# Interpolation

by

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In this paper we shall give detailed proofs of the following two theorems.

**Theorem 1.** Let D denote the Dirichlet space (i.e. the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel, $\frac{1}{\bar{\lambda} z} \ln \frac{1}{1-\bar{\lambda} z}$), and let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $z_1, \ldots, z_n \in \mathbb{C}$ There exists a multiplier $\phi$ of D with $\phi(\lambda_i) = z_i$ for each $i \leqslant n$ and of norm (as a multiplier) less than or equal to one if and only if the $n \times n$ matrix

$$\left( (1 - \bar{z}_i z_j) \frac{1}{\bar{\lambda}_i \lambda_j} \ln \frac{1}{1 - \bar{\lambda}_i \lambda_j} \right)$$

is positive semidefinite.

**Theorem 2.** Let $H^\infty(\mathbb{D}^2)$ denote the bounded analytic functions on the bidisc, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}^2$ and $z_1, \ldots, z_n \in \mathbb{C}$ There exists a function $\phi \in H^\infty(\mathbb{D}^2)$ with $\phi(\lambda_i) = z_i$ for each $i \leqslant n$ and with $sup|\phi(\lambda)| \leqslant 1$ if and only if there exist two positive semidefinite matrices $(a_{ij}^1)$ and $(a_{ij}^2)$ such that

$$1 - \bar{z}_i z_j = (1 - \bar{z}_{i1} \lambda_{j1}) a_{ij}^1 + (1 - \bar{\lambda}_{i2} \lambda_{j2}) a_{ij}^2$$

for all $k, j \leqslant n$.

## I. An Abstract Interpolation Theorem.

Fix $n$, a positive integer and set $N = \{i \mid i \text{ is an integer and } 1 \leqslant i \leqslant n\}$. If $k: N \times N \to \mathbb{C}$ we say that $k$ is *positive semidefinite* (more briefly $k \geqslant 0$) if

$$\sum_{i,j \in N} k(i,j) c_j \overline{c_i} \geqslant 0$$

whenever $c = (c_i) \in \mathbb{C}^n$. Clearly, if $k: N \times N \to \mathbb{C}$ and $k \geqslant 0$, then

$$k(i,j) = 0 \text{ if either } k(i,i) = 0 \text{ or } k(j,j) = 0.$$

Define the *support of* $k$, $\mathrm{spt}(k)$, by

$$spt(k) = \{i \in N \mid k(i,i) \neq 0\}.$$

By a *kernel on* $N$ we shall mean a map $k: N \times N \to \mathbb{C}$ such that $k \geqslant 0$ and with the additional property that

$$\left.\begin{array}{c} \text{if } c \in \mathbb{C}^n \text{ and } \sum_{i,j \in N} k(i,j) c_j \overline{c_i} = 0, \text{ then} \\ c_i = 0 \text{ whenever } i \in spt(k). \end{array}\right\} \tag{1.1}$$

If $k$ is a kernel on $N$ we may define for each $i \in N$, $k_i: N \to \mathbb{C}$ by requiring that

$$k_i(j) = k(i,j).$$

We may then define an inner product on formal linear combinations $\sum_{i \in spt(k)} c_i k_i$, by

$$< \sum_{i \in spt(k)} c_i k_i , \sum_{i \in spt(k)} c_i k_i >_k = \sum_{i,j \in spt(k)} c_j \overline{c_i} .$$

Let $\mathbf{H}_k$ denote the resulting Hilbert space. If $\nu(k)$ denotes the cardinality of spt(k) then it is clear that

$$dim\ \mathbf{H}_k = \nu(k).$$

Now let $I \subseteq N$. For k a kernel on N define $\mathbf{H}_k(I) \subseteq \mathbf{H}_k$ by

$$\mathbf{H}_k(I) = \left[ k_i \mid i \in I \cap spt(k) \right],$$

where throughout this paper if H is a Hilbert space, I is an index set, and $f_i \in \mathbf{H}$ whenever $i \in I$, then $[f_i \mid i \in I]$ denotes the closed linear span of $\{f_i \mid i \in I\}$. It is clear that if $I_0 \subseteq I_1 \subseteq N$, then $\mathbf{H}_k(I_0) \subseteq \mathbf{H}_k(I_1)$.

If H is a Hilbert space let L(H) denote the bounded linear transformations of H . If $I \subseteq N$ and $z: I \to \mathbb{C}$ define $T_{k,z} \in \mathrm{L}(\mathbf{H}_k(I))$ by requiring that

$$T_{k,z} k_i = \overline{z(i)} k_i \quad , \ i \in I \cap spt(k).$$

If $I_0 \subseteq I_1 \subseteq N$, $z_0: I_0 \to \mathbb{C}$, and $z_1: I_1 \to \mathbb{C}$, we say that $z_1$ *is an extension of* $z_2$ if $z_1 = z_2 | I_0$. It is easy to check that if $z_2$ is an extension of $z_1$ then $\mathbf{H}_k(I_0)$ is invariant for $T_{k,z_1}$ and

$$T_{k,z_0} = T_{k,z_1} \mid \mathbf{H}_k(I_0).$$

Now let K be a collection of kernels on N. We shall say that K is a *kernel structure* if it satisfies the four properties described below.

**Property 1.2.** The elements of K are normalized in the following way. If $k \in K$ and $i \in spt\ k$, then $k_i(i) = k(i,i) = 1$. Observe that if k is *any* kernel on $N$ and $(c_i) \in \mathbb{C}^n$ is defined by

$$c_i = \begin{cases} \dfrac{1}{\sqrt{k_i(i)}} & \text{if} \quad k_i(i) \neq 0 \\ 0 & \text{if} \quad k_i(i) = 0 \end{cases}$$

then the kernel $\tilde{k}$ defined by

$$\tilde{k}(i,j) = \overline{c_i}\, c_j\, k(i,j) \qquad i,j \in N$$

satisfies the normalization conditions.

**Property 1.3.** If $k, k' \in K$, $c, c' \in \mathbb{C}^n$, then

want $|c_i|^2 + |c_i'|^2 = 1\ \forall i$ or fails Property 1.2 !

$$\left(\overline{c_i}\, c_j k_i(j) + \overline{c_i}'\, c_j' k_i'(j)\right) \in K .$$

**Property 1.4.** K is endowed with a metric,

$$d(k,k') = \sum_{i,j \in N} |\, k_i(j) - k_i'(j)| \ ,$$

and K is compact in this metric.

**Property 1.5.** K possesses a Carleson constant. Observe that (1.1) implies that if k is a kernel, then

$$(k_i(j))_{i,j \in spt(k)} > 0 \ ,$$

i.e. $(k_i(j))_{i,j \in spt\ k}$ is strictly positive definite. To say K possesses a Carleson constant means that there exists a constant $\delta > 0$ such that

$$(k_i(j))_{i,j\in spt(k)} \geq \delta$$

for all $k \in \mathbf{K}$.

