ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1985

THE UNIQUENESS OF THE DIRICHLET SPACE AMONG MOBIUS-INVARIANT HILBERT SPACES

BY

J. ARAZY AND S.D. FISHER

The Dirichlet space D on the unit disc $\Delta = \{z\colon |z| < 1\}$ consists of those analytic functions $f(z)$ on Δ for which the semi-norm

$$\rho_0(f) = \left(\frac{1}{\pi}\int_\Delta\int |f'(z)|^2\,dx\,dy\right)^{1/2} \tag{1}$$

is finite. Equivalently if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the power series of $f(z)$ valid in Δ, then

$$\rho_0(f) = \left(\sum_{n=1}^{\infty} n|a_n|^2\right)^{1/2} \tag{2}$$

so that D can be viewed as well as a weighted l^2 space. The Dirichlet space has this fundamental property: if $\varphi(z)$ is a Möbius function mapping the disc Δ into itself,

$$\varphi(z) = \lambda\frac{z-\alpha}{1-\bar{\alpha}z}, \quad |\alpha| < 1, |\lambda| = 1, \tag{3}$$

then

$$\rho_0(f\circ\varphi) = \rho_0(f). \tag{4}$$

Property (4) follows from (1) by replacing f by $f\circ\varphi$ and using the usual change of variables formula. In this paper we show that property (4) actually characterizes D. Indeed, we show that if H is a Hilbert space of analytic functions on Δ, continuously contained in the Bloch space, with the property that the Mobius group acts continuously and boundedly on H by composition, then, in fact, $H = D$ with equivalent norms. The details follow.

Let $\mathscr{M}$ denote the group of all Möbius functions of the form (3); $\mathscr{M}$ is topologized by making the bijection $\varphi \leftrightarrow (\lambda, \alpha)$ of $\mathscr{M}$ onto $T\times\Delta$ a homeomorphism. Let $\mathscr{B}$ denote the Bloch space on Δ; this consists of all analytic

Received April 18, 1983.

functions $g(z)$ on Δ for which the semi-norm

$$\rho_{\mathscr{B}}(g) = \sup_{z \in \Delta} (1 - |z|^2)|g'(z)| \tag{5}$$

is finite. The space $\mathscr{B}$ includes the algebra H^∞ of bounded analytic functions on Δ as well as the Dirichlet space. A linear space $\mathscr{H}$ of analytic functions on Δ is *Möbius invariant* if it satisfies the following condition:

$$f \circ \varphi \in \mathscr{H} \text{ whenever } f \in \mathscr{H} \text{ and } \varphi \in \mathscr{M}. \tag{6}$$

We shall suppose that there is a semi inner product $(\cdot, \cdot)$ on $\mathscr{H}$; that is, a map of $\mathscr{H} \times \mathscr{H}$ into $\mathbf{C}$ which satisfies all the usual axioms of an inner product with the exception that $(f, f) = 0$ need not imply that $f = 0$. Let

$$\rho(f) = (f, f)^{1/2} \geq 0, \quad f \in \mathscr{H}. \tag{7}$$

We suppose further that $\mathscr{H}$ is a linear subspace of the Bloch space $\mathscr{B}$ and that there is a constant A with

$$\rho_{\mathscr{B}}(f) \leq A\rho(f), \quad f \in \mathscr{H} \tag{8}$$

It follows from (8) that the kernel of the semi-norm ρ is either $\{0\}$ or $\mathbf{C}$. We define a norm on $\mathscr{H}$ by

$$\|f\| = \rho(f) \quad \text{if } \rho^{-1}(0) = \{0\} \tag{9}$$

or

$$\|f\| = \sqrt{\rho^2(f) + |f(0)|^2} \quad \text{if } \rho^{-1}(0) = \mathbf{C}. \tag{10}$$

We make two more assumptions:

(11) $\mathscr{H}$ is complete in the norm given in (9) or (10);

(12) for each $f \in \mathscr{H}$, the mapping $\varphi \mapsto f \circ \varphi$ is continuous from $\mathscr{M}$ into $\mathscr{H}$. It is almost immediate and certainly quite easy that the Dirichlet space D satisfies (6)–(12), with (10). For that matter so does the space of those functions f with f' in the Hardy space H^2 and a number of other Hilbert spaces of analytic functions on Δ. What we shall show in this paper, however, is that among all such Hilbert spaces *only* D satisfies (4) or even a substantial weakening of (4).

