Biparameter singular integral representation theorem
Difficulties raised and overcome in biparameter setting

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Motivation: There is a dichotomy between multiple variables and parameters. Harmonic analysis using objects defined by multiple parameters, such as rectangles, raise new difficulties. Starting with the theme of the maximal function, we can see how the strong maximal function defined with respect to rectangles is different from the usual Hardy-Littlewood maximal function.

1. Strong maximal function $M_s = sup_R \frac{1}{|R|} \int_R |f|$.
2. Biparameter Hilbert transform $H_1 H_2 = * \frac{1}{x_1 x_2}$
3. Generalizing Calderon-Zygmund conditions takes some work
4. Two extremes to defining $\langle Tf, g \rangle$ – vector valued (Journe) or tensor products

We'll prove a new representation theorem for the desired $\langle Tf, g \rangle$ in terms of simplier shift operators, through a new characterization of Calderon-Zygmund operators, inspired by [3], without resorting to vector valued techniques or a priori boundedness estimates.

A Calderon-Zygmund operator T is an operator with the following representation:

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x,y)f(y)g(x)dxdy \qquad (1)$$

where f and g are from L^2 (though eventually we'll expand on this using $T(1)$ type theorems). The many requirements of the objects involved include: $f = f_1 f_2 = f_1 \otimes f_2$, $supp(f_1 \cap g_1) = \emptyset$.
And for K:

decay condition:

$$|K(x,y)| \le C \frac{1}{|x_1 - y_1|^n} \frac{1}{|x_2 - y_2|^m}$$

Holder smoothness

$$|K(x,y) - K(x,(y_1,y_2')) - K(x,(y_1',y_2)) + K(x,y')| \le C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|y_1 - y_1'| \le |x_1 - y_1|/2$ and $|y_2 - y_2'| \le |x_2 - y_2|/2$.

$$|K(x,y) - K((x_1,x_2'),y) - K((x_1',x_2),y) + K(x',y)| \le C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|x_1 - x_1'| \le |x_1 - y_1|/2$ and $|x_2 - x_2'| \le |x_2 - y_2|/2$.

The mixed smoothness:

$$|K(x,y) - K((x_1,x_2'),y) - K(x,(y_1',y_2)) + K((x_1',x_2),(y_1',y_2))|$$

$$\leq C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|y_1 - y_1'| \leq |x_1 - y_1|/2$ and $|x_2 - x_2'| \leq |x_2 - y_2|/2$.

$$|K(x,y) - K(x,(y_1,y_2')) - K((x_1',x_2),y) + K((x_1,x_2),(y_1,y_2'))|$$

$$\leq C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$ and $|y_2 - y_2'| \leq |x_2 - y_2|/2$.

Also, we have the combined decay/smoothness discussion (conditions provided to account for decay in one variable and Holder in the other variable).

$$|K(x,y) - K(x,(y_1,y_2'))| \leq C \frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{1}{|x_1 - y_1|^m}$$

whenever $|y_1 - y_1'| \leq |x_1 - y_1|/2$.

$$|K(x,y) - K((x_1,x_2'),y)| \leq C \frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{1}{|x_1 - y_1|^m}$$

whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$.

$$|K(x,y) - K((x_1,x_2'),y)| \leq C \frac{1}{|x_2 - y_2|^n} \frac{|x_2 - x_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|x_2 - x_2'| \leq |x_2 - y_2|/2$.

$$|K(x,y) - K(x,(y_1,y_2'))| \leq C \frac{1^\delta}{|x_2 - y_2|^n} \frac{|y_2 - y_2'|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever $|y_2 - y_2'| \leq |x_2 - y_2|/2$.

First we assume the CZ structure in each parameter – the "sliced" kernel representation:

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{f_1,g_1}(x_1,y_1) f_1(y_1) g_1(x_1)\, dx_1\, dy_1$$

with decay

$$|K_1(x_1,y_1)| \le C(f_2,g_2)\frac{1}{|x_1-y_1|^n}$$

and smoothness in x and y

$$|K_1(x_1,y_1) - K(x_1',y_1)| \leq C(f_2,g_2)\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}$$

$$|K(x_1,y_1) - K(x_1,y_1')| \leq C(f_2,g_2)\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}$$

when $|x_1 - x_1'| << |x_1 - y_1|/2$ and $|y_1 - y_1'| < |x_1 - y_1|/2$.
Here $C(f,g)$ is a constant where we require small control over the diagonal: $C(\chi_v,\chi_v) + C(\chi_v,u_v) + C(u_v,\chi_v) \leq C|V|$ where u_v is V adapted with zero mean, that is $supp(u_v) \subset V, |u_v| \leq 1$ and $\int u_v = 0$.