Now if $\mathbf{K}$ is a kernel structure on N, $I \subseteq N$, and $z: I \to \mathbb{C}$ we may define

$$\|z\| = \sup_{k\in \mathbf{K}} \|T_{k,z}\| . \tag{1.6}$$

**Definition 1.7.** We say a kernel structure $\mathbf{K}$ on N is an NP *structure* if for all $I \subseteq N$ and all $z: I \to N$ we have that

$$\inf_{\substack{\tilde z: N\to \mathbb{C}\\ \tilde z | I = z}} \|\tilde z\| = \|z\| . \tag{1.8}$$

We now prove four lemmas which hopefully will shed some light on the nature of kernel structures.

**Lemma 1.9.** *Let* k *be a kernel on* N *with* spt(k) = N, *and assume that* $z: N \to \mathbb{C}$ *Let* $\mathbb{C}^n$ *denote the canonical Hilbert space of dimension* n *and define* $D_z \in \mathbf{L}(\mathbb{C}^n)$ *by*

$$D_z e_i = \overline{z(i)} e_i \quad , \quad i \in N \quad ,$$

*where* $\{e_i\}$ *is the standard orthonormal basis in* $\mathbb{C}^n$. *Define an* $n \times n$ *matrix* $A_k$ *by setting*

$$A_k = (k_j(i))_{i,j\in N} .$$

*Then* $T_{k,z}$ *and* $A^{\frac{1}{2}} D_z A_k^{-\frac{1}{2}}$ *are unitarily equivalent.*

Proof: Observe that because k is a kernel and $\mathrm{spt}(k) = \mathbb{N}$, $A_k > 0$, so that $A_k^{\frac{1}{2}}$ and $A_k^{-\frac{1}{2}}$ are well defined. Also, note that if we define $L_k\colon \mathrm{H}_k \to \mathbb{C}^n$ by the formula

$$L_k\left(\sum_{i\in N} c_i k_i\right) = A_k^{\frac{1}{2}}\left(\sum_{i\in N} c_i e_i\right),$$

then $L_k$ is a Hilbert space isomorphism. Calculating, one obtains that

$$L_k T_{k,z} = \left(A_k^{\frac{1}{2}} D_z A_k^{-\frac{1}{2}}\right) L_k ,$$

which establishes the lemma.

The next two lemmas assert that the optima in (1.6) and (1.8) are attained.

Lemma 1.10. *Let* $\mathbb{K}$ *be a kernel structure on* $\mathbb{N}$, *let* $I \subseteq N$, *and let* $z\colon I \to \mathbb{C}$ *There exists* $k_0 \in \mathbb{K}$ *such that*

$$\|T_{k_0,z}\| = \sup_{k\in\mathbb{K}} \|T_{k,z}\| .$$

Proof: For $J \subseteq N$, let

$$\mathbb{K}(J) = \{k \in \mathbb{K} \mid spt(k) \subseteq J\}.$$

From Property 1.4 it is obvious that each $\mathbb{K}(J)$ is both open and closed in $\mathbb{K}$. Hence there exists $J \subseteq N$ such that

$$\|T_{k_0,z}\| = \sup_{k\in\mathbb{K}(J)} \|T_{k,z}\| .$$

Now set $I_0 = J \cap I$ and note that if $k \in \mathbf{K}(J)$, then $\mathbf{H}_k(I) = \mathbf{H}_k(I_0)$. Hence if $z_0: I_0 \to \mathbb{C}$ is defined by setting $z_0 = z | I_0$ and $k \in \mathbf{K}(J)$, then $T_{k,z} = T_{k,z_0}$. In particular, since $\|T_{k,z}\| = \|T_{k,z_0}\|$ we see that

$$\|T_{k_0,z}\| = \sup_{k \in \mathbf{K}(J)} \|T_{k,z_0}\| .$$

Now observe that if $k \in \mathbf{K}(J)$ and $\bar{k}$ is the kernel on $I_0$ defined by

$$\bar{k}_i(j) = k_i(j) \quad , \quad i, j \in I_0 \quad ,$$

then by an obvious relabeling

$$T_{\bar{k},z_0} \doteq T_{k,z_0} .$$

Thus, by Lemma 1.9,

$$\|T_{k_0,z}\| = \sup_{\bar{k} \in \mathbf{K}(J)^{\cdot}} \left\| A_{\bar{k}}^{\frac{1}{2}} D_{z_0} A_{\bar{k}}^{-\frac{1}{2}} \right\| .$$

Since the objective in this last optimum is continuous the optimum is attained. Unraveling the reductions yields the lemma.

Note that the sets $\mathbf{K}(J)$ defined in the proof of Lemma 1.10 are actually connected. This follows immediately from Property 1.3. Hence the components of $\mathbf{K}$ are precisely the subsets $\mathbf{K}(J)$, for $J \subseteq N$.

**Lemma 1.11.** *Let* $\mathbf{K}$ *be a kernel structure on* $\mathrm{N}$, *let* $I \subseteq N$, *and let* $z_0: I \to \mathbb{C}$ *There exists* $z_1: N \to \mathbb{C}$ *with* $z_1 | I = z$ *and*

$$\|z_1\| = \inf_{\substack{z: N \to \mathbb{C} \\ z | I = z_0}} \|z\| .$$

**Proof**: Choose a sequence $z_m$ such that

$$\lim_{m \to \infty} \|z_m\| = \inf_{z|I = z_0} \|z\| .$$

Clearly we may assume $z_m(i) = 0$ if $i \notin spt(k)$. Also, since $\|z_m\| \geq \max_{i \in spt(k)} |z_m(i)|$, $\{z_m(i)\}_{m=1}^{\infty}$ is bounded for each $i \in spt(k)$. Passing to a subsequence if necessary we may assume that there exists $z_1 : N \to \mathbb{C}$ with

$$\lim_{m \to \infty} z_m(i) = z_1(i)$$

and it can be deduced via the compactness of K and Lemmas 1.9 and 1.10 that $z_1$ has the desired property.