THEOREM 1. *Let* $\mathscr{H}$ *satisfy* (6)–(12). *If*

$$\rho(f) = \rho(f \circ \varphi), \quad f \in \mathscr{H}, \varphi \in \mathscr{M} \tag{13}$$

then there is a positive constant λ *with*

$$\rho(f) = \lambda\rho_0(f), \quad f \in \mathscr{H} \tag{14}$$

and hence $\mathscr{H}$ *is exactly D.*

THEOREM 2. *Let* $\mathscr{H}$ *satisfy* (6)–(12). *If there is a positive constant* δ, $0 < \delta < 1$, *with*

$$\delta\rho(f) \le \rho(f \circ \varphi) \le \frac{1}{\delta}\rho(f), \quad f \in \mathscr{H}, \varphi \in \mathscr{M} \tag{15}$$

then there is a positive constant ν *with*

$$\nu\rho_0(f) \le \rho(f) \le \frac{1}{\nu}\rho_0(f), \quad f \in \mathscr{H} \tag{16}$$

and hence $\mathscr{H}$ *is exactly D.*

Theorem 1 is quite direct and is proved in Section 1. The proof of Theorem 2 is considerably more involved and it is contained in Section 2. Section 3 contains several examples which show that Theorem 2 is "best possible" in a number of ways.

Remark. It is worth pointing out explicitly here that conditions (6) and (15) force (8) to hold if there is at least one linear functional L on $\mathscr{H}$ which satisfies

$$|L(f)| \le M \sup\{|f(z)| : f \in K\} \tag{17}$$

for some constant M, some compact set K in Δ, and all $f \in \mathscr{H}$. This is a theorem of L.A. Rubel and R.M. Timoney [5]. Moreover, (17) is a natural condition if norm convergence is to imply uniform convergence on compact sets in Δ. Hence, we may as well assume (8) initially.

We begin by obtaining several conclusions from the hypotheses (6)–(12) and (15). First let f be any non-constant function in $\mathscr{H}$ and consider the integral

$$u_k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}z)e^{-ik\theta}\,d\theta, \quad k = 1, 2, \dots .$$

u_k is an element of $\mathscr{H}$ (by (6) and (12)) and a simple computation gives $u_k(z) = a_k z^k$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since f is non-constant there is at least one $n \ge 1$ for which $a_n \ne 0$. Hence, $z^n \in \mathscr{H}$ for some $n \ge 1$. Thus, $(z - r)^n/(1 - rz)^n$ lies in $\mathscr{H}$ for every r, $r \in (-1, 1)$. There are many choices of r

for which

$$0 \neq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{it} - r}{1 - re^{it}} \right)^n e^{-it} \, dt$$

and hence $z \in \mathscr{H}$ by the argument above. This shows that $(z - r)/(1 - rz)$ lies in $\mathscr{H}$ for all $r \in (-1, 1)$ and yields

$$(1 - r^2) r^{k-1} z^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{ze^{i\theta} - r}{1 - re^{i\theta} z} \right) e^{-ik\theta} \, d\theta, \quad k = 1, 2, \dots .$$

Consequently, z^k lies in $\mathscr{H}$ for all $k = 1, 2, \dots$. Further, by taking semi-norms of both sides we find that

$$(1 - r^2) r^{k-1} \rho(z^k) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho \left(\frac{ze^{i\theta} - r}{1 - re^{i\theta} z} \right) d\theta \leq \frac{1}{\delta} \rho(z), \quad -1 < r < 1.$$

If we choose $r^2 = (k - 1)/k$ we obtain

$$\rho(z^k) k^{-1} (1 - 1/k)^{(k-1)/2} \leq (1/\delta) \rho(z)$$

This yields the estimate

$$\rho(z^k) \leq A'k, \quad k = 1, 2, \dots \tag{18}$$

for some constant A' (and shows as well that $\rho(z) \neq 0$.) We see that any function analytic on a neighborhood of $|z| \leq 1$ is in $\mathscr{H}$ and also that the series $\sum_{k=0}^{\infty} r^k z^k$ is absolutely convergent in $\mathscr{H}$ for any r, $|r| < 1$. Hence, in the semi inner product $(\cdot, \cdot)$ we can bring summation from the inside to the outside and assert, for instance, that

$$\left(\frac{1}{1 - rz}, \frac{1}{1 - sz} \right) = \sum_{j,k=0}^{\infty} r^j s^k (z^j, z^k)$$

if $-1 < r, s < 1$.

