaces via tent spaces*

In this section, we shall use the tent spaces, Carleson measures and duality to obtain the last part of the chains of desired inequalities.

Proposition 12. *Under the assumptions of Proposition 5, we have* $\|f\|_{H^1} \lesssim \|sf\|_1$ *whenever the right-hand side is finite and* $f \in L^1(\Omega)$. *Here,* H^1 *is* $H^1(\mathbb{R}^n)$ *if* $\Omega = \mathbb{R}^n, H^1_r(\Omega)$ *under DBC and* $H^1_{\mathrm{CW}}(\Omega)$ *under NBC.*

This will complete chains (a) and (b) of inequalities in Proposition 5 and for chain (c), we further need the following lemma

Lemma 13. *For any strongly Lipschitz domain, one has* $H^1_{\mathrm{CW}}(\Omega) \subset H^1_z(\Omega)$ *with* $\|f\|_{H^1_z(\Omega)} \lesssim \|f\|_{H^1_{\mathrm{CW}}(\Omega)}$.

To prove the lemma, it suffices to show that any $H^1_{\mathrm{CW}}(\Omega)$-atom a has its zero extension in $H^1(\mathbb{R}^n)$. Call again a this extension. By assumption, there is a cube Q of $\mathbb{R}^n$, centered in Ω with $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$ containing the support of a and $\|a\|_\infty \leqslant |Q \cap \Omega|^{-1}$. Furthermore, a has mean value zero. We conclude with the following geometric claim proved in Appendix B.

Claim 1. *There is a constant* $C > 0$ *such that if* Q *is a cube centered in* Ω *with* $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$ *then* $|Q \cap \Omega| \geqslant C|Q|$.

Let us begin the proof of Proposition 12. We first use some ideas from the H^∞-functional calculus for L (this need not the assumption (G_∞)). We start from the equality

$$\int_0^{+\infty} (-tz^{1/2}e^{-tz^{1/2}})(-tz^{1/2}e^{-tz^{1/2}})\frac{dt}{t} = \frac{1}{4} \tag{11}$$

valid for all $z \neq 0$ in a sector $|\arg z| < \mu$ with $\mu \in]\omega, \pi[$. As a consequence, one has

$$\mathrm{Id} = 4\int_0^{+\infty} (tL^{1/2}P_t)(tL^{1/2}P_t)\frac{dt}{t},$$

where the integral converges strongly in $L^2(\Omega)$. Note that $tL^{1/2}P_t = -t\partial_t P_t$. Thus, if $f, \phi \in L^2(\Omega)$ one has

$$\int_\Omega f(y)\overline{\phi(y)}\,dy = 4\int_{\mathbb{R}^n}\int_0^{+\infty} (t\partial_t P_t)(f)(y)\overline{(t\partial_t P_t^*)(\phi)(y)}\frac{dy\,dt}{t}. \tag{12}$$

Now let us assume that $f \in L^2(\Omega)$ with $\|sf\|_1 < \infty$ and ϕ is continuous with compact support in Ω. Well-known arguments from the theory of tent spaces on spaces of homogeneous type (see [11,25]) show that the right-hand side of (12) is bounded by

$$C\|sf\|_{L^1(\Omega)}\|T\phi\|_{L^\infty(\Omega)},$$

where $T\phi$ is defined as follows: When Q is a cube of $\mathbb{R}^n$ with center $x_Q \in \Omega$ and radius $r = \ell(Q)/2 \leqslant \operatorname{diam}(\Omega)$, define the tent over Q by

$$T(Q) = \{(y,t) \in \Omega \times]0, +\infty[; |y - x_Q| < r - t\}.$$

We recall that $|\ |$ is the sup norm on $\mathbb{R}^n$. When ϕ is locally integrable on Ω with $\phi(y)(1+|y|)^{-n-1}$ integrable on Ω, set

$$T\phi(x) = \left(\sup_{Q \ni x} \frac{1}{|Q \cap \Omega|} \int_{T(Q)} |\partial_t P_t^* \phi(y)|^2 t \, dy \, dt \right)^{1/2},$$

where the supremum is taken over all the above cubes Q that contain x.

The next step is to control the operator T. We need to introduce some BMO spaces on Ω. Recall that a locally integrable function f on $\mathbb{R}^n$ is said to be in $BMO(\mathbb{R}^n)$ if

$$||f||_{BMO(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < +\infty$$

where the supremum is taken over all the cubes Q of $\mathbb{R}^n$. Here, $f_E = \frac{1}{|E|} \int_E f(x) \, dx$ is the mean of f over E.

Next, the space $BMO_z(\Omega)$ is defined as being the space of all functions in $BMO(\mathbb{R}^n)$ supported in $\overline{\Omega}$, equipped with the norm $||f||_{BMO_z(\Omega)} = ||f||_{BMO(\mathbb{R}^n)}$.

The space $BMO_r(\Omega)$ is defined as being the space of all restrictions to Ω of functions in $BMO(\mathbb{R}^n)$. If $f \in BMO_r(\Omega)$ set

$$||f||_{BMO_r(\Omega)} = \inf ||F||_{BMO(\mathbb{R}^n)},$$

the infimum being taken over all the functions $F \in BMO(\mathbb{R}^n)$ such that $F|_\Omega = f$.

A locally integrable function ϕ on Ω is in $BMO_{\mathrm{CW}}(\Omega)$ if

$$||\phi||_{BMO_{\mathrm{CW}}(\Omega)} = \sup \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} |\phi(x) - \phi_{Q \cap \Omega}| \, dx < +\infty,$$

where the supremum is taken over all cubes of $\mathbb{R}^n$ centered in Ω with $\ell(Q) \leqslant 2 \operatorname{diam}(\Omega)$. This is the space defined in [12]. A slight variation is that the indicator function of Ω is not supposed in $BMO_{\mathrm{CW}}(\Omega)$ when Ω has finite measure.

Proposition 14. *Assume that* $L = (A, \Omega, V)$ *satisfies* (3).

(a) *If* $\Omega = \mathbb{R}^n$, *then for all* $\phi \in BMO(\mathbb{R}^n)$, $||T\phi||_{L^\infty(\mathbb{R}^n)} \lesssim ||\phi||_{BMO(\mathbb{R}^n)}$.
(b) *Under DBC and* ${}^c\Omega$ *unbounded, for all* $\phi \in BMO_z(\Omega)$, $||T\phi||_{L^\infty(\Omega)} \lesssim ||\phi||_{BMO_z(\Omega)}$.
(c) *Under NBC, for all* $\phi \in BMO_{\mathrm{CW}}(\Omega)$, $||T\phi||_{L^\infty(\Omega)} \lesssim ||\phi||_{BMO_{\mathrm{CW}}(\Omega)}$.

The proofs of the three assertions are similar, and we concentrate on (b). First, the H^∞-functional calculus for L^* implies for all $f \in L^2(\Omega)$ (see [19])

$$\left(\int_0^{+\infty} \|t\partial_t P_t^* f\|_2^2 \frac{dt}{t}\right)^{1/2} \leqslant c\|f\|_2. \tag{13}$$

Consider $\phi \in BMO_z(\Omega)$ with $\|\phi\|_{BMO_z(\Omega)} \leqslant 1$, $x \in \Omega$ and a cube Q containing x, centered in Ω and $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$. Write

$$\phi = \phi_{2Q\cap\Omega} + (\phi - \phi_{2Q\cap\Omega})\mathscr{X}_{2Q\cap\Omega} + (\phi - \phi_{2Q\cap\Omega})\mathscr{X}_{\Omega-2Q}.$$

Using the square function estimate (13) for the local part $(\phi - \phi_{2Q\cap\Omega})\mathscr{X}_{2Q\cap\Omega}$, the decay of the kernel of $\partial_t P_t^*$ and BMO inequalities for the non-local part $(\phi - \phi_{2Q\cap\Omega})\mathscr{X}_{\Omega-2Q}$, one obtains classically

$$\frac{1}{|Q\cap\Omega|}\int_{T(Q)} |\partial_t P_t^*(\phi - \phi_{2Q\cap\Omega})(y)|^2 t\,dy\,dt \lesssim \|\phi\|_{BMO_{\mathrm{CW}}(\Omega)}^2.$$

It remains to show

$$\|\phi\|_{BMO_{\mathrm{CW}}(\Omega)} \lesssim \|\phi\|_{BMO_z(\Omega)} \tag{14}$$

and

$$I_Q = \frac{1}{|Q\cap\Omega|}\int_{T(Q)} |\partial_t P_t * (\phi_{2Q\cap\Omega})(y)|^2 t\,dy\,dt \lesssim 1. \tag{15}$$

We use the following lemma.