Our final observation on kernel structures is that the requirement of Property 1.4, that K be compact, the requirement of Property 1.5, that K possess a Carleson constant and finally, the requirement that (1.6) actually defines a norm are not independent. Note that in any case, if $z : I \to \mathbb{C}$, then since $\{\overline{z(i)} \mid i \in I \cap spt(k)\}$ are the eigenvalues of $T_{k,z}$,

$$\max_{i \in I \cap spt(k)} |z(i)| \leq \|T_{k,z}\| .$$

Hence (1.6) defines a norm if and only if there exists a constant C such that

$$\|T_{k,z}\| \leq C \max_{i \in I \cap spt(k)} |z(i)| .$$

**Lemma 1.12.** *Let* K *be a set of kernels on* N. *Assume that the elements of* K *are normalized as in Property 1.2 and* K *has a metric defined on it as in Property 1.4. Then among the following conditions, (a) implies (c), and (b) and (c) are equivalent.*

(a) K *is compact.*

(b) K *possesses a Carleson constant*

(c) *there exists an absolute constant* C *such that for all* $I \subseteq N$ *and all* $z: I \to \mathbb{C}$,

$$\|z\| \leqslant C \max_{i \in I} |z(i)| .$$

**Proof:** We first show that (a) implies (c). Thus, assume that **K** is compact, let $I \subseteq N$, and fix $z: I \to \mathbb{C}$ Because **K** is compact it should be clear that there is a constant C, depending only on K such that if $A_k$ is defined as in Lemma 1.9 then,

$$\|A_k^{\frac{1}{2}}\| \, \|A_k^{-\frac{1}{2}}\| \leqslant C .$$

Lemma 1.9 thus implies (c).

To see that (c) implies (b) is equally elementary. Let $k \in \mathbf{K}$ and assume $c \in \mathbb{C}^n$ with $\|c\| = 1$ and $c_i = 0$ if $i \notin spt(k)$. We will show that

$$<(k_i(j))c, c> \geqslant \frac{1}{C\sqrt{n}} ,$$

which will establish (b). Since $\|c\| = 1$, there exists $i_0 \in spt(k)$ with $|c_{i_0}|^2 \geqslant \frac{1}{n}$. Define $z: spt(k) \to \mathbb{C}$ by

$$z(i) = \begin{cases} 0 & \text{if } i \neq i_0 \\ \frac{1}{\overline{c}_{i_0}} & \text{if } i = i_0 . \end{cases}$$

Then $\max_{i \in spt(k)} |z(i)| = \frac{1}{|c_{i_0}|} \leqslant \sqrt{n}$. Hence by (c), $\|z\|_\infty \leqslant C\sqrt{n}$. Consequently,

$$1 = \|k_{i_0}\|^2 = \|T_{k,z}(\sum c_i k_i)\|^2 \leqslant C\sqrt{n}\ \|\sum c_i k_i\|^2$$
$$\leqslant C\sqrt{n}\ < k_i(j)c, c>$$

which establishes (b). That (b) implies (c) follows by a similar argument.

Roughly, NP structures correspond to the situations where one can reduce interpolation problems of Nevanlinna-Pick type to questions involving only the data. To illustrate this last assertion more concretely we now study NP structures that are generated by a single kernel.

*Definition 1.13.* We say a kernel k on N is an NP kernel if for all $I \subseteq N$ and all $z: I \to N$ we have that

$$\inf_{\substack{\tilde{z}: N \to \mathbb{C} \\ \tilde{z}|I = z}} \|T_{k,\tilde{z}}\| = \|T_{k,z}\| .$$

Before proceeding to Proposition 1.15 we isolate a simple fact which we will use repeatedly in the paper.

**Lemma 1.14.** *Let* k *be a kernel on* N. *Let* $I \subseteq N$ *and assume* $z: I \to N$. *Then*

$$\|T_{k,z}\|^2 = \inf\left\{\rho \,\middle|\, \left((\rho-\overline{z(i)}z(j))k_i(j)\right)_{i,j\in I} \geqslant 0\right\}.$$

In particular, if $\rho = \|T_{k,z}\|^2$, then

$$\left((\rho-\overline{z(i)}z(j))k_i(j)\right)_{i,j\in I} \geqslant 0 , \text{ and}$$

*there exist* $c_i \in \mathbb{C}$, $i \in I$ *such that not all* $c_i = 0$ *and*

$$\sum_{i,j\in I} (\rho - \overline{z(i)}z(j))c_j\overline{c_i} = 0 \ .$$

**Proof:** Analyze

$$\|T_{k,z}\left(\sum_{i\in I} \overline{c_i} k_i\right)\|^2 \ .$$

**Proposition 1.15.** Let k be an NP kernel on N. If

$$\mathbf{K} = \left\{\left(a_{i_j} k_i(j)\right)_{i,j\in N} \mid (a_{i_j}) \geqslant 0\right\} ,$$

then K is an NP structure. Also if $I \subseteq N$ and $z: I \to N$, then

$$\|z\| = \|T_{k,z}\| \ . \tag{1.16}$$

**Proof:** To see that K is a kernel structure we need to verify the four properties. Observe that Property 1.2 is satisfied tautologically. Also, Properties 1.3 and 1.4 are immediate. Because K is compact, if K does not possess a Carleson constant, then there exists a matrix $(a_{i_j}) \geqslant 0$ and constants $c_i \in \mathbb{C}$, with $c_{i_0} \neq 0$ for some $i_0 \in spt\left((a_{ij} k_i(\bar{j}))\right)$ such that

$$\sum_{i,j} a_{i_j k_i(j) c_i \overline{c_j}} = 0 \ .$$

Let $I = spt\left((a_{ij} k_i(j))\right)$, and set

$$a_{ij} = \sum_l a_i^l a_j^T \ .$$

Then because $spt\left((a_{ij} k_i(j))\right) = spt\left((a_{ij} k_i(j))\right)$,

$$0 = \sum_{i,j\in I} a_{ij} k_i(j) c_i \overline{c_j} = \sum_{l} \sum_{i,j\in I} k_i(j)(a_i^l c_i)(a_j^T c_j).$$

Since $k_i(j)$ is a kernel, (1.1) holds and thus $a_i^l c_i = 0$ for all $i \in spt(k)$ and all $l$. Hence since $i_0 \in I$ and $I \subseteq spt(k)$ we conclude that $a_{i_0}^l = 0$ for all $l$. But $i_0 \in I \subseteq spt(a_{i_j})$ so that $\sum_l |a_{i_0}^l|^2 \neq 0$, a contradiction which establishes Property 1.5.

Now observe from Definitions 1.7 and 1.13 that since K is a kernel structure and k is an NP kernel. K will be an NP structure if (1.6) holds. To establish (1.16) observe first that trivially,

$$||z|| \geqslant ||T_{k,z}||.$$

Thus, there remains to show that

$$||z|| = \sup_{g\in \mathrm{K}} ||T_{g,z}|| \leqslant ||T_{k,z}|| .$$

Fix $g = (a_{ij} k_i(j)) \in \mathrm{K}$. Let $\rho = ||T_{k,z}||^2$. Lemma 1.14 implies that

$$\left((\rho - \overline{z(i)} z(j)) k_i(j)\right)_{i,j\in I} \geqslant 0.$$

Because the pointwise product of positive semidefinite matrices is positive semidefinite we conclude that

$$\left((\rho - \overline{z(i)} z(j)) a_{ij} k_i(j)\right)_{i,j\in I} \geqslant 0.$$

Since $g_i(j) = \overline{c_i} c_j a_{ij} k_i(j)$ for appropriately chosen $c_i$'s we conclude that

$$\left((\rho - \overline{z(i)} z(j)) g_i(j)\right)_{i,j\in I} \geqslant 0 .$$

Finally, by Lemma 1.4 we conclude that

$$\|T_{g,z}\|^2 \leqslant \rho ,$$

which concludes the proof of Proposition 1.15.