1. Besides the constants' dependence, these are the same as for the single parameter case.
2. These sliced conditions are what you would expect for a tensor product generalization of singular integrals.
3. Here we see that these are just some of the requirements, as boundedness is much more complex.

Journe dealt with multiparameter operators using vector valued inequalities – defining an operator valued kernel:

Definition

Let B is a Banach space and $0 < \delta < 1$. Journe's vector valued kernerl is a continous function $K : \mathbb{R}^2/\Delta \to B$ such that:

$$\|K(x,t)\|_B \leq C\frac{1}{|x-t|^\delta}$$

and

$$\|K(x,t) - K(x',t')\|_B \leq C\frac{(|x-x'|+|t-t'|)^\delta}{|x-t|^{1+\delta}},$$

when $(|x-x'|+|t-t'| < |x-t|)/2$.

This definition encodes much complexity. Additionally, we have a priori boundedness assumptions mentioned earlier.

To get the desired L^2 bound here, we have a $T(1)$ type theorem assumptions: $T(1)$, $T^*(1)$, $T_1(1)$, $T_1^*(1) \in BMO_p$ and a WBP: $|\langle T(\chi_K \otimes \chi_V), \chi_K \otimes \chi_V \rangle| \leq C|K||V|$ for all cubes $K \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ where T_1 is the partial adjoint.

The BMO_p is the product BMO space. We define dyadically:

Definition

We say $f \in BMO_p$ if

$$\sup_{\Omega} \sum_{I \times J \in \Omega} \langle f, h_I \otimes h_J \rangle^2 \leq C|\Omega|$$

where Ω is any open set. The $\{h_I\}$ are Haar functions $h_I = |I|^{-1/2}(\chi_l - \chi_r)$, where χ_l is the characteristic function of the left half of a dyadic interval, and χ_r is the right half. The h_I form a basis of L^2 as well as many other Banach spaces, as long as we add the noncancellative constant function 1.

The Haar functions form a localized basis, which naturally fit with the dyadic structure of a space. We can then define the sqaure function via Haar:

Definition

The square function is

$$Sq(f) = [\sum_{K \in D_n} \sum_{V \in D_m} |\langle f, h_K \otimes u_V \rangle|^2 \frac{\chi_K \otimes \chi_V}{|K||V|}]^{1/2}$$

Then f is in the product Hardy space H^1 if and only if $\|S_q(f)\|_{L^1} < \infty$.

We also need a few "diagonal" BMO conditions (using characteristic functions and some adapted functions).

The main theorem of this paper is

Theorem

We have

$$\langle Tf, g\rangle = C_T E_{w_n} E_{w_m} \sum_{i,j\in\mathbb{Z}} 2^{-max(i_1,i_2)\delta/2} 2^{-max(j_1,j_2)\delta/2} \langle S^{i,j} f, g\rangle \quad (2)$$

where the shifts S are taken with respect to the pair of dyadic grdis $(\mathcal{D}_n, \mathcal{D}_m)$ and $w_n \in \{\{0,1\}^n\}^{\mathbb{Z}}$ corresponds to a random shift.

A key concept to the proof of 4 is the use of random dyadic grids. A basic averaging property with regards to these grids will be proved, allowing us to rewrite the desired decomposition.

Good and Bad cubes We call a dyadic cube $I \in \mathbb{R}^n$ bad if there is another cube J such that both $l(J) \geq 2^r l(I)$ and $d(I, \partial(J)) \leq 2l(I)^\gamma l(J)^{1-\gamma}$ where $\gamma = \delta/(2n + 2\delta)$.

Definition

Given nonnegative integers $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}^2$, define

$$S^{(i,j)} f = \sum_{K \in D_a} \sum_{V \in D_b} A_{KV} f \tag{3}$$

where

$$A_{KV} f = \sum_{I_1, I_2 \subseteq K} \sum_{J_1, J_2 \subseteq V} a_{I,K,J,K} \langle f, h_{I_1} \otimes u_{J_1} \rangle h_{I_2} \otimes u_{J_2}$$

and $l(I_1) = 2^{-i_1} l(K)$ and similarity for i_2, j_1 and j_2. Moreover, we have that $a \leq \frac{|I_1||I_2||J_1||J_2|}{|K||V|}$. and the subshifts (where $K \in A, V \in B$) are L^2 bounded with a maximum norm of 1.