Lemma 15. *Assume ${}^c\Omega$ unbounded. Let $\phi \in BMO_z(\Omega)$ with $\|\phi\|_{BMO_z(\Omega)} \leqslant 1$ and Q be a cube centered in Ω with $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$. Then*

$$|\phi_{Q\cap\Omega}| \lesssim \sup\left(\ln\left(\frac{\delta}{\ell}\right), 1\right),$$

where $\ell = \ell(Q)$ and $\delta \geqslant 0$ its distance to the boundary of Ω.

Proof. Denote still by ϕ the zero extension of ϕ to $\mathbb{R}^n$, so that $\phi \in BMO(\mathbb{R}^n)$. Notice first that, for any cube Q of $\mathbb{R}^n$, one has

$$|\phi_Q - \phi_{2Q}| \lesssim 1. \tag{16}$$

We use the following geometric claim whose proof is presented in Appendix B:

Claim 2. *There exists $\rho \in]0, +\infty]$, such that if Q is a cube with $2Q \subset \Omega$ but $4Q \cap \partial\Omega \neq \emptyset$, and $\ell(Q) < \rho$, there exists a cube $\tilde{Q} \subset {}^c\Omega$ such that $|\tilde{Q}| \sim |Q|$ and the distance from $\tilde{Q}$ to Q is comparable to $\ell(Q)$. Furthermore, $\rho = \infty$ if ${}^c\Omega$ is unbounded.*

Hence, for such cubes Q, one can find a larger cube of comparable size and containing Q and $\tilde{Q}$. It follows from (16) that

$$|\phi_Q| = |\phi_Q - \phi_{\tilde{Q}}| \lesssim 1.$$

Next, assume that Q is such that $4Q \subset \Omega$, and let k be the smallest integer such that $2^{k+1}Q \cap \partial\Omega \neq \emptyset$. Then, one has

$$|\phi_Q| \leqslant \sum_{i=0}^{k-1} |\phi_{2^iQ} - \phi_{2^{i+1}Q}| + |\phi_{2^kQ}| \leqslant C(k+1) \lesssim \ln\left(\frac{\delta}{\ell}\right).$$

It remains to look at cubes centered in Ω with $4Q \cap \partial\Omega \neq \emptyset$. Take a Whitney decomposition of $Q \cap \Omega$ with respect to $\partial\Omega$,

$$Q \cap \Omega = \bigcup_k Q_k,$$

where, for each k, Q_k is a cube with $2Q_k \subset \Omega$ but $4Q_k \cap \partial\Omega \neq \emptyset$. Therefore, one has

$$|\phi_{Q\cap\Omega}| \leqslant \sum_k \frac{|Q_k|}{|Q \cap \Omega|} |\phi_{Q_k}| \lesssim 1,$$

and the lemma is proved. □

Let us come back to the proof of Proposition 14. Lemma A.6 (see Appendix A) shows that, for all $y \in Q \cap \Omega$,

$$\left| \int_\Omega \partial_t p_t(z,y)\, dz \right| \leqslant \frac{C}{t}\left(1 + \frac{\delta(y)}{t}\right)^{-1},$$

where $\delta(y)$ is the distance from y to the boundary of Ω. It is then fairly easy, using that Ω is strongly Lipschitz, to show that

$$\frac{1}{|Q\cap\Omega|} \int_{T(Q)} \frac{C}{t^2}\left(1 + \frac{\delta(y)}{t}\right)^{-2} t\, dy\, dt \lesssim \inf\left(\frac{\ell^2}{\delta^2}, 1\right).$$

See [5, Lemma 29] for details in a related situation. This and Lemma 15 prove that $I_Q \lesssim 1$.

Assertion (b) will be complete after we establish (14). Let Q be cube centered in Ω with $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$. If $Q \subset \Omega$ there is nothing to do. If Q intersects $\partial\Omega$, since ϕ

vanishes outside Ω,

$$\frac{1}{|Q\cap\Omega|}\int_{Q\cap\Omega}|\phi(x)-\phi_{Q\cap\Omega}|\,dx=\frac{|Q|}{|Q\cap\Omega|}\frac{1}{|Q|}\int_{Q}|\phi(x)-\phi_{Q}|\,dx$$
$$\leqslant\frac{|Q|}{|Q\cap\Omega|}||\phi||_{BMO}\lesssim||\phi||_{BMO},$$

where we have used the geometric claim 1.

To prove assertions (a) and (c), decompose ϕ as above. Since $\partial_t P_t$ annihilates constants $I_Q=0$ and only the other terms arise with similar treatment. Proposition 14 is proved. $\square$

Let us summarize what we have obtained so far: combining (12), the tent space inequality and Proposition 14, we have inequalities of the form

$$\left|\int_{\Omega}f(y)\overline{\phi(y)}\,dy\right|\lesssim||sf||_1||\phi||_{BMO} \tag{17}$$

with the appropriate BMO space in each case for $f\in L^2(\Omega)$ with $||sf||_1<\infty$ and ϕ is continuous with compact support in Ω.

Next, we use duality. For an open set D of $\mathbb{R}^n$, let $C_c(D)$ be the space of continuous compactly supported functions in D. Define $VMO(\mathbb{R}^n)$ as the closure of $C_c(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, $VMO_z(\Omega)$ and $VMO_{\mathrm{CW}}(\Omega)$ as the closure of $C_c(\Omega)$ in $BMO_z(\Omega)$ and $BMO_{\mathrm{CW}}(\Omega)$, respectively. Note that $VMO(\mathbb{R}^n)$ is defined in the sense of Coifman and Weiss [12] and is different from the one considered by Sarason [21], which is the closure of the space of all uniformly continuous BMO-functions on $\mathbb{R}^n$. See the recent work of Bourdaud for clarifications [6]. It is well-known that $H^1(\mathbb{R}^n)$ is the dual of $VMO(\mathbb{R}^n)$ and also that $H^1_{\mathrm{CW}}(\Omega)$ is the dual space of $VMO_{\mathrm{CW}}(\Omega)$ [12,13]. That $H^1_r(\Omega)$ is the dual space of $VMO_z(\Omega)$ is an easy consequence of duality for quotient spaces knowing the one on $\mathbb{R}^n$ (see [7]). Hence, we arrive at

$$||f||_{H^1}\lesssim||sf||_1 \tag{18}$$

when $f\in L^2(\Omega)$ and $||sf||_1<\infty$.

It remains to replace the assumption $f\in L^2(\Omega)$ by $f\in L^1(\Omega)$. Indeed, assume that $f\in L^1(\Omega)$ and $||sf||_1<\infty$. Define $f_k=P_{2^{-k}}f, k\geqslant 0$. Then

- $f_k\in L^2(\Omega)$.
- $||sf_k||_1\leqslant||sf||_1$.
- for all $\phi\in C_c(\Omega), \lim_{k\to+\infty}\langle f_k,\phi\rangle=\langle f,\phi\rangle$.