Let $f \in \mathscr{H}$ and let $r \in (0, 1)$. We set $f_r(z) = f(rz)$; we actually have

$$f_r(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi_\theta(z)) P_r(\theta) \, d\theta$$

where P_r is the Poisson kernel and $\psi_\theta(z) = e^{i\theta} z$. Since $f \circ \psi_\theta$ is a continuous function of θ we see that f_r lies in $\mathscr{H}$ and, further, $f_r \to f$ in the norm of $\mathscr{H}$ as $r \to 1$. Hence, the functions analytic on a neighborhood of $|z| \leq 1$ are dense in $\mathscr{H}$.

1. The proof of Theorem 1

Note first that (13) actually gives us

$$(f \circ \varphi, g \circ \varphi) = (f, g), \quad f, g \in \mathscr{H}, \varphi \in \mathscr{M}.$$

First take φ to be $\psi_\theta(z) = e^{i\theta}z$. Then

$$(z^k, z^n) = (z^k \circ \psi_\theta, z^n \circ \psi_\theta) = (e^{ik\theta}z^k, e^{in\theta}z^n) = e^{i(k-n)\theta}(z^k, z^n)$$

and so

$$(z^k, z^n) = 0 \quad \text{if } k \neq n. \tag{19}$$

Next, take φ to be $\varphi_r(z) = (z - r)/(1 - rz)$, $-1 < r < 1$. Then

$$\begin{aligned}(1,1) + r^2(z, z) &= (1 + rz, 1 + rz)\\ &= (1 + r\varphi_r, 1 + r\varphi_r)\\ &= (1 - r^2)^2 \sum_{k=0}^{\infty} r^{2k}(z^k, z^k).\end{aligned}$$

The coefficient of r^2 gives $(z, z) = -2(1,1) + (z, z)$ so that $(1,1) = 0$. Further, the coefficient of r^{2n}, $n \geq 2$, gives

$$0 = (z^n, z^n) - 2(z^{n-1}, z^{n-1}) + (z^{n-2}, z^{n-2}), \quad n \geq 2,$$

so that by solving recursively we obtain

$$(z^n, z^n) = n(z, z), \quad n = 2, 3, 4, \ldots . \tag{20}$$

Setting $\lambda^2 = (z, z)$ we thus have

$$\rho^2(f) = (f, f) = \lambda^2 \sum_{n=1}^{\infty} n|a_n|^2 = \lambda^2\rho_0^2(f) \tag{21}$$

as we wished to show.

2. Proof of Theorem 2

We begin by introducing two new semi inner products on $\mathscr{H}$ which produce semi-norms equivalent to ρ. The first is

$$[f, g] = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f \circ \psi_\theta, g \circ \psi_\theta)\, d\theta \tag{22}$$

and the second is

$$\langle f, g\rangle = m(f \circ \varphi_r, g \circ \varphi_r) \tag{23}$$

where, as before

$$\psi_\theta(z) = e^{i\theta}z, \quad -\pi \leq \theta < \pi,$$
$$\varphi_r(z) = \frac{z-r}{1-rz}, \quad -1 < r < 1,$$

and m is an invariant mean on the abelian group $\mathscr{G} = \{\varphi_r\colon -1 < r < 1\}$; see [4]. The semi inner product in (22) is rotation invariant

$$[f \circ \psi_\theta, g \circ \psi_\theta] = [f, g], \quad -\pi \leq \theta \leq \pi \tag{24}$$

while the semi-inner product in (23) is invariant under the group $\mathscr{G}$ in the sense that

$$\langle f \circ \varphi_r, g \circ \varphi_r\rangle = \langle f, g\rangle, \quad -1 < r < 1. \tag{25}$$