Note that we can rewrite the shift S in the handy kernel representation:

$$Sf(x) = \sum_{K,V} A_{KV} f(x) = \sum_{K,V} \frac{1}{|K \times V|} \int_{K \times V} K_{AV}(x,y) f(y) dy$$

$$= \int_{\mathbb{R}^{n+m}} K_S(x,y) f(y) dy$$

As an example, we can consider shifts where $i_1 = i_2 = j_1 = j_2 = 1$, which give rise to kernels constant on quarters of "Haar rectangles".

Lemma

We have

$$\langle Tf, g\rangle =$$

$$C\mathbb{E}\sum_{I_1,I_2\in D_n}\sum_{J_1,J_2\in D_m}\chi_{good}(sm(I))\chi_{good}(sm(J))\langle T(h_{I_1}\otimes u_{J_1}), h_{I_2}\otimes u_{J_2}\rangle\langle f, h_{I_1}\otimes u_{J_1}\rangle\langle g, h_{I_2}\otimes u_{J_2}\rangle$$

where $C = 1/(\pi^n_{good}\pi^m_{good})$, $\mathbb{E} = \mathbb{E}_{w_n}\mathbb{E}_{w_m}$, and the summation over all the $2^n - 1$ or $2^m - 1$ cancellative Haar functions is suppressed.

1. Use trivial representation with expectation - we'll show this
2. Now take original $\langle Tf, g \rangle$ can expand f in $\mathbb{R}^n$
3. multiply and divide by π^n_{good}, use independence, and fact that π^n_{good} is independent of I
4. expand g in $\mathbb{R}^n$, split into two sums by lengths of I_1 and I_2
5. Use independence again when $l(I_2) > l(I_1)$, compare $l(I_1) \leq l(I_2)$ to trivial sum
6. start over, expand g first, compare $l(I_1) > l(I_2)$ piece
7. Now we have expansion for $\mathbb{R}^n$
8. Now expand along $\mathbb{R}^m$: expand $h_{I_1} \otimes \langle f, h_{I_1} \rangle_1$
9. multiply and divide, use independence, expand g part, split sum

Thanks to the lemma, to prove 4 we now separate the sums over good cubes only.

$$\sum_{l(I_1)\leq l(I_2)} \sum_{l(J_1)\leq l(J_2)} \langle T(h_{I_1} \otimes u_{J_1})), h_{I_2} \otimes u_{J_2}\rangle \langle f, h_{I_1} \otimes u_{J_1}\rangle \langle g, h_{I_2} \otimes u_{J_2}\rangle$$

This separates into:

$$\sum_{l(I_1)\leq l(I_2)} = \sum_{d(I_1,I_2)>l(I_1)^{\gamma} l(I_2)^{1-\gamma}} + \sum_{I_1\subseteq I_2} + \sum_{I_1=I_2} + \sum_{d(I_1,I_2)\leq l(I_1)^{\gamma} l(I_2)^{1-\gamma}, I_1\cap I_2=\emptyset}$$

where the four sums are denoted separate, in, equal, and near.

We must do this for when $l(I_1) \leq l(I_2)$ and $l(J_1) \leq l(J_2)$ resulting the mixed types . Remember that we have suppressed the critical fact that all cubes are good

The proof of the main theorem is a case by case analysis of each scenario mentioned above. By extimation, each piece can be estimated by a sum of the simple shifts with good decay factors, allowing us the representation 2 and hence the following corollary.

Corollary

The singular integral operator defined in 2 is L^2 bounded.

A recent application of the importance of this representation is the famous A_2 conjecture, proved by Hytonen and Lerner, though with contributions by many, many others [4],[1]. Both the techniques leading to the proof of 4 and the applications of the result are fascinating! The type of representation in 4 and the lemmas used to prove it are in the flavor of [4]. And there are many other open questions that can possibly be attacked using such a representation, as it translates singular intergral operator questions to dyadic questions.

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- Lerner, Andrei. A simple proof of the A_2 conjecture. ArXiv, Feb 2012.
- Chang and Fefferman. A continuous version of duality of H^1 with BMO on the Bidisc. The Annals of Math, Jul 1980, p 179-201.
- Pott and Villarroya. A T(1) theorem on product Spaces. ArXiv, 10 Aug 2011.
- Hytonen, Tuomas. Representation of singular integrals by dyadic operators, and the A_2 Theorem. ArXiv, Aug 2011.