The first two points are obvious. The third can be done using arguments analogous to [2, p. 776], which rely on the decay of the kernel of $P_{2^{-k}}$.

For this sequence we have (in each of the three cases)

$$||f_k||_{H^1}\lesssim||sf_k||_1\leqslant||sf||_1$$

uniformly in k. Since H^1 is in each case the dual of a space in which $C_c(\Omega)$ is dense, there are $g \in H^1$ and a subsequence f_{k_j} such that for all $\phi \in C_c(\Omega)$,

$$\lim_{k \to +\infty} \langle f_{k_j}, \phi \rangle = \langle g, \phi \rangle.$$

It follows that $g = f$ and $\|f\|_{H^1} \lesssim \|sf\|_1$.

Remark 16. Let us discuss the relevance of the assumption $f \in L^1$. As the maximal spaces are contained in L^1, it is natural to make it in view of proving Theorem 1. However, it is interesting to have an intrinsic characterization of the space defined by $\|sf\|_1 < \infty$ on its own (especially if one is interested in H^p spaces, p below 1). The first task is to say to which f we apply the square functions. Basically, consider $y \to p_t(x, y)$, $x \in \Omega$, as a set of test functions so that it suffices that f be a linear functional acting boundedly on this set. The analyticity of the semigroup allows one to take derivatives with respect to t. Having done this, sf is well defined but it is still is not clear whether $\|sf\|_1$ finite implies a priori $f \in L^1$. One way around is to restrict even more the range of admissible f by imposing that $P_t f$ tends "weakly" to f if $t \to 0$, to 0 if $t \to \infty$. Then the approximations f_k of the previous argument are changed to $P_{2^{-k}} f - P_{2^k} f$. This allows one to obtain the Hardy space embedding and as a corollary one gets $f \in L^1$. See [1] for further developments.

4. Local Hardy and BMO spaces on strongly Lipschitz domains

We now give localized versions of the previous results.

4.1. Strategy

We first recall the definition of $h^1(\mathbb{R}^n)$ and its atomic decomposition from [15].

Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. For all $t > 0$, define $\phi_t(x) = t^{-n}\phi(x/t)$. A locally integrable function f on $\mathbb{R}^n$ is said to be in $h^1(\mathbb{R}^n)$ if the maximal function

$$mf(x) = \sup_{0<t<1} |\phi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^n)$. If it is the case, define

$$\|f\|_{h^1(\mathbb{R}^n)} = \|mf\|_1.$$

One has $H^1(\mathbb{R}^n) \subset h^1(\mathbb{R}^n)$. It should be noted that a function in $h^1(\mathbb{R}^n)$ does not necessarily have zero integral. We note that other maximal functions $\sup_{0<t<\delta} |\phi_t * f(x)|$ with $\delta > 0$ would lead to an equivalent norm.

Replacing $t>0$ by $0<t<1$ in (1), one obtains a characterization of $h^1(\mathbb{R}^n)$ in terms of a non-tangential maximal function associated with the heat or the Poisson semigroup generated by Δ (see [15]).

A function a is an $h^1(\mathbb{R}^n)$-atom if it is supported in a cube Q, satisfies $||a||_\infty \leqslant |Q|^{-1}$ and has mean-value zero if $\ell(Q)<1$. Then, $f \in h^1(\mathbb{R}^n)$ if and only if $f = \sum_Q \lambda_Q a_Q$, where the a_Q's are $h^1(\mathbb{R}^n)$-atoms and $\sum_Q |\lambda_Q| < \infty$. Moreover, $||f||_{h^1(\mathbb{R}^n)}$ is comparable with the infimum of $\sum_Q |\lambda_Q|$ taken over all such decompositions.

The spaces $h^1_r(\Omega)$ and $h^1_z(\Omega)$ are defined in the same way as $H^1_r(\Omega)$ and $H^1_z(\Omega)$, replacing $H^1(\mathbb{R}^n)$ by $h^1(\mathbb{R}^n)$. Observe that $h^1_z(\Omega)$ is a strict subspace of $h^1_r(\Omega)$ (see [8, Proposition 6.4]).

An $h^1_{\mathrm{CW}}(\Omega)$-atom is a function a supported in $Q \cap \overline{\Omega}$, where Q is a cube of $\mathbb{R}^n$ centered in Ω with $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$ (but not necessarily included in Ω) with

$$||a||_\infty \leqslant \frac{1}{|Q \cap \Omega|}, \quad \text{and} \quad \text{if } \ell(Q)<\delta, \int a(x)\,dx = 0.$$

A function f is in $h^1_{\mathrm{CW}}(\Omega)$ if it can be written

$$f = \sum_Q \lambda_Q a_Q.$$

where the a_Q's are $h^1_{\mathrm{CW}}(\Omega)$-atoms and $\sum_Q |\lambda_Q| < \infty$. The norm is defined as usual. Changing δ yields an equivalent norm provided there is at least one atom with non-zero mean. To simplify the exposition, we assume and fix $\delta = 1$ for now on.

Remark that $h^1_{\mathrm{CW}}(\Omega) \subset h^1_z(\Omega) \subset h^1_r(\Omega)$.

Remark 17. Each global Hardy space is contained in the corresponding local space. If Ω is bounded, one can see that $h^1_r(\Omega) = H^1_r(\Omega)$, $h^1_{\mathrm{CW}}(\Omega) = H^1_{\mathrm{CW}}(\Omega) + \mathbb{C}\mathscr{X}_\Omega$, and $h^1_z(\Omega) = H^1_z(\Omega) + \mathbb{C}\mathscr{X}_\Omega$. Here $\mathscr{X}_\Omega$ is the indicator function of Ω. All these facts but the inclusion $h^1_z(\Omega) \subset H^1_z(\Omega) + \mathbb{C}\mathscr{X}_\Omega$ are easy to prove. For the latter one uses the following observation which is easy to prove using maximal characterizations: if $f \in h^1(\mathbb{R}^n)$ has compact support and vanishing mean, then $f \in H^1(\mathbb{R}^n)$. Details are left to the reader.

If $L = (A, \Omega, V)$ is a second-order elliptic operator in divergence form and if $f \in L^1_{\mathrm{loc}}(\Omega)$ with $y \mapsto |y|^{-n-1} f(y) \in L^1(\Omega)$, we say that $f \in h^1_{\max,L}(\Omega)$ if

$$f^*_{\mathrm{loc},L}(x) = \sup_{|y-x|<t\leqslant 1} |e^{-tL^{1/2}} f(y)| \in L^1(\Omega).$$

Define

$$||f||_{h^1_{\max,L}} = ||f^*_{\mathrm{loc},L}||_1.$$

It is evident that $H^1_{\max,L}(\Omega)\subset h^1_{\max,L}(\Omega)$. Define also the local square functions $s_{\mathrm{loc}}f$ and $S_{\mathrm{loc}}f$ by truncating the cones at $t=1$.