Thus, if k is an NP kernel, then the kernel structure generated by k is an NP structure. Our next question is: how do you recognize an NP kernel? The question is answered by Proposition 1.18 below.

If k is a kernel on $N$, $I_0 \subseteq I_1 \subseteq N$, and $z: I_0 \to N$ define $\mathbf{H}_k(I_0, I_1) \subseteq \mathbf{H}_k$ by

$$\mathbf{H}_k(I_0,I_1) = \mathbf{H}_k(I_1) \ominus \mathbf{H}_k(I_1 \backslash I_0)$$

and define $T_{k,z}(I_1) \in \mathbf{L}(\mathbf{H}_k(I_0,I_1))$ by

$$T_{k,z}(I_1) = P \; T_{k,\tilde{z}} \mid \mathbf{H}_k(I_0,I_1) ,$$

where P is the orthogonal projection of $\mathbf{H}_k$ onto $\mathbf{H}_k(I_0,I_1)$ and $\tilde{z}$ is any extension of z to $I_1$.

Our first observation is that the definition of $T_{k,z}(I_1)$ does not depend upon the choice of extension $\tilde{z}$. To see this let

$$f = \sum_{i \in I_1 \cap spt(k)} c_i k_i \in \mathbf{H}_k(I_0,I_1) .$$

Evidently,

$$f = \sum_{i \in I_0 \cap spt(k)} c_i k_i + \sum_{i \in (I_1 \backslash I_0) \cap \, spt\, k} c_i k_i$$

so that

$$T_{k,z}(I_1)f = P \sum_{i \in I_0 \cap spt(k)} c_i \overline{z(i)} k_i$$

regardless of how $\overline{z}$ is chosen. Our next observation concerning $T_{k,z}(I_1)$ is summarized in the following lemma.

**Lemma 1.7.** *Let* k *be a kernel on* N. *Let* $I_0 \subseteq I_1 \subseteq N$ *and let* $z: I_0 \to \mathbb{C}$ *Then* $T_{k,z}(I_1)$ *is similar to* $T_{k,z}$. *In particular,* $T_{k,z}(I_1)$ *and* $T_{k,z}$ *have the same eigenvalues.*

**Proof:** Define $L: \mathrm{H}_k(I_0) \to \mathrm{H}_k(I_0, I_1)$ by setting

$$L = P | \mathrm{H}_k(I_0) ,$$

where P is the orthogonal projection of $\mathrm{H}_k$ onto $\mathrm{H}_k(I_0, I_1)$. We claim that L is 1 - 1 and onto. To see this it is sufficient to show that L is 1 - 1 for

$$\begin{aligned} dim\ \mathrm{H}_k(I_0) &= \mathrm{card}\,(I_0 \cap spt(k)) \\ &= card(I_1 \cap spt(k)) - card((I_1 \setminus I_0) \cap spt(k)) \\ &= dim\ \mathrm{H}_k(I_0, I_1) . \end{aligned}$$

Let $f = \sum_{i \in I_0 \cap spt(k)} c_i k_i \in \mathrm{H}_k(I_0)$ and assume that L(f) = 0. Then

$$\sum_{i \in I_0 \cap spt(k)} c_i k_i \in \mathrm{H}_k(I_1 \setminus I_0) .$$

Hence there exist $d_i \in \mathbb{C}$, $i \in (I_1 \setminus I_0) \cap spt(k)$, with

$$\sum_{i \in I_0 \cap spt(k)} c_i k_i - \sum_{i \in (I_1 \setminus I_0) \cap spt(k)} d_i k_i = 0.$$

Since k is a kernel we conclude from (1.1) that $c_i = 0$ whenever $i \in I_0 \cap spt(k)$. Hence f = 0 and L is a bijection.

Now let $\tilde{z}$ be an extension of z to $I_1$ so that $T_{k,z}(I_1) = PT_{k,\tilde{z}} \mid \mathrm{H}_k(I_0, I_1)$. Since $\mathrm{H}_k(I_0,I_1) = \mathrm{H}_k(I_1) \ominus \mathrm{H}_k(I_1 \backslash I_0)$ and $\mathrm{H}_k(I_1 \backslash I_0)$ is invariant for $T_{k,\tilde{z}}$ we see that

$$PT_{k,\tilde{z}}(1-P) = 0.$$

Hence if $f \in \mathrm{H}_k(I_0)$,

$$\begin{aligned} T_{k,z}(I_1)Lf &= PT_{k,\tilde{z}}Pf = PT_{k,\tilde{z}}f \\ &= PT_{k,z}f = LT_{k,z}f \quad , \end{aligned}$$

and L implements a similarity between $T_{k,z}$ and $T_{k,z}(I_1)$.

**Proposition 1.18.** Let k be a kernel on N. k is an NP kernel if and only if

$$\|T_{k,z}(I_1)\| \leqslant \|T_{k,z}\| \tag{1.19}$$

whenever $I_0 \subseteq I_1 \subseteq N$ and $z: I_0 \to \mathbb{C}$

**Proof:** We first establish the necessity of the condition. Thus, assume that k is an NP kernel. Let $I_0 \subseteq I_1 \subseteq N$ and assume $z: I_0 \to \mathbb{C}$ From the NP condition we see that there exists $\tilde{z}: N \to \mathbb{C}$ such that $\tilde{z} \mid I_0 = z$ and $\|T_{k,\tilde{z}}\| = \|T_{k,z}\|$. If we let $z_1 = \tilde{z} \mid I_1$ notice that $T_{k,z}(I_1) = P\ T_{k,z_1} \mid \mathrm{H}_k(I_0,I_1)$ where P is the orthogonal projection of $\mathrm{H}_k$ onto $\mathrm{H}_k(I_0,I_1)$. Thus, since $\mathrm{H}_k(I_1)$ is invariant for $T_{k,\tilde{z}}$,

$$\begin{aligned} \|T_{k,z}(I_1)\| &= \|PT_{k,z_1} \mid \mathrm{H}_k(I_0,I_1)\| \\ &\leqslant \|T_{k,z_1}\| \leqslant \|T_{k,\tilde{z}}\| = \|T_{k,z}\|. \end{aligned}$$

Now assume that (1.19) holds whenever $I_0 \subsetneq I_1 \subsetneq N$ and $z: I_0 \to \mathbb{C}$ Fix $I \subsetneq N$, $z: I \to \mathbb{C}$ Let $i' \in N \setminus I$ and set $I' = I \cup \{i'\}$. We shall show that

$$\inf_{\substack{\tilde{z}: I' \to \mathbb{C} \\ \tilde{z} | I = z}} \| T_{k,\tilde{z}} \| = \| T_{k,z} \| .$$