Further, because of (15) we have

$$\delta\rho(f) \leq [f, f]^{1/2} \leq \frac{1}{\delta}\rho(f), \quad f \in \mathscr{H},$$
$$\delta\rho(f) \leq \langle f, f\rangle^{1/2} \leq \frac{1}{\delta}\rho(f), \quad f \in \mathscr{H},$$

so that the semi-norms produced by $[\ ,\ ]$ and $\langle\ ,\ \rangle$ are equivalent to ρ. Thus, we may, and we will, work with the semi inner products (22) and (23) without altering our space $\mathscr{H}$. We can not assume that the inner product *simultaneously* satisfies (24) and (25) unless we assume that $\rho(f \circ \varphi) = \rho(f)$ for all $\varphi \in \mathscr{M}$; this is just the situation handled in Section 1.

Using the rotation invariant semi-norm defined in (22) we find that

$$[f, f] = \sum_{n=0}^{\infty} |a_n|^2 [z^n, z^n]$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an element of $\mathscr{H}$. Hence, to show that $\mathscr{H}$ is just the Dirichlet space we need only show that there is a positive constant c such that

$$cn \leq [z^n, z^n] \leq \frac{1}{c}n, \quad n = 1, 2, \ldots$$

or, equivalently,

$$c'n \leq \langle z^n, z^n\rangle \leq \frac{1}{c'}n \tag{26}$$

for some constant c'.

The semi inner product $\langle \ , \ \rangle$ is invariant under the group $\mathscr{G}$ by (25) so that for $n, k \geq 0$ we have

$$\langle z^n, z^k \rangle = \langle (\varphi_r)^n, (\varphi_r)^k \rangle, \quad -1 < r < 1. \tag{27}$$

The functions $(\varphi_r)^n$ and $(\varphi_r)^k$ are real analytic functions of r and because of (18) we know that the right-hand side of (27) is also a real analytic function of r. We differentiate the right-hand side of (27) with respect to r and then set $r = 0$; by doing this we obtain

$$0 = n\left\langle z^{n-1}(z^2 - 1), z^k \right\rangle + k\left\langle z^n, z^{k-1}(z^2 - 1)\right\rangle. \tag{28}$$

Set

$$\alpha_{j,k} = \langle z^j, z^k \rangle \quad \text{and} \quad \beta_k = \alpha_{k,k} = \langle z^k, z^k \rangle.$$

From (28) we get

$$0 = n\{\alpha_{n+1,k} - \alpha_{n-1,k}\} + k\{\alpha_{n,k+1} - \alpha_{n,k-1}\}. \tag{29}$$

In (29), take $n = k + 1$ and add the resulting expressions from $k = 0$ to $k = N$. The result is

$$N\beta_{N+1} = \beta_0 + 2\sum_{k=1}^{N} \beta_k - (N + 1)\alpha_{N+2,N}.$$

Thus,

$$\frac{\beta_{N+1}}{N + 1} = \frac{\beta_0 + 2\sum_{k=1}^{N} \beta_k}{N(N + 1)} - \frac{\langle z^{N+2}, z^N \rangle}{N}. \tag{30}$$

In order to estimate β_n/n we need to get good estimates on the rate of growth of $S_N = \sum_{k=1}^{N}\beta_k + \frac{1}{2}\beta_0$. Set $b_k = [z^k, z^k]$, $k = 0, 1, 2, \dots$. For $r \in (-1, 1)$, we have

$$\begin{aligned} \delta^{-4}\beta_1 = \delta^{-4}\langle z, z \rangle &= \delta^{-4}\langle \varphi_r, \varphi_r \rangle \geq [\varphi_r, \varphi_r] \\ &= r^2 b_0 + (1 - r^2)^2 \sum_{k=0}^{\infty} r^{2k} b_{k+1} \\ &\geq (1 - r^2)^2 \sum_{k=0}^{N} r^{2k} b_{k+1}. \end{aligned}$$

Choose $r^2 = N/(N + 1)$; we find that

$$S_N \leq CN^2 \tag{31}$$

for some constant C and all N. This is enough to actually prove that $\beta_n \le C'n$ as we now show. We begin with (30).