Remark 18. When Ω is bounded and L satisfies (G_1), $h^1_{\max,L}(\Omega)=H^1_{\max,L}(\Omega)$. Indeed, one inclusion holds. For the converse, consider $x\in\Omega$, $t>1$ and $y\in\Omega$ satisfying $|y-x|<t$. Inequality (7) yields

$$|P_t f(y)|\lesssim\int_\Omega\frac{t}{t+|y-z|}\,|f(z)|\,dz\lesssim\|f\|_1.$$

As a consequence, for all $x\in\Omega$,

$$|f^*_L(x)|\lesssim(|f^*_{\mathrm{loc},L}(x)|+\|f\|_1)$$

and

$$\|f^*_L\|_1\lesssim\|f^*_{\mathrm{loc},L}\|_1+\|f\|_1\lesssim\|f^*_{\mathrm{loc},L}\|_1.$$

Theorem 2 can be proved following the same strategy as Theorem 1 by means of

Proposition 19. *Assume that L satisfies (G_∞). Let $f\in L^1(\Omega)$.*

(a) *If $\Omega=\mathbb{R}^n$, one has*

$$\|f\|_{h^1(\mathbb{R}^n)}\lesssim\|s_{\mathrm{loc}}f\|_1\leqslant\|S_{\mathrm{loc}}f\|_1\lesssim\|f\|_{h^1_{\max,L}(\mathbb{R}^n)}\lesssim\|f\|_{h^1(\mathbb{R}^n)}.$$

(b) *under DBC, one has*

$$\|f\|_{h^1_r(\Omega)}\lesssim\|s_{\mathrm{loc}}f\|_1\leqslant\|S_{\mathrm{loc}}f\|_1\lesssim\|f\|_{h^1_{\max,L}(\Omega)}\lesssim\|f\|_{h^1_r(\Omega)}.$$

(c) *Under NBC one has*

$$\|f\|_{h^1_z(\Omega)}\lesssim\|f\|_{h^1_{\mathrm{CW}}(\Omega)}\lesssim\|s_{\mathrm{loc}}f\|_1\leqslant\|S_{\mathrm{loc}}f\|_1\lesssim\|f\|_{h^1_{\max,L}(\Omega)}\lesssim\|f\|_{h^1_z(\Omega)}.$$

Furthermore, if Ω is bounded, statements (b) *and* (c) *hold for when L satisfies (G_1). The implicit constants depend neither on f, nor its L^1 norm.*

For instance, this result applies when the coefficients of A are real-valued functions or complex-valued BUC functions (and more generally, in the closure of BUC in *bmo*) on C^1 domains (see [4]).

4.2. Proof of Proposition 19

That f is in a local Hardy space h^1 implies $f^*_{\mathrm{loc},L}\in L^1$ can be proved exactly as in the global case (Section 3.2). Details are left to the reader.

Next, the fact that $f^*_{\mathrm{loc},L} \in L^1$ implies $S_{\mathrm{loc}} f \in L^1$ is as in Section 3.3.

It remains to pass from the square functions $s_{\mathrm{loc}} f \in L^1$ to f in a local Hardy space via tent spaces and duality (vmo, h^1). The result is the local analog of Proposition 12. One needs the *bmo* spaces $bmo(\mathbb{R}^n)$, $bmo_z(\Omega)$ and $bmo_{\mathrm{CW}}(\Omega)$.

Recall that a locally integrable function f on $\mathbb{R}^n$ is said to be in $bmo(\mathbb{R}^n)$ if

$$||\phi||_{bmo(\mathbb{R}^n)} = \sup\left(\sup_{\ell(Q)\leqslant 1} \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|\, dx, \sup_{\ell(Q)>1} \frac{1}{|Q|} \int_Q |\phi(x)|\, dx \right) < +\infty.$$

Define $vmo(\mathbb{R}^n)$ as the closure of $C_c(\mathbb{R}^n)$ in $bmo(\mathbb{R}^n)$. It is well-known that $bmo(\mathbb{R}^n)$ is the dual of $h^1(\mathbb{R}^n)$, which is the dual of $vmo(\mathbb{R}^n)$ (see [6]).

Define $bmo_z(\Omega)$, $vmo_z(\Omega)$ analogously to the corresponding global BMO_z or VMO_z spaces, replacing $BMO(\mathbb{R}^n)$ by $bmo(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n)$ by $vmo(\mathbb{R}^n)$.

A locally integrable function ϕ defined on Ω is in $bmo_{\mathrm{CW}}(\Omega)$ if

$$\begin{aligned} &||\phi||_{bmo_{\mathrm{CW}}(\Omega)} \\ &\quad = \sup\left(\sup_{\ell(Q)<1} \frac{1}{|Q\cap\Omega|} \int_{Q\cap\Omega} |\phi(x) - \phi_{Q\cap\Omega}|\, dx, \sup_{\ell(Q)>1} \frac{1}{|Q\cap\Omega|} \int_{Q\cap\Omega} |\phi| \right) \\ &\quad < +\infty, \end{aligned}$$

where the cubes have center in Ω and sidelength less than the diameter of Ω. The space $vmo_{\mathrm{CW}}(\Omega)$ is defined as the closure of $C_c(\Omega)$ in $bmo_{\mathrm{CW}}(\Omega)$. Its dual space is $h^1_{\mathrm{CW}}(\Omega)$.

Let us now discuss the proof of this step. Since the square functions use truncated cones, one must do something else in (11). A computation shows that this formula rewrites as

$$\int_0^1 (-tz^{1/2}e^{-tz^{1/2}})(-tz^{1/2}e^{-tz^{1/2}}) \frac{dt}{t} + \frac{1}{16}(2z^{1/2}+1)e^{-2z^{1/2}} = \frac{1}{4}$$

so that any $f \in L^2(\Omega)$[1] writes $f = f_1 + f_2$ where

$$f_1 = 4\int_0^1 (tL^{1/2}P_t)(tL^{1/2}P_t f) \frac{dt}{t}$$

and

$$f_2 = \tfrac{1}{4}(2L^{1/2} + I)P_1 P_1 f.$$

[1] If Ω is bounded and $V = W^{1,2}(\Omega)$ then the formula holds if $\int_\Omega f = 0$. If the mean of f is not zero, then it applies to $\tilde{f} = f - c\mathscr{X}_\Omega$ with the constant c so that the mean of f is zero. Since $\mathscr{X}_\Omega \in h^1_{\mathrm{CW}}(\Omega)$, it suffices to apply the argument to $\tilde{f}$.

For f_1, we proceed as in the global case using the tent spaces and duality in the local spaces and then eliminate the requirement that $f \in L^2(\Omega)$ to obtain $f_1 \in h^1(\mathbb{R}^n)$ or $h^1_r(\Omega)$ under DBC or $h^1_{\mathrm{CW}}(\Omega)$ under NBC.

Let us consider f_2. The idea is to prove that $f_2 \in h^1_{\mathrm{CW}}$ in each case. Since $h^1_{\mathrm{CW}}(\mathbb{R}^n) = h^1(\mathbb{R}^n)$ and $h^1_{\mathrm{CW}}(\Omega) \subset h^1_z(\Omega) \subset h^1_r(\Omega)$ the conclusion follows.

Here is the argument. Assume first $\Omega = \mathbb{R}^n$. Observe that $P_1 f = g$ is bounded by $f^*_{\mathrm{loc},L}$ which is in $L^1(\mathbb{R}^n)$. Also since $L^{1/2}P_1 = -\partial_t P_t|_{t=1}$ the subordination formula (see Appendix A) yields that the kernel $K(x,y)$ of $\frac{1}{4}(2L^{1/2} + I)P_1$ is bounded by $ck(x,y)$ with $k(x,y) = (1 + |x-y|)^{-n-1}$.