Since I, z, and $i'$ are arbitrary the condition of Definition 1.13 will then follow by an obvious iteration. We argue by contradiction. Thus, let

$$\rho_0 = \| T_{k,z} \|^2, \ \rho_1 = \inf_{\substack{\tilde{z}: I' \to \mathbb{C} \\ \tilde{z} | I = z}} \| T_{k,\tilde{z}} \|^2 ,$$

and assume that

$$\rho_0 < \rho_1 . \tag{1.20}$$

Fix $\tilde{z}_0: I' \to \mathbb{C}$ with $\tilde{z}_0 | I = z$ and

$$\rho_1 = \| T_{k,\tilde{z}_0} \|^2 .$$

We claim that

$$dim \ ker\left( \rho_1 - T^*_{k,\tilde{z}_0} \ T_{k,\tilde{z}_0} \right) = 1. \tag{1.21}$$

For suppose (1.21) were false. Then there would exist a unit vector $\gamma \in \mathbf{H}_k(I)$ with

$$\| T_{k,\tilde{z}_0} \gamma \| = \| T_{k,\tilde{z}_0} \| .$$

Hence we would have

$$\rho_1 = \|T_{k,\bar{z}_0}\gamma\|^2 = \|T_{k,z}\gamma\|^2 \leqslant \|T_{k,z}\| = \rho_0 \tag{1.22}$$

contradicting (1.20).

Choose a unit vector $\gamma \in ker\left(\rho_1 - T'_{k,\bar{z}_0}\ T_{k,\bar{z}_0}\right)$. Let $\omega \in \mathbf{H}_k(I_1) \ominus \mathbf{H}_k(I_0)$ with $<\omega, k_{i_0}> = 1$; $\omega$ exists, for otherwise, $i' \notin spt(k)$ and $T_{k,z} = T_{k,\bar{z}_0}$ contradicting (1.20). Now note that if $t \in \mathbf{R}$ and $c \in \mathbf{C}$, then

$$\phi_c(t) = \|T_{k,\bar{z}_0} + t\ c\ k_{k'} \otimes \omega\|^2$$

is a candidate for the infimum that defines $\rho_1$. Furthermore, since (1.21) holds, for each fixed c, $\phi_c(t)$ is differentiable when t is close to 0. We conclude that $\varphi_c'(0) = 0$ for all c. Calculating $\phi_c'(0)$ explicitly we obtain that

$$2\ \mathrm{Re}\ c\ <(k_{i'} \otimes \omega)\gamma,\ T_{k,\bar{z}_0}\gamma> = 0$$

for all c. Finally, we deduce that

$$<\gamma,\omega>\ <k_{i'},\ T_{k,\bar{z}_0}\gamma> = 0. \tag{1.23}$$

Our first observation is that $<\gamma,\omega> \neq 0$. For suppose $<\gamma,\omega> \neq 0$. Then since $\omega$ is a nonzero vector in $\mathbf{H}_k(I_1) \ominus \mathbf{H}_k(I_0)$, we have that $\gamma \in \mathbf{H}_k(I_0)$. But then line (1.22) holds, contradicting (1.20). Hence $<\gamma,\omega> \neq 0$ and we conclude from (1.23) that in fact,

$$<K_{i'},\ T_{k,\bar{z}_0}\gamma> = 0. \tag{1.24}$$

Now, since $\gamma \in ker\left(\rho_1 - T'_{k,\bar{z}_0} \; T_{k,\bar{z}_0}\right)$,

$$
\begin{aligned}
<k_1 \cdot, \gamma> &= \rho_1^{-1} < k_1 \cdot, \; T'_{k,\bar{z}_0} T_{k,\bar{z}_0} \gamma> \\
&= \rho_1^{-1} < T_{k,\bar{z}_0} k_1 \cdot, \; T_{k,\bar{z}_0} \gamma> \\
&= \rho_1^{-1} \bar{z}_0(i') <k_1 \cdot, \; T_{k,\bar{z}_0} \gamma> = 0
\end{aligned}
$$

and we see that also

$$
<k_1 \cdot . \gamma> = 0. \tag{1.25}
$$

We now are able to contradict (1.19) and thus complete the proof of Proposition 1.18. Observe that (1.25) implies that $\gamma \in \mathbf{H}_k(I,I')$ and that (1.24) asserts that $T_{k,\bar{z}_0} \gamma \in \mathbf{H}_k(I,I')$. Hence

$$
\begin{aligned}
\|T_{k,z}(I')\|^2 &\geqslant \|T_{k,z}(I')\gamma\|^2 \\
&= \|T_{k,\bar{z}_0} \gamma\|^2 = \rho_1 > \rho_0 = \|T_{k,z}\|^2
\end{aligned}
$$

contradicting (1.19) and concluding the proof of the proposition.

We now illustrate the value of Proposition 1.18 by giving a proof of the classical Nevanlinna-Pick Theorem. The heart of the matter is the following lemma.

**Lemma 1.26.** *Let* $\lambda_1, \ldots, \lambda_n \in D$ and define a kernel k on $N = \{1, \ldots, n\}$ by setting

$$k_i(j) = \frac{1}{1 - \overline{\lambda}_i \lambda_j} \quad , \quad i, j \in N.$$

Then k is an NP kernel.

We now show how Lemma 1.26 implies the Nevanlinna-Pick theorem.

**Theorem 1.27.** *Let* $\lambda_1, \ldots, \lambda_n \in D$ *and let* $z_1, \ldots, z_n \in \mathbb{C}$ *There exists* $\phi \in H^\infty$ *with* $\|\phi\|_\infty \leqslant 1$ *and* $\phi(\lambda_i) = z_i$ *whenever* $1 \leqslant i \leqslant n$ *if and only if*

$$\left( \frac{1 - \overline{z}_i z_j}{1 - \overline{\lambda}_i \lambda_j} \right) \geqslant 0 \ .$$

Proof. The necessity of the condition follows in the usual way. To prove sufficiency extend $\{z_i\}_{i=1}^n$ to an infinite sequence $\{z_i\}_{i=1}^\infty$ which is a set of uniqueness for $H^2$. Fix $m \geqslant 1$. Define a kernel on $I_m = \{1, .., m\}$ by setting

$$k_m = \left( \frac{1}{1 - \overline{\lambda}_i \lambda_j} \right)_{i,j \in I_m} . \tag{1.28}$$