$$\frac{\beta_{N+1}}{N+1} = \frac{2S_N}{N(N+1)} - \frac{\langle z^N, z^{N+2}\rangle}{N}$$
$$= \frac{2S_N}{N(N+1)} - \frac{1}{2N}\langle z^N + z^{N+2}, z^N + z^{N+2}\rangle + \frac{1}{2N}(\beta_N + \beta_{N+2}).$$

Using the equivalence of the semi-norms we find that

$$\frac{\delta^8}{2N}\{\beta_N + \beta_{N+2}\} \le \frac{1}{2N}\langle z^N + z^{N+2}, z^N + z^{N+2}\rangle$$
$$= \frac{2S_N}{N(N+1)} + \frac{1}{2N}\{\beta_N + \beta_{N+2}\} - \frac{\beta_{N+1}}{N+1}.$$

Rearranging we have

$$2\frac{\beta_{N+1}}{N+1} \le A'' + (1-\delta^8)\left\{\frac{\beta_N}{N} + \frac{\beta_{N+2}}{N+2}\right\} \tag{32}$$

where A'' is a constant which incorporates the upper bound of CN^2 on S_N and the bounded term

$$(\beta_{N+2})\left(\frac{1}{N} - \frac{1}{N+2}\right).$$

Let

$$\gamma_n = \frac{\beta_n}{n} - \frac{A''}{2\delta^8}.$$

Then (32) is equivalent to

$$2\gamma_{N+1} \le (1-\delta^8)\{\gamma_N + \gamma_{N+2}\} \tag{32$'$}$$

and we know from (31) that $\gamma_N \le C'N^2$ for all N. Suppose that there is some integer, say M, such that

$$\gamma_{M+1} + \gamma_M > 0, \quad \gamma_{M+1} - \gamma_M > 0.$$

Then

$$\gamma_{M+1} - \gamma_M \le \gamma_{M+1} - \delta^8\gamma_M + (1-\delta^8)\gamma_{M+2} - 2\gamma_{M+1}$$
$$= (1-\delta^8)(\gamma_{M+2} - \gamma_{M+1}) - \delta^8(\gamma_M + \gamma_{M+1})$$
$$< (1-\delta^8)(\gamma_{M+2} - \gamma_{M+1}).$$

Consequently, $\gamma_{M+2} - \gamma_{M+1}$ and $\gamma_{M+2} + \gamma_{M+1}$ are both positive and the argument can be repeated indefinitely yielding

$$\gamma_{M+1} - \gamma_M \leq (1 - \delta^8)^k \{\gamma_{M+k+1} - \gamma_{M+k}\}, \quad k = 1, 2, \ldots .$$

This surely implies that the sequence $\{\gamma_n\}$ grows exponentially fast, contradicting (31). Hence, for every n we have either

$$(\gamma_{n+1} + \gamma_n)(\gamma_{n+1} - \gamma_n) \leq 0 \tag{33}$$

or

$$\gamma_{n+1} + \gamma_n \leq 0 \quad \text{and} \quad \gamma_{n+1} - \gamma_n \leq 0. \tag{34}$$

In either case $\gamma_{n+1} \leq |\gamma_n|$ for all n. Since $\gamma_n \geq -A''/2\delta^8$ for all n we see that $\{\gamma_n\}$ remains bounded and so

$$\beta_n \leq c'n, \quad n = 1, 2, 3, \ldots \tag{35}$$

for all n, where c' is some constant.

We next establish the lower bound $s_N \geq cN^2$. We have

$$\begin{aligned}
\delta^8 r^2[z, z] &\leq \delta^8[1 + rz, 1 + rz] \\
&\leq \delta^4\langle 1 + rz, 1 + rz\rangle \\
&= \delta^4\langle 1 + r\varphi_r, 1 + r\varphi_r\rangle \\
&= \delta^4(1 - r^2)^2\left\langle \frac{1}{1 - rz}, \frac{1}{1 - rz}\right\rangle \\
&\leq (1 - r^2)^2\left[\frac{1}{1 - rz}, \frac{1}{1 - rz}\right] \\
&= (1 - r^2)2\sum_{k=0}^{\infty} r^{2k} b_k \\
&\leq (1 - r^2)2\left\{\sum_{k=0}^{mN} b_k + \sum_{mN+1}^{\infty} r^{2k} b_k\right\}
\end{aligned}$$