Take (Q_k) be a covering of $\mathbb{R}^n$ by cubes with size 1 obtained by translation from the unit cube $[0,1]^n$. Let (η_k) be a smooth partition of unity associated with this covering so that η_k is supported in $2Q_k$. Then one has

$$f_2(x) = \sum_k b_k(x),$$

where

$$b_k(x) = \eta_k(x) \int_{\mathbb{R}^n} K(x,y) g(y)\, dy.$$

Observe that b_k is supported in $2Q_k$. Set $\lambda_k = |2Q_k| \|b_k\|_\infty$. Note that $k(x,y) \leqslant c \inf_{|x-z| \leqslant 1} k(z,y)$ for all $x, y \in \mathbb{R}^n$. For $x \in 2Q_k$, since $\ell(Q_k) \leqslant 1$, $k(x,y) \leqslant c \inf_{z \in 4Q_k} k(z,y)$ for all y. Hence

$$\lambda_k \leqslant c|2Q_k| \int_{\mathbb{R}^n} \inf_{z \in 4Q_k} k(z,y)|g(y)|\, dy \leqslant \int\int \mathscr{X}_{4Q_k}(x) k(x,y)|g(y)|\, dy\, dx$$

and it follows from the finite overlap property of the family $(4Q_k)$ that

$$\sum_K \lambda_k \leqslant c \int\int k(x,y)|g(y)|\, dy\, dx \leqslant c\|g\|_1 \leqslant c\|f\|_{h^1_{\max,L}(\mathbb{R}^n)}.$$

If we set $a_k = \lambda_k^{-1} b_k$ when $\lambda_k \neq 0$, then a_k is an $h^1(\mathbb{R}^n)$-atom. Thus, $f_2 \in h^1(\mathbb{R}^n)$ with $\|f_2\|_{h^1(\mathbb{R}^n)} \leqslant c\|f\|_{h^1_{\max,L}(\mathbb{R}^n)}$.

Assume now that Ω is strongly Lipschitz and L satisfies either boundary condition. Let (Q_k) be a covering of Ω with cubes of $\mathbb{R}^n$ such that $\ell(Q_k) = 1$. We keep only those cubes which intersect Ω. If Q_k has center in Ω, we set $\lambda_k = |2Q_k \cap \Omega| \|b_k\|_\infty$. If Q_k has center outside of Ω, then we replace Q_k by $\tilde{Q}_k$ with center in $Q_k \cap \Omega$ and $\ell(\tilde{Q}_k) = 2\ell(Q_k)$ and define $\lambda_k = |2\tilde{Q}_k \cap \Omega| \|b_k\|_\infty$. With these modifications of λ_k, we see from the same argument that a_k is an $h^1_{\mathrm{CW}}(\Omega)$-atom and that $\sum |\lambda_k| \leqslant c\|f\|_{h^1_{\max,L}(\Omega)}$, remarking that all integrals should take place on Ω.

It remains to relax the condition (G_∞) to (G_1) when Ω is bounded. The same arguments work because of small time decay estimates for the Poisson kernel (see Lemma 4 and Appendix A for proofs).

5. Other maximal functions

It is possible to obtain a characterization of our maximal Hardy space with other maximal functions, such as the vertical and the non-tangential maximal functions associated with e^{-tL}. More precisely, the following holds:

Theorem 20. *Let $L = (A, \Omega, V)$. Assume that $\Omega = \mathbb{R}^n$ or that Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ under DBC with ${}^c\Omega$ unbounded or under NBC. Assume also that L satisfy (G_∞). The following are equivalent:*

$$\sup_{t>0} |e^{-tL}f(x)| \in L^1(\Omega), \tag{19}$$

$$\sup_{|x-y|<\sqrt{t}} |e^{-tL}f(y)| \in L^1(\Omega), \tag{20}$$

$$f \in H^1_{\max,L}(\Omega), \tag{21}$$

One also has the analogous local statement, replacing $H^1_{\max,L}$ by $h^1_{\max,L}$ and $t>0$ by $0<t<t_0$ for any $t_0>0$ without restriction on Ω. Moreover, if Ω is bounded then (G_1) suffices.

We write the proof for global spaces, under DBC when ${}^c\Omega$ is unbounded for example. The other cases are similar.

We have already proved (Theorem 1) the implication (21) $\Rightarrow f \in H^1_r(\Omega)$ (when ${}^c\Omega$ is unbounded) and the implication $f \in H^1_r(\Omega) \Rightarrow$ (19) is an easy consequence of the estimates for K_t and follows the proof of $f \in H^1_r(\Omega) \Rightarrow f \in H^1_{\max,L}(\Omega)$. We therefore turn to the proofs of (19) $\Rightarrow$ (20) and (20) $\Rightarrow$ (21).

The argument for (19) $\Rightarrow$ (20) relies upon the comparison between the L^1 norms of two maximal functions and is inspired by Fefferman and Stein [13, p. 185]. For all $\alpha>0$ and $v : \Omega\times]0,+\infty[\to\mathbb{C}$ set

$$v^*_\alpha(x) = \sup_{|y-x|<\alpha\sqrt{t}} |v(y,t)|.$$

If $f \in L^1_{\mathrm{loc}}$ with slow growth (ie $y \mapsto (1+|y|)^{-n-1}f(y) \in L^1(\Omega)$), set

$$u(x,t) = e^{-tL}f(x),\ u^+(x) = \sup_{t>0} |u(x,t)|,\ u^*(x) = u^*_1(x).$$

Recall that we assume (G_∞) so that the slow growth condition insures that u is well defined.

Finally, for all $\varepsilon>0$, all $N\in\mathbb{N}$ and all $x\in\Omega$, consider

$$u^*_{\varepsilon,N}(x)=\sup_{|y-x|<\sqrt{t}<\varepsilon^{-1}}|u(y,t)|\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}$$

and

$$U^*_{\varepsilon,N}(x)=\sup_{\substack{|y-x|<\sqrt{t}<\varepsilon^{-1},\\ |y'-x|<\sqrt{t}<\varepsilon^{-1}}}\left(\frac{\sqrt{t}}{|y-y'|}\right)^\mu|u(y,t)-u(y',t)|\times\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}$$

for some $\mu>0$ to be chosen later.

We intend to show the following proposition:

Proposition 21. *There exists $C>0$ such that, for all $f\in L^1_{\mathrm{loc}}$, $||u^*||_1\leqslant C||u^+||_1$.*

Notice first that the L^1-norm of u^*_α is controlled by the L^1-norm of u^*. More precisely, the following holds (see [13, Lemma 1, p. 166]):

Lemma 22. *There exists C such that, for all continuous function v on $\Omega\times]0,+\infty[$ and all $\alpha>0$,*

$$||v^*_\alpha||_1\leqslant C\alpha^n||v^*||_1.$$

Note that this inequality holds if v is truncated for $t>t_0$.

The proof of Proposition 21 relies on the following observation:

Lemma 23. *Assume that $u^*_{\varepsilon,N}\in L^1$. Then*

$$||U^*_{\varepsilon,N}||_1\leqslant C||u^*_{\varepsilon,N}||_1,$$

where C is independent on ε, N and u.

Fix $x\in\Omega$ and consider y,y' and t such that $|y-x|<\sqrt{t}$ and $|y'-x|<\sqrt{t}$. Define also

$$v(y,t)=u(y,t)(1+\varepsilon|y|)^{-N}\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N\mathscr{X}_{]0,1[}(\varepsilon\sqrt{t})$$

so that $v^*_1=u_{\varepsilon,N}$. Start from

$$u(y,t)-u(y',t)=\int_\Omega(K_{t/2}(y,z)-K_{t/2}(y',z))u(z,t/2)\,dz=I_0+\sum_{k\geqslant1}I_k,$$

where

$$I_0 = \int_{|z-y|\leqslant\sqrt{t}} |K_{t/2}(y,z) - K_{t/2}(y',z)||u(z,t/2)|\,dz$$

and

$$I_k = \int_{2^{k-1}\sqrt{t}<|z-y|\leqslant 2^k\sqrt{t}} |K_{t/2}(y,z) - K_{t/2}(y',z)||u(z,t/2)|\,dz.$$