By Lemma 1.26, $k_m$ is an NP kernel for every m. The condition

$$\left( \frac{1 - \overline{z}_i z_j}{1 - \overline{\lambda}_i \lambda_j} \right) \geqslant 0$$

guarantees that if we define $z_n : I_n \to \mathbb{C}$ by $z_n(i) = z_i$, $i \in I_n$, then

$$\| T_{k_n, z_n} \| \leqslant 1 \ .$$

Applying Definition 1.13 repeatedly, we conclude there exists an infinite sequence $z_{n+1}, z_{n+2}, \dots$ such that if we define $z_m : I_m \to \mathbb{C}$ by

$$z_m(i) = z_i \quad , \quad i \in I_m ,$$

then

$$\|T_{k_m, z_m}\| \leqslant 1 \quad \text{for } \textit{all} \quad m. \tag{1.29}$$

Now for $m \geqslant 1$ let

$$H_m^2 = \left[ k_{\lambda_i} \mid i \leqslant i \leqslant m \right] \subseteq H^2$$

and define $T_m \in \mathbf{L}(H_m^2)$ by requiring that

$$T_m k_{\lambda_i} = \overline{z_i} k_{\lambda_i} \quad , \quad 1 \leqslant i \leqslant m.$$

It is easy to check that (1.28) guarantees that $T_m$ and $T_{k_m, z_m}$ are unitarily equivalent for each $m \geqslant 1$. Thus, (1.29) implies that $\|T_m\| \leqslant 1$ for all m. But $T_{m_1} = T_{m_2} | H_{m_1}^2$ whenever $m_1 \leqslant m_2$ and $\bigcup_{m=1}^{\infty} H_m^2$ is dense in $H^2$ because $\{\lambda_i\}_{i=1}^{\infty}$ is a set of uniqueness. Thus, there exists $T \in \mathbf{L}(H^2)$ such that

$$\|T\| \leqslant 1 \text{ and } T k_{\lambda_i} = \overline{z_i} k_{\lambda_i} \text{ for } \textit{all} \quad i.$$

Now observe that if $M_z$ denotes the operator

$$M_z f(z) = z f(z) \quad , \quad f \in H^2,$$

then

$$M_z' k_\lambda = \overline{\lambda} k_\lambda \quad whenever \quad \lambda \in \mathbf{D}.$$

Hence T commutes with $M_z'$. Since the commutant of $M_z$ is $H^\infty$ we conclude that there exists a $\phi \in H^\infty$ such that

$$T = M_\phi' .$$

In particular $\|\phi\|_\infty \leqslant 1$ and since $T_n k_{\lambda_i} = \overline{z_i} k_{\lambda_i}$, $\phi(\lambda_i) = z_i$ when $1 \leqslant i \leqslant n$. This concludes the proof.

**Theorem 1.30.** *Let* K *be a kernel structure on* N. K is an NP structure if and only if for all $I_0 \subseteq I_1 \subseteq N$ and all $z: I_0 \to N$,

$$\sup_{k \in} \|T_{k,z}(I_1)\| \leqslant \sup_{k \in} \|T_{k,z}\| . \tag{1.31}$$

The proof of Theorem 1.30 will occupy the remainder of this section. The necessity of condition (1.31) follows exactly as in the proof of Proposition 1.18. Now assume that condition (1.31) holds whenever $I \subseteq N$ and $z: I \to N$. Let $I_0 \subseteq N$ and let $z: I_0 \to \mathbb{C}$ Let

$$\rho_0 = \|z\|^2 = \sup_{k \in \mathbf{K}} \|T_{k,z}\|^2 .$$

Fix $i_0 \in N \backslash I_0$ and let $I_1 = I_0 \cup \{i_0\}$. Let

$$\rho_1 = \inf_{\substack{\tilde{z}: I_1 \to \mathbb{C} \\ \tilde{z} | I_0 = z}} \sup_{k \in \mathbf{K}} \|T_{k,\tilde{z}}\|^2.$$

Using the iterative process employed in the proof of Proposition 1.18, it is clear that Theorem 1.30 will follow if we can prove that

$$\rho_0 = \rho_1 . \tag{1.32}$$

To simplify the notation somewhat let us agree to set $c = \bar{z}(i_0)$, $T_k(c) = T_{k,\bar{z}}$, and $\rho(c,k) = \|T_k(c)\|^2$. Thus, we have that

$$\rho_1 = \inf_{c\in\mathbb{C}} \sup_{k\in\mathrm{K}} \rho(c,k) .$$

**Lemma 1.33.** *$\rho$ is continuous on $\mathbb{C}\times\mathrm{K}$.*

Proof: Let $k_m \in \mathrm{K}$ and $c_m \in \mathbb{C}$ with $k_m \to k$ in $\mathrm{K}$ and $c_m \to c$ in $\mathbb{C}$. Observe that Property 1.4 in the definition of a kernel structure implies that there exist $I \subseteq N$ and a positive integer $m_0$ such that

$$m > m_0 \text{ implies } spt(k_m) = I.$$

Let $z_m : I \cap I_1 \to \mathbb{C}$ be defined by

$$z_m(i) = \begin{cases} z(i) & \text{if} \quad i \in I_0 \cap I \\ c_m & \text{if} \quad i = i_0 \text{ and } i_0 \in I. \end{cases}$$

Let $\rho_m = \|T_{k_m,z_m}\|^2$. Since $\|T_{k_m,z_m}\|^2 \leqslant \rho_m$ we have that if $m > m_0$, then

$$\left((\rho_m - \overline{z_m(i)}z_m(j))(k_m)_i(j)\right)_{i,j\in I\cap I_1} \geqslant 0 \ . \tag{1.34}$$

Also since, $\|T_{k_m,z_m}\|$ actually equals $\rho_m$, for each $m > m_0$ and each $i \in I \cap I_1$, there exist $r_{mi} \in \mathbb{C}$ with $\sum_{i\in I\cap I_1} |r_{mi}|^2 = 1$ and

$$\sum_{i,j\in I\cap I_1} \left(\rho_m - \overline{z_m(i)}z_m(j)\right)(k_m)_i(j)\overline{r}_{mi}r_{mj} = 0. \tag{1.35}$$

Passing to a subsequence if necessary we may assume that there exist $r_i \in \mathbb{C}$ with $r_{mi} \to r_i$ for each $i \in I \cap I_1$. If $\delta$ is the Carleson constant of we know that

$$(k_m)_{i,j \in I \cap I_i} \geq \delta$$

whenever $m > m_0$. Consequently, $\rho_m$ is bounded. Furthermore, (1.34) implies that

$$\|T_k(c)\| \leq \lim_{m \to \infty} \rho_m$$

and (1.35) implies that

$$\overline{\lim_{m \to \infty}} \rho_m \leq \|T_k(c)\| .$$

Thus, since $\|T_{k_m}(c_m)\| = \rho_m$ when $m > m_0$ we conclude that $\rho$ is continuous.