where m is an integer to be determined shortly. Take $r^2 = N/(N + 1)$ and use the fact that $b_k \leq c''k$ from (35). Thus,

$$\begin{aligned}
\sum_{mN+1}^{\infty}\left(\frac{N}{N + 1}\right)^k b_k &\leq c''\sum_{mN+1}^{\infty} k\left(\frac{N}{N + 1}\right)^k \\
&< c''\int_{mN}^{\infty} xe^{px}\,dx
\end{aligned}$$

where $p = \log(N/(N+1))$. The integral has the value

$$\left\{\frac{-mN}{p} + \frac{1}{p^2}\right\}\left(\frac{N}{N+1}\right)^{mN}. \tag{36}$$

Multiply the expression in (36) by $(1-r^2)^2 = 1/(N+1)^2$ and let $N \to \infty$; the limit is $(m+1)e^{-m}$. Choose m so big that

$$c''(m+1)e^{-m} \le \tfrac{1}{3}\delta^8 b_1.$$

Hence,

$$\delta^8\left(\frac{N}{N+1}\right)b_1 \le (N+1)^{-2}\left\{\sum_0^{mN} b_k\right\} + \tfrac{1}{3}\delta^8 b_1, \quad N \text{ large},$$

and so $\sum_0^{mN} b_k \ge \eta N^2$ for some positive constant η. Thus,

$$\sum_0^N b_k \ge \eta' N^2, \quad \text{all } N \tag{37}$$

as we wished to show. Combining the estimate in (37) with (30) we obtain

$$\eta' \le \frac{\beta_{N+1}}{N+1} + \left(\frac{\beta_N}{N}\right)^{1/2}\left(\frac{\beta_{N+2}}{N}\right)^{1/2}. \tag{38}$$

We shall show below that there is a constant γ with

$$\beta_{n+1} \le \gamma\beta_n, \quad n = 1, 2, \dots. \tag{39}$$

Using this in (38) we find that $\eta' \le 2\gamma\beta_N/N$ which together with (35) gives (26). Next, we must prove (39). To do this we write

$$\begin{aligned} b_n &= [z^n, z^n] \\ &\ge \delta^4\langle z^n, z^n\rangle \\ &= \delta^4\langle \varphi_r^n, \varphi_r^n\rangle \\ &\ge \delta^8[\varphi_r^n, \varphi_r^n] \\ &= \delta^8 \sum_{j=1}^{\infty} |c_{j,n}(r)|^2 b_j \\ &\ge \delta^8 |c_{n+1,n}(r)|^2 b_{n+1} \end{aligned}$$

where we have written

$$\varphi_r^n(z) = \left(\frac{z-r}{1-rz}\right)^n = \sum_{j=0}^{\infty} c_{j,n}(r) z^j.$$

Now

$$c_{n+1,n}(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{it}-r}{1-re^{it}}\right)^n e^{-i(n+1)t}\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1-re^{-it}}{1-re^{it}}\right)^n e^{-it}\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1+\frac{2ri\sin t}{1-re^{it}}\right)^n e^{-it}\,dt.$$

Take $r = 1/n$ and let $n \to \infty$; the integral converges to the value

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2i\sin t}e^{-it}\,dt.$$

This integral can be easily evaluated by the residue theorem; its value is

$$\sum_{j=0}^{\infty}(-1)^j\frac{1}{j!(j+1)!} = 1 - \frac{1}{2} + \frac{1}{12} - \frac{1}{144} + \cdots > 1/2$$

and so (39) is proved. The last step of the proof of Theorem 2 is to show that $b_0 = \beta_0 = 0$. Suppose, to the contrary, that $[1,1] = b_0 \neq 0$; we shall derive a contradiction. If $f \in \mathscr{H}$, then

$$\rho^2(f) \geq \delta^2[f,f] = \delta^2\left\{b_0|f(0)|^2 + \sum_1^{\infty}|a_n|^2 b_n\right\}.$$