Using $(1+\varepsilon|z|)^N \leqslant (1+\varepsilon|y|)^N(1+2^k)^N$ if $|z-y|\leqslant 2^k\sqrt{t}$ and $\varepsilon\sqrt{t}<1$ and

$$\int_{2^{k-1}\sqrt{t}<|z-y|\leqslant 2^k\sqrt{t}} |K_{t/2}(y,z) - K_{t/2}(y',z)|\,dz \leqslant c\left(\frac{|y-y'|}{\sqrt{t}}\right)^{\mu} e^{-\alpha 2^{2k}}$$

for some $\mu>0$ and $\alpha>0$ from (G_∞) for L^*, we easily get

$$\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N \frac{|u(y,t)-u(y',t)|}{(1+\varepsilon|y|)^N} \leqslant c\left(\frac{|y-y'|}{\sqrt{t}}\right)^{\mu}\left(v_2^*(x) + \sum_{k\geqslant 1} e^{-\alpha 2^{2k}}(1+2^k)^N v_{2^{k+1}}^*(x)\right).$$

Therefore,

$$U_{\varepsilon,N}^*(x) \leqslant c\left(v_2^*(x) + \sum_{k\geqslant 1} e^{-\alpha 2^{2k}}(1+2^k)^N v_{2^{k+1}}^*(x)\right).$$

Lemma 23 follows at once from Lemma 22. □

We now prove Proposition 21, following [13, p. 186]. Consider f such that $u^+\in L^1$ and $N\in\mathbb{N}$ large enough, so that one easily derives that $u_{\varepsilon,N}^*\in L^1$ for all $\varepsilon>0$. Define $G_{\varepsilon,N} = \{x\in\Omega;\, U_{\varepsilon,N}^*(x)\leqslant Bu_{\varepsilon,N}^*(x)\}$ for some $B>0$ to be chosen. Then, one has

$$\begin{aligned}\int_{\Omega\setminus G_{\varepsilon,N}} u_{\varepsilon,N}^*(x)\,dx &\leqslant \frac{1}{B}\int_{\Omega\setminus G_{\varepsilon,N}} U_{\varepsilon,N}^*(x)\,dx\\ &\leqslant \frac{1}{2}\int_{\Omega} u_{\varepsilon,N}^*(x)\,dx\end{aligned}$$

provided that B is large enough.

Moreover, for almost all $x\in G_{\varepsilon,N}$, one has $u_{\varepsilon,N}^*(x)\leqslant CM(x)$, where

$$M(x) = \sup_{Q\ni x}\left(\frac{1}{|Q\cap\Omega|}\int_{Q\cap\Omega} u^+(y)^r\,dy\right)^{1/r},$$

with $0<r<1$ (in this definition, the cubes are centered in Ω and have $\ell(Q)\leqslant 2\,\mathrm{diam}(\Omega)$). Indeed, let $x\in G_{\varepsilon,N}$ for which $u_{\varepsilon,N}^*(x)<\infty$. There exist y,t such

that $|y-x|<\sqrt{t}<\varepsilon^{-1}$ and

$$|u(y,t)|\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}\geqslant\frac{1}{2}u^*_{\varepsilon,N}(x).$$

Since $x\in G_{\varepsilon,N}$, if $|z-x|<\sqrt{t}$ and $|z'-x|<\sqrt{t}$, one has

$$\begin{aligned}&\left(\frac{\sqrt{t}}{|z-z'|}\right)^\mu|u(z,t)-u(z',t)|\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N(1+\varepsilon|z|)^{-N}\\&\quad\leqslant 2B|u(y,t)|\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}\end{aligned}$$

hence,

$$\left(\frac{\sqrt{t}}{|z-z'|}\right)^\mu|u(z,t)-u(z',t)|\leqslant c|u(y,t)|.$$

It follows that

$$|u(z,t)|\geqslant\tfrac{1}{2}|u(y,t)|$$

when $z\in A=\{w;|w-x|<\sqrt{t}\text{ and }|w-y|<\frac{\sqrt{t}}{2C}\}$. Therefore, when $z\in A$, one has

$$|u(z,t)|\geqslant\frac{1}{2}|u(y,t)|\left(\frac{\sqrt{t}}{\sqrt{t}+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}\geqslant\frac{1}{4}u^*_{\varepsilon,N}(x).$$

Hence,

$$\begin{aligned}M(x)^r&\geqslant\frac{c}{|B(x,2\sqrt{t})|}\int_{B(x,2\sqrt{t})}u^+(z)^r\,dz\\&\geqslant\frac{c}{|B(x,2\sqrt{t})|}\int_{B(x,2\sqrt{t})}|u(z,t)|^r\,dz\\&\geqslant c\left(\frac{1}{4}u^*_{\varepsilon,N}(x)\right)^r\frac{|A|}{|B(x,2\sqrt{t})|}\\&\geqslant cu^*_{\varepsilon,N}(x)^r.\end{aligned}$$

Finally, using the fact that $1/r>1$, one obtains that

$$\begin{aligned}\int_\Omega u^*_{\varepsilon,N}(x)\,dx&\leqslant 2\int_{G_{\varepsilon,N}}u^*_{\varepsilon,N}(x)\,dx\\&\leqslant C\int_{G_{\varepsilon,N}}M(x)\,dx\end{aligned}$$

$$\leqslant C \int_\Omega M(x)\,dx$$
$$\leqslant C \int_\Omega u^+(x)\,dx,$$

where C does not depend on ε. Letting $\varepsilon \to 0$ yields $||u^*||_1 \leqslant C||u^+||_1$ and (20) is proved.

We end the proof of Theorem 20 by noting that (20) $\Rightarrow$ (21) follows easily from the subordination formula (22) and Lemma 22.

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Appendix A. Kernel estimates

In this appendix, we derive some consequences of the Gaussian upper bounds (3) which we assume to hold for $0 < t < \tau$. The first consequence is that an estimate of the form (3) holds for $t\partial_t K_t(x,y)$ by analyticity of the semigroup (see [3, Chapter I, Lemma 19]).

We first claim the following:

Lemma A.1. *Assume that $\tau = 1$. Then, for all $t > 1$ and all $x, y \in \Omega$, one has*

$$|K_t(x,y)| \leqslant Ce^{-\alpha\frac{|x-y|^2}{t}}.$$

The proof relies on the following L^2-maximum principle (see [16]):

Proposition A.2. *Assume that $A \in \mathscr{A}(c)$. Let $u(x,t)$ be a function on $\Omega \times]0,+\infty[$ satisfying $\partial_t u(x,t) + Lu(x,t) = 0$ on Ω. Then, if $\xi : \Omega \times]0,+\infty[\to \mathbb{R}$ is locally Lipschitz and satisfies the relation*

$$\partial_t \xi(x,t) + \alpha|\nabla \xi(x,t)|^2 \leqslant 0,$$

where $\alpha = \frac{1}{2c^3}$, the function

$$I(t) = \int_\Omega |u(x,t)|^2 e^{\xi(x,t)}\,dx$$

is non-increasing in $t > 0$.

Indeed, for all $t>0$, one has

$$\begin{aligned}
I'(t) &= 2\mathrm{Re}\int_\Omega \partial_t u(x,t)\overline{u(x,t)}e^{\xi(x,t)}\,dx + \int_\Omega u(x,t)\overline{u(x,t)}\partial_t\xi(x,t)e^{\xi(x,t)}\,dx\\
&= -2\mathrm{Re}\int_\Omega Lu(x,t)\overline{u(x,t)}e^{\xi(x,t)}\,dx - \alpha\int_\Omega u(x,t)\overline{u(x,t)}|\nabla_x\xi(x,t)|^2e^{\xi(x,t)}\,dx\\
&= -2\mathrm{Re}\int_\Omega A(x)\nabla u(x,t)\overline{\nabla u(x,t)}e^{\xi(x,t)}\,dx\\
&\quad -2\mathrm{Re}\int_\Omega A(x)\nabla u(x,t)\nabla\xi(x,t)\overline{u(x,t)}e^{\xi(x,t)}\,dx\\
&\quad -\alpha\int_\Omega u(x,t)\overline{u(x,t)}|\nabla_x\xi(x,t)|^2e^{\xi(x,t)}\,dx\\
&\leqslant -2c\int_\Omega |\nabla u(x,t)|^2e^{\xi(x,t)}\,dx + 2c^{-1}\int_\Omega |\nabla u(x,t)||\nabla\xi(x,t)||u(x,t)|e^{\xi(x,t)}\,dx\\
&\quad -\alpha\int_\Omega |u(x,t)|^2|\nabla_x\xi(x,t)|^2e^{\xi(x,t)}\,dx\\
&\leqslant (-2c+c^{-1}/\varepsilon)\int_\Omega |\nabla u(x,t)|^2e^{\xi(x,t)}\,dx\\
&\quad +(\varepsilon c^{-1}-\alpha)\int_\Omega |u(x,t)|^2|\nabla_x\xi(x,t)|^2e^{\xi(x,t)}\,dx\\
&= 0.
\end{aligned}$$