Now let $c_0$ be chosen so that

$$\rho_1 = \sup_{k \in} \rho(c_0, k) .$$

For each $c \in \mathcal{C}$, let

$$\mathrm{K}_c = \left\{ k \in \mathrm{K} \mid \rho(c,k) = \sup_{g \in \mathrm{K}} \rho(c,g) \right\}.$$

Evidently, Lemma 1.33 guarantees that $c_0$ exists and that $\mathrm{K}_c$ is compact and nonempty for all $c \in \mathcal{C}$

The proof of Theorem 1.30 depends inherently on the fact that $\rho(c,k)$ is differentiable in c whenever $\rho(c,k) > \rho_0$. The next four lemmas explore this phenomenon. For $c \in \mathcal{C}$ and $k \in \mathrm{K}$ define

$$dim(c,k) = dim\Big(ker(\rho(c,k) - T_k(c)^* T_k(c))\Big) ;$$

**Lemma 1.36.** *Let $c \in \mathbb{C}$ and $k \in \mathrm{K}$. If $\rho(c,k) > \rho_0$, then* $\dim(c,k) = 1$.

**Proof:** Suppose $\dim(c,k) > 1$. Then there exists $r \in {}_k(I_0)$ such that $||r|| = 1$ and

$$||T_k(c)r||^2 = ||T_k(c)||^2 .$$

But

$$\begin{aligned} ||T_k(c)r||^2 &\leqslant ||T_k(c)\,|\,{}_k(I_0)||^2 \\ &= ||T_{k,z}||^2 \leqslant \sup_{k\in\mathrm{K}} ||T_{k,z}||^2 = \rho_0 . \end{aligned}$$

We conclude that in fact

$$\rho(c,k) = ||\,T_k(c)||^2 \leqslant \rho_0 ;$$

which establishes Lemma 1.36.

**Lemma 1.37.** *Let $\bar{\phi}_1 > \rho_0$, then there exists $\epsilon > 0$ such that if*

$$D_\epsilon = \{c \mid |c-c_0| < \epsilon\}$$

*and*

$$G = \bigcap_{k\in\mathrm{K}_{\epsilon_0}} \{g \in \mathrm{K} \mid d(k,g) < \epsilon\},$$

then

$$dim\,(c,k) = 1 \quad whenever \quad (c,k) \in D_\epsilon \times G_\epsilon .$$

**Proof.** Suppose the lemma is false. For each $m \geqslant 1$ choose $(c_m,k_m) \in D_{\frac{1}{m}} \times G_{\frac{1}{m}}$ with $dim\,(c_m,k_m) > 1$. Lemma 1.36 implies that $\rho(c_m,k_m) \leqslant \rho_0$. Since $c_m \in D_{\frac{1}{m}}$, $c_m \to c_0$. Also, passing to a subsequence if necessary we may assume that there exists $k \in K_{c_0}$ with $k_m \to k$. Hence by Lemma 1.33,

$$\rho_1 = \rho(c_0, k) = \lim_{m \to \infty} \rho(c_m, k_m) \leqslant \rho_0,$$

a contradiction which establishes the lemma.

For fixed $k \in K$ and $\delta \in \mathbb{C}$ define

$$D_\delta(c,k) = \lim_{t \to 0} \frac{1}{t} \left(\rho(c - t\delta, k) - \rho(c,k)\right)$$

whenever the limit exists.

**Lemma 1.38.** *If* $\dim(c,k) = 1$, *then* $D_\delta(c,k)$ *exists for every* $\delta \in \mathbb{C}$ *Furthermore, if* $\dim(c,k) = 1$ *and* $\gamma$ *is a unit vector in* $\ker(\rho(c,k) - T_k(c)^* T_k(c))$, *then*

$$D_\delta(c,k) = 2\,\mathrm{Re}\left(\overline{\delta} < \gamma, \omega > < k_{i_0}, T_k(c)\gamma >\right)$$

*where* $\omega \in H_k(I_1) \ominus H_k(I_0)$ *and* $< k_{i_0}, \omega > = 1$. *(Note:* $\omega$ *will not exist only if* $i_0 \notin spt(k)$ *in which case* $D_\delta(c,k) = 0$).

**Proof:** First observe that

$$T_k(c + t\delta) = T_k(c) + t\overline{\delta} k_{i_0} \otimes \omega.$$

Thus,

$$\rho(c + t\delta, k) = \|T_k(c + t\delta)\|^2 = \lambda_{\max}(T_k(c + t\delta)^* \; T_k(c + t\delta))$$
$$= \lambda_{\max}\Big(T_k(c)^* T_k(c) + t\Big(\delta(k_{i_0} \otimes \omega)^* \; T_k(c) + \overline{\delta} T_k(c)^* (k_{i_0} \otimes \omega)\Big)$$

$$+ t^2 |\delta|^2 (k_{i_0} \otimes \omega)^* (k_{i_0} \otimes \omega) \Big) .$$

Thus,

$$D_\delta(c,k) = \frac{d}{dt} \Big( \rho(c + t\delta, k) \Big) (0)$$

$$= < \Big( \delta (k_{i_0} \otimes \omega)^* T_k(c) + \overline{\delta} T_k(c)^* (k_{i_0} \otimes \omega) \Big) \gamma, \gamma >$$

$$= 2 \operatorname{Re} \Big( \overline{\delta} < \gamma, \omega > < k_{i_0}, T_k(c) \gamma > \Big) ,$$

which establishes Lemma 1.38.

By Lemma 1.37 and Lemma 1.38 if $\rho_1 > \rho_0$, then there exists $\epsilon > 0$ such that

$$(c,k) \in D_\epsilon \times G_\epsilon \text{ implies } D_\delta(c,k) \text{ exists for all } \delta .$$

Thus, the following results is not too surprising.

**Lemma 1.39.** *If $\rho_1 > \rho_0$, then there exists $\epsilon > 0$ such that $D_\delta(c,k)$ exists and is continuous on $D_\epsilon \times G_\epsilon$.*

We omit the proof of Lemma 1.39 which is based on changing variables in the expression for $D_\delta(c,k)$ in Lemma 1.38 to a fixed Hilbert space by employing the Hilbert space isomorphism of Lemma 1.9.