Now if we write $f(\varphi(z)) = f(\varphi(0)) + \sum_1^{\infty}a'_n z^n$ then we know that

$$[f\circ\varphi, f\circ\varphi] = b_0|f(\varphi(0))|^2 + \sum_1^{\infty}|a'_n|^2 b_n.$$

The numbers b_n satisfy

$$n\nu \leq b_n \leq \frac{1}{\nu}n, \quad n = 1,2,\ldots$$

and hence

$$\sum_1^{\infty}|a'_n|^2 b_n \geq \nu\sum_1^{\infty}|a'_n|^2 n$$
$$= \nu\rho_0(f\circ\varphi)$$
$$= \nu\rho_0(f)$$
$$= \nu\sum_1^{\infty}|a_n|^2 n$$
$$\geq \nu^2\sum_1^{\infty}|a_n|^2 b_n.$$

Thus

$$\begin{aligned} b_0|f(0)|^2 + \sum_1^\infty |a_n|^2 b_n &= [f,f] \\ &\geq \delta^8 [f\circ\varphi, f\circ\varphi] \\ &= \delta^8 \left[b_0 |f(\varphi(0))|^2 + \sum_1^\infty |a_n'|^2 b_n \right] \\ &\geq \delta^8 \left[b_0 |f(\varphi(0))|^2 + \nu^2 \sum_1^\infty |a_n|^2 b_n \right]. \end{aligned}$$

Hence,

$$b_0|f(0)|^2 + (1-\delta^8\nu^2)\sum_1^\infty |a_n|^2 b_n \geq \delta^8 b_0 |f(\varphi(0))|^2.$$

Since $\varphi(0)$ can be any point of Δ we see that $f(z)$ is bounded in Δ by an expression equivalent to its norm. Thus, every element of the Hilbert space $\mathscr{H}$ is bounded. However, we already know that D is $\mathscr{H}$ and since D contains unbounded functions we must have $b_0 = 0$.

3. Examples

The Möbius group $\mathscr{M}$ is composed of the two (abelian) subgroups

$$\mathscr{G} = \{\varphi_r\colon -1 < r < 1\} \quad \text{and} \quad \mathscr{R} = \{\psi_\theta\colon -\pi \leq \theta \leq \pi\}$$

in the following sense. If

$$\varphi(z) = \lambda \frac{z-\alpha}{1-\bar{\alpha}z}, \qquad \lambda = e^{ib}, \alpha = re^{it},$$

then

$$\varphi(z) = \psi_{b+t}(\varphi_r(\psi_{-t}(z))), \quad z \in \Delta.$$

We now give two examples which show that a semi-norm can be invariant under either of the groups $\mathscr{G}$ and $\mathscr{R}$ (but not both) and yet still not be equivalent to ρ_0.

Example 1. Let $\mathbf{w} = \{w_n\}$ be a sequence of non-negative numbers and let $H_{\mathbf{w}}$ consist of all analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$ on Δ for which

$$\rho_{\mathbf{w}}(f) = \left(\sum_0^\infty |a_n|^2 w_n \right)^{1/2} < \infty. \tag{40}$$

The semi-inner product

$$(f, g)_{\mathrm{w}} = \sum_0^\infty a_n \bar{b}_n w_n \tag{41}$$

satisfies (7) and, further, $(f \circ \psi_\theta, g \circ \psi_\theta)_{\mathrm{w}} = (f, g)_{\mathrm{w}}$ for all θ and all $f, g \in H_{\mathrm{w}}$. Furthermore, the mapping $\tau(\theta) = f \circ \psi_\theta$ is a continuous map from the unit circle into H_{w} for each $f \in H_{\mathrm{w}}$. The form (41) is an inner product precisely when $w_n > 0$ for all n. If $w_1 = 0$ but $w_2 > 0$ then $\mathscr{H}_{\mathrm{w}}$ is not equivalent to $\mathscr{D}$.