In the previous computation, $\varepsilon=\alpha c=\frac{1}{2c^2}$. Proposition 25 is therefore proved. □

In order to prove Lemma A.1, observe that, for $\beta>0$ small enough (namely, $\beta\leqslant\frac{1}{4\alpha}$), the function $\xi(x,t)=\beta\frac{|x-y|^2}{t}$ satisfies the assumptions of Proposition A.2. Then, write

$$\begin{aligned}
|K_t(x,y)| &= \left|\int_\Omega K_{t/2}(x,z)e^{\frac{\alpha}{2}\frac{|x-z|^2}{t}}K_{t/2}(z,y)e^{\frac{\alpha}{2}\frac{|z-y|^2}{t}}e^{-\frac{\alpha}{2}\frac{|x-z|^2+|z-y|^2}{t}}\,dz\right|\\
&\leqslant \left(\int_\Omega |K_{t/2}(x,z)|^2e^{\alpha\frac{|x-z|^2}{t}}\,dz\right)^{1/2}\left(\int_\Omega |K_{t/2}(z,y)|^2e^{\alpha\frac{|z-y|^2}{t}}\,dz\right)^{1/2}e^{-\frac{\alpha}{4}\frac{|x-y|^2}{t}}\\
&\leqslant \left(\int_\Omega |K_{1/2}(x,z)|^2e^{\alpha|x-z|^2}\,dz\right)^{1/2}\left(\int_\Omega |K_{1/2}(z,y)|^2e^{\alpha|z-y|^2}\,dz\right)^{1/2}e^{-\frac{\alpha}{4}\frac{|x-y|^2}{t}}\\
&\leqslant Ce^{-\frac{\alpha}{4}\frac{|x-y|^2}{t}}.
\end{aligned}$$

As a consequence of the upper bounds for K_t, we get the following estimates for the Poisson kernel:

Lemma A.3. (a) *Assume that* $\tau = +\infty$. *Then, for all* $t>0$, *all* $x, y \in \Omega$,

$$|p_t(x,y)| \leqslant \frac{Ct}{(t+|x-y|)^{n+1}}.$$

(b) *Assume now that* $\tau = 1$ and Ω *is bounded. Then, for all* $0<t<1$, *all* $x, y \in \Omega$,

$$|p_t(x,y)| \leqslant \frac{Ct}{(t+|x-y|)^{n+1}}.$$

For all $t>1$, *all* $x, y \in \Omega$,

$$|p_t(x,y)| \leqslant \frac{Ct}{(t+|x-y|)}.$$

By analyticity, the same estimates hold for $t\partial_t p_t(x,y)$.

This follows from the subordination formula (see [24]):

$$p_t(x,y) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} K_{\frac{t^2}{4u}}(x,y) e^{-u} u^{-1/2} du. \tag{A.1}$$

Details are left to the reader.

This proves the part of Lemma 4 concerned with upper bounds. The part concerned with Hölder regularity is left to the reader.

We now summarize L^2-estimates for $\nabla K_t(x,y)$ that follow from assumption (3) and the Caccioppoli inequality (see [5, Proposition 15]):

Proposition A.4. (a) *Assume that* $\tau = +\infty$. *For all* $x \in \Omega$, *all* $t>0$ *and all* $r>0$,

$$\left(\int_{r \leqslant |x-y| \leqslant 2r} |\nabla_y K_t(y,x)|^2 \, dy \right)^{1/2} \leqslant c C_G t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{n-2}{2}} e^{-\beta \frac{r^2}{t}}.$$

(b) *Assume that* $\tau = 1$. *Then, for all* $x \in \Omega$, *all* $0<t<1$ *and all* $r>0$,

$$\left(\int_{r \leqslant |x-y| \leqslant 2r} |\nabla_y K_t(y,x)|^2 \, dy \right)^{1/2} \leqslant c C_G t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{n-2}{2}} e^{-\beta \frac{r^2}{t}}.$$

For all $x \in \Omega$, *all* $t>1$ *and all* $r>0$,

$$\left(\int_{r \leqslant |x-y| \leqslant 2r} |\nabla_y K_t(y,x)|^2 \, dy \right)^{1/2} \leqslant c C_G t^{-\frac{1}{2}} r^{\frac{n-2}{2}} e^{-\beta \frac{r^2}{t}}.$$

As a consequence of Proposition A.4, the following holds:

Lemma A.5. *For all $x \in \Omega$, denote by $\delta(x)$ the distance from x to $\partial\Omega$.*

(a) *Under NBC, for all $x \in \Omega$,*

$$\int_\Omega \partial_t K_t(y,x)\,dy = 0.$$

(b) *Under DBC, for all $x \in \Omega$ for all $0 < t < \tau$,*

$$\left|\int_\Omega \partial_t K_t(y,x)\,dy\right| \leqslant \frac{C}{t} e^{-\frac{\beta\delta(x)^2}{4t}}.$$

Under DBC, if Ω is bounded and $\tau = 1$, for all $x \in \Omega$ and all $t > 1$,

$$\left|\int_\Omega \partial_t K_t(y,x)\,dy\right| \leqslant \frac{C}{t}.$$

Under NBC, one has $e^{-tL}1 = 1$, whence assertion (a) holds.

To prove the first part of assertion (*b*), choose $\psi_1 \in C_0^\infty(\Omega)$ such that $\psi_1(z) = 1$ if $d(z,y) \leqslant \delta/4, \psi_1(z) = 0$ if $d(z,y) \geqslant \delta/2$ and $\|\nabla\psi_1\|_\infty \leqslant C/\delta$. Here $\delta = \delta(x)$. Define $\psi_2 = 1 - \psi_1$. Then, one has

$$\int_\Omega \partial_t K_t(y,x)\,dy = \int_\Omega \partial_t K_t(y,x)\psi_1(y)\,dy + \int_\Omega \partial_t K_t(y,x)\psi_2(y)\,dy.$$

But Lemma A.5 shows that

$$\begin{aligned}
\left|\int_\Omega L_y K_t(y,x)\psi_1(y)\,dy\right| &= \left|\int_\Omega A\nabla_y K_t(y,x)\nabla_y\psi_1(y)\,dy\right| \\
&\leqslant C\int_{\frac{\delta}{4}\leqslant d(z,y)\leqslant\frac{\delta}{2}} |\nabla_y K_t(y,x)||\nabla_y\psi_1(y)|\,dy \\
&\leqslant Ct^{-\frac{1}{2}-\frac{n}{4}}\left(\frac{\delta}{\sqrt{t}}\right)^{\frac{n-2}{2}} e^{-\beta\frac{\delta^2}{t}}\delta^{\frac{n-2}{2}} \\
&= \frac{C}{t}\left(\frac{\delta}{\sqrt{t}}\right)^{n-2} e^{-\beta\frac{\delta^2}{t}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\int \partial_t K_t(x,y)\psi_2(y)\,dy\right| &\leqslant \int_{|y-x|\geqslant\delta/2} |\partial_t K_t(x,y)|\,dy \\
&\leqslant \frac{C}{t} e^{-\beta\frac{\delta^2}{t}}.
\end{aligned}$$

For the second part of assertion (b), we have

$$\int_\Omega |\partial_t K_t(x,y)|\,dy \leqslant \frac{C|\Omega|}{t}. \qquad \square$$

We can deduce the following:

Lemma A.7. *For all $x \in \Omega$, denote again by $\delta(x)$ the distance from x to $\partial\Omega$.*

(a) *Under NBC, for all $x \in \Omega$,*

$$\int_\Omega \partial_t p_t(x,y)\,dy = 0.$$

(b) *Under DBC, if $\tau = +\infty$, for all $x \in \Omega$,*

$$\left|\int_\Omega \partial_t p_t(x,y)\,dy\right| \leqslant \frac{C}{t}\left(1 + \frac{\delta(x)}{t}\right)^{-1}.$$

Under DBC, if Ω is bounded and $\tau = 1$, for all $x \in \Omega$ and $0 < t < 1$

$$\left|\int_\Omega \partial_t p_t(x,y)\,dy\right| \leqslant \frac{C}{t}\left(1 + \frac{\delta(x)}{t}\right)^{-1}.$$

Let us prove the second point of part (b). By differentiating the subordination formula, one has

$$\int_\Omega \partial_t p_t(x,y)\,dy = \frac{1}{\sqrt{\pi}} \int_\Omega \int_0^{+\infty} \frac{t}{2u} \partial_s K_s(x,y)|_{s=\frac{t^2}{4u}} e^{-u} u^{-1/2}\,du\,dy.$$

Break the integral at $u = t^2/4$. By the preceeding lemma, (b), the part for $u \geqslant t^2/4$ is controlled by $\frac{C}{t}(1 + \frac{\delta(x)}{t})^{-1}$. The part for $u \leqslant t^2/4$ is bounded by $\frac{c}{t}\int_0^{t^2/4} e^{-u} u^{-1/2}\,du \leqslant c$. Since $t + \delta(x) \leqslant 1 + \operatorname{diam}(\Omega)$, we obtain $c \leqslant \frac{C}{t}(1 + \frac{\delta(x)}{t})^{-1}$. This concludes the proof. $\square$

Appendix B. Elementary geometry of Lipschitz domains

A strongly Lipschitz domain is by definition a domain in $\mathbb{R}^n$ whose boundary is covered by a finite number of parts of Lipschitz graphs (up to rotations) at most one them being unbounded. A special Lipschitz domain is the domain above the graph of a Lipschitz function defined on $\mathbb{R}^{n-1}$.

Let Ω be a strongly Lipschitz domain.

1. There exists a finite covering of $\mathbb{R}^n$ by open sets $U_1, U_2, \ldots, U_s$ with at most one of them being infinite such that for each k either $U_k \cap \Omega = \emptyset$ or there is

a special Lipschitz domain Ω_k and a rotation R_k in $\mathbb{R}^n$ such that $U_k \cap \Omega = U_k \cap R_k(\Omega_k)$.

2. There exists a cube Q_0 such that either $\Omega \subset Q_0$ or there is a rotation R and a special Lipschitz domain Ω_0 such that ${}^cQ_0 \cap \Omega = {}^cQ_0 \cap R(\Omega_0)$.
3. There is a constant $C > 0$ such that if Q is a cube centered in Ω with $\ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$ then $|Q \cap \Omega| \geqslant C|Q|$.
4. There exists $\rho \in]0, +\infty]$, such that if Q is a type (b) cube and $\ell(Q) < \rho$, there exists a cube $\tilde{Q} \subset {}^c\Omega$ such that $|\tilde{Q}| = |Q|$ and the distance from $\tilde{Q}$ to Q is comparable to the side length of Q. Furthermore, $\rho = \infty$ is ${}^c\Omega$ is unbounded.
5. Assume Ω is unbounded. Let Q be a cube with $\ell(Q) \geqslant 1$, centered in Ω with $4Q \cap \Omega \neq \emptyset$. There exists a cube Q' with $4Q' \subset \Omega$, $\ell(Q') = \ell(Q)$ and the distance between Q and Q' is comparable to $\ell(Q)$.

Point 1: Its proof is classical and skipped.

Point 2: Take Q_0 as the smallest cube containing the bounded U_k's in point 1.

Point 3: When Ω is a special Lipschitz domain, this is classical using "vertical" reflection. Localisation gives us $\rho > 0$ such that $\ell(Q) \leqslant \rho$ implies $|Q \cap \Omega| \geqslant C|Q|$. If Ω is bounded and $\rho \leqslant \ell(Q) \leqslant 2\,\mathrm{diam}(\Omega)$ then if $\tilde{Q}$ has same center as Q and $\ell(\tilde{Q}) = \rho$, then $|Q \cap \Omega| \geqslant |\tilde{Q} \cap \Omega| \geqslant C|\tilde{Q}| \geqslant C\rho^n(2\,\mathrm{diam}(\Omega))^{-n}|Q|$.

To obtain $\rho = \infty$ when Ω is unbounded, we argue as follows: let Q_0 be the cube of point 2. Let Q be a cube centered in Ω with $\ell(Q) > \rho$. If $\ell(Q) \leqslant \lambda\ell(Q_0)$ for some $\lambda > 4$ to be chosen, then for $\tilde{Q} = \frac{\rho}{\ell(Q)}Q$ we have $|Q \cap \Omega| \geqslant |\tilde{Q} \cap \Omega| \geqslant C|\tilde{Q}| \geqslant C\frac{\rho^n}{(\lambda\ell(Q_0))^n}|Q|$. If $\ell(Q) \geqslant \lambda\ell(Q_0)$ and the center of Q belongs to $R(\Omega_0)$, then $|Q \cap \Omega| \geqslant |Q \cap \Omega \cap {}^cQ_0| = |Q \cap R(\Omega_0) \cap {}^cQ_0| \geqslant |Q \cap R(\Omega_0)| - |Q_0| \geqslant C_0|Q| - \lambda^{-n}|Q|$ where C_0 is the constant obtained for the domain $R(\Omega_0)$. If $\ell(Q) \geqslant \lambda\ell(Q_0)$ and the center of Q does not belong to $R(\Omega_0)$, then this center belongs to Q_0 by construction of Q_0. Hence $Q_0 \subset \frac{2}{\lambda}Q$. Let $y \in Q_0 \cap R(\Omega_0)$. One checks that the cube centered at y with sidelength $4\ell(Q)/\lambda$ is contained in Q. We apply the above argument to that cube with λ replaced by $\lambda/4$ and choose $\lambda^n = 4^n 2/C_0$.

Point 4: This is well-known if Ω is special Lipschitz or bounded. See [9, p. 304]. By the same argument, one can see it holds for some ρ finite for all strongly Lipschitz domains. It remains to show that one can take drop the finiteness of ρ if ${}^c\Omega$ is unbounded. In that case, let Q be a type (b) cube contained in Ω with $\ell(Q) \geqslant \rho$. Pick Q_0, R and Ω_0 of point 2. In a basis $(e_1, \ldots, e_n), \Omega_0$ is $x_n > \varphi(x_1, \ldots, x_{n-1})$. We take $\tilde{Q} = Q - c\ell(Q)R(e_n)$ for some appropriately chosen c that depends only on the domain Ω. We leave to the reader the care of verifying that such a choice is possible.

Point 5: Let Q is a cube of size greater than 1, centered in Ω with $4Q \cap \Omega \neq \emptyset$. Arguing as above, we take $Q' = Q + c\ell(Q)R(e_n)$ where, since Ω is unbounded, one can pick c large enough and independent of Q such that Q' enjoys the desired properties. Details are left to the reader.

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