Example 2. In the Dirichlet space D consider the operator

$$(Tf)(z) = (z^2 - 1)f'(z).$$

T is the infinitesimal generator of the one parameter group

$$\tilde{G} = \{C_r\colon -1 < r < 1\} \quad \text{where } C_r(f) = f \circ \varphi_r.$$

That is,

$$Tf = \frac{d}{dr}(f \circ \varphi_r)|_{r=0}$$

where differentiation is taken in the strong topology of D. T is an unbounded, closed, densely defined, purely imaginary operator (see [3; Chapter XII.6] and [2]).

Let u be a non-negative, measurable function on the imaginary axis and let $u(T)$ be the positive operator (usually unbounded) corresponding to u by the usual operational calculus (see [3; Chapter XII]). Let H_u be the domain of $u(T)$ and set

$$(f, g)_u = (u(T)f, f)_D, \quad f, g \in H_u.$$

This semi inner-product is invariant under the group $\tilde{G}$ since $u(T)$ commutes with C_r for all r.

If $u > 0$ a.e., then $(\cdot,\cdot)_u$ is an inner product; this inner product is equivalent to that of D if and only if u is an invertible element of L^∞. Thus, a non-invertible positive element u of L^∞ will produce a Hilbert space of analytic functions on Δ which is invariant under $\mathscr{G}$ but which is not the Dirichlet space D.

Example 3. It is possible for both the groups $\mathscr{G}$ and $\mathscr{R}$ to act on a space simultaneously in such a way that (6)–(12) hold and yet the space is not D. One simple example is the Hilbert space H consisting of all analytic functions $f(z) = \sum_0^\infty a_n z^n$ on Δ with

$$\rho^2(f) = \sum_0^\infty n^2 |a_n|^2 = \|f'\|_{H^2}^2 < \infty$$

where H^2 is the Hardy space. Here, $\rho(f \circ \psi_\theta) = \rho(f)$ for all $f \in H$ and all $\theta \in [-\pi, \pi]$ and further

$$\rho(f \circ \varphi_r) \leq \frac{2}{(1-r)^2}\rho(f), \quad 0 < r < 1, f \in H$$

so that $\mathscr{G}$ acts continuously on H. But exactly because the action of $\mathscr{G}$ is not uniformly bounded the conclusion of Theorem 2 can not, and does not, hold.

Remarks. (1) The Möbius group $\mathscr{M}$ is not amenable [4] so the proof of Theorem 2 can not be reduced to the case of Theorem 1 by averaging over $\mathscr{M}$ as we averaged over $\mathscr{G}$ in (23).

(2) The arguments that showed that s_N is comparable to N^2 are really more general and can easily be adapted to prove the following:

THEOREM. *Suppose* $\{b_n\}$ *is a sequence of non-negative numbers and let* $s_k = \Sigma_0^k b_n$, $k = 1, 2, \ldots$ *be the sequence of partial sums. Assume that* $A(t) = \Sigma_0^\infty b_n t^n$ *converges for all* $0 < t < 1$. *Then the following are equivalent*:

(i) *There is a constant K such that* $K^{-1} \leq (1-t)^2 A(t) \leq K$, $0 \leq t < 1$.

(ii) *There is a constant L such that* $L^{-1} \leq n^{-2}s_n \leq L$, $n = 1, 2, \ldots$.

A closely related result appears in [1; Theorem 1.10a]; the authors thank the referee for bringing this reference to their attention.

Acknowledgment. J. Arazy wishes to express his appreciation to L.A. Rubel for introducing him to the subject of Möbius invariant spaces and for several valuable discussions on the subject.

REFERENCES

1. G. Bennet, D. Stegenga and R. Timoney, *Coefficients of Bloch and Lipshitz functions*, Illinois J. Math., vol. 25 (1981), pp. 520–531.
2. P. L. Butzer and H. Behrens, Semi-groups of operators and approximation, Springer Verlag, New York, 1967.
3. N. Dunford and J. T. Schwartz, *Linear operators* II, Interscience, New York 1963.
4. F. Greenleaf, *Invariant means on topological groups*, Van Nostrand, New York, 1969.
5. L.A. Rubel and R.M. Timoney, *An extremal property of the Bloch space*, Proc. Amer. Math. Soc., vol. 75 (1979), pp. 45–49.

University of Haifa
Haifa, Israel
Northwestern University
Evanston, Illinois
The Technion
Haifa, Israel