**Lemma 1.40.** *Fix $c \in \mathbb{C}$ Assume $k, k' \in \mathrm{K}$ and assume that*

$$\rho(c,k) = \rho = \rho(c,k') .$$

*Assume that $\dim(c,k) = \dim(c,k') = 1$. Let $\gamma$ and $\gamma'$ be unit vectors in the kernels of $\rho - T_k(c)^* T_k(c)$ and $\rho - T_{k'}(c)^* T_{k'}) = 1$ respectively. Choose $\alpha + \alpha' = 1$. Let $\delta \in \mathbb{C}$ Let $\gamma = \sum_{i \in I_1} \overline{\tau_i} k_i$ and $\gamma' = \sum_{i \in I_1} \overline{\tau_i}' k_i$. Define $k'' \in \mathrm{K}$*

*by*

$$(k_i''(j)) = (\alpha \overline{r_i} r_j k_i(j) + \alpha' \overline{r_i} r_j k_i(j))^- .$$

*Under the above assumptions,*

$$\rho(c, k'') = \rho ,$$

*and if* $dim(c, k'') = 1$, *then*

$$D_\delta(c, k'') = \alpha D_\delta(c, k) + \alpha' D_\delta(c, k') .$$

Proof: Choose

$$\omega \in \mathbf{H}_k(I_1) \ominus \mathbf{H}_k(I_0)$$
$$\omega' \in \mathbf{H}_{k'}(I_1) \ominus \mathbf{H}_k(I_0)$$

with $\langle k_{i_0}, \omega \rangle_k = 1 \langle k_{i_0}', \omega \rangle_{k'}$. Fix $c$, $\delta$, $\alpha$, $\alpha'$. Consider

$$T(c) = \begin{bmatrix} T_k(c) & 0 \\ 0 & T_{k'}(c) \end{bmatrix} \mid \mathbf{M}$$

where $\mathbf{M} \subseteq \mathbf{H}_k(I_1) \oplus \mathbf{H}_{k'}(I_1)$ is defined by

$$\mathbf{M} = \left[ \begin{pmatrix} \sqrt{\alpha} T_k^j(c) r \\ \sqrt{\alpha'} T_{k'}(c)^j r' \end{pmatrix} \mid j \geq 0 \right].$$

Since $\rho(c,k) = \rho(c,k') = \rho$ it is obvious that

$$\|T(c)\|^2 \leqslant \rho .$$

$\begin{pmatrix} \sqrt{\alpha}\, r \\ \sqrt{\alpha'}\, r' \end{pmatrix} \in \mathbf{M}$, we deduce that in fact $\|T(c)\|^2 = \rho$.

Also notice that an alternate description of $\mathbf{M}$ is:

$$\mathbf{M} = \left[ \begin{pmatrix} \sqrt{\alpha\gamma_i}\, k_i \\ \sqrt{\alpha'\gamma_i'}\, k_i' \end{pmatrix} \mid i \in I_1 \right] .$$

Since $\begin{pmatrix} \sqrt{\alpha\gamma_i}\, k_i \\ \sqrt{\alpha'\gamma_i'}\, k_i' \end{pmatrix}$ are eigenfunctions for $T(c)$ (with corresponding eigenvalues $\overline{z(i)}$ if $i \in I_0$ and $\overline{c}$ if $i = i_0$) it is easy to check that

$$T(c) \cong T_{k''}(c).$$

Thus, $\rho(c,k'') = \|T_{k''}(c)\|^2 = \|T(c)\|^2 = \rho$.

Now assume $dim(c,k'') = 1$. Define

$$\rho(t) = \lambda_{\max}\Big( T(c+t\delta)^* \, T(c+t\delta) \Big).$$

Since $\rho(t) = \rho(c+t\delta, k'')$ it is clear that the lemma will be established if we can show that

$$\rho'(0) = \alpha D_\delta(c,k) + \alpha' D_\delta(c,k') .$$

Now note that

$$\begin{bmatrix} k_{i_0} \otimes \omega & 0 \\ 0 & k_{i_0}' \otimes \omega' \end{bmatrix} \mathbf{M} \subseteq \mathbf{M} .$$

Thus,

$$T(c+t\delta) = T(c) + t\bar{\delta}\begin{bmatrix} k_{i_0}\otimes\omega & 0 \\ 0 & k_{i_0}'\otimes\omega' \end{bmatrix}.$$

Thus,

$$\rho(t) = \lambda_{\max}\left(P_{\mathbf{M}}\left(\begin{bmatrix} T_k(c) & 0 \\ 0 & T_{k'}(c) \end{bmatrix} + t\bar{\delta}\begin{bmatrix} k_{i_0}\otimes\omega & 0 \\ 0 & k_{i_0}'\otimes\omega' \end{bmatrix}\right)^* \right.$$
$$\left.\left(\begin{bmatrix} T_k(c) & 0 \\ 0 & T_{k'}(c) \end{bmatrix} + t\bar{\delta}\begin{bmatrix} k_{i_0}\otimes\omega & 0 \\ 0 & k_{i_0}'\otimes\omega' \end{bmatrix}\right) \Big| \mathbf{M}\right).$$

Consequently,

$$\rho'(0) = 2\,\mathrm{Re}\,\bar{\delta}\left( < \begin{bmatrix} k_{i_0}\otimes\omega & 0 \\ 0 & k_{i_0}'\otimes\omega' \end{bmatrix}\begin{pmatrix} \sqrt{\alpha}\gamma \\ \sqrt{\alpha'}\gamma' \end{pmatrix}, \begin{bmatrix} T_k(c) & 0 \\ 0 & T_{k'}(c) \end{bmatrix}\begin{pmatrix} \sqrt{\alpha}\gamma \\ \sqrt{\alpha'}\gamma \end{pmatrix} > \right)$$

$$= \alpha\, D_\delta(c,k), - \alpha'\, D_\delta(c,k'),$$

which establishes Lemma 1.40.

Now recall that we are trying to establish (1.32). We argue by contradiction. Thus, assume that $\rho_1 > \rho_0$. Exploiting Lemma 1.37 and 1.39 there exists $\epsilon_0 > 0$ such that

$$dim(c,k) = 1 \text{ and } D_\delta(c,k). \tag{1.41}$$

Now observe that if $\delta \in \mathbb{C}$ and $k \in \mathbf{K}_{c_0+\delta}$, then

$$\phi(c_0+\delta,k) - \phi(c_0,k) \geqslant 0. \tag{1.42}$$

To prove (1.42) choose $g \in K_{c_0}$ and note that

$$\phi(c_0+\delta,k) - \phi(c_0,k)$$
$$= \Big(\phi(c_0+\delta,k) - \phi(c_0,g)\Big) + \Big(\phi(c_0,g) - \phi(c_0,k)\Big).$$

But,

$$\rho(c_0+\delta,k) - \rho(c_0,g) = \sup_{h\in K} \rho(c_0+\delta,h) - \inf_{c} \sup_{h\in K} \rho(c,h) \geqslant 0,$$

and

$$\rho(c_0,g) - \rho(c_0,k) = \sup_{h\in K} \rho(c_0,h) - \rho(c_0,k) \geqslant 0.$$

We claim there exists a constant $\epsilon_1 > 0$ such that

$$|\delta| < \epsilon_1 \text{ implies } (c_0+t\delta,k) \in D_{\epsilon_0} \times G_{\epsilon_0} \quad (1.43)$$
$$\text{for all } 0 \leqslant t \leqslant 1 \text{ and all } k \in K_{c_0-\delta}.$$

To prove (1.43) we argue by contradiction. Thus, assume that there exists a sequence $\delta_m \to 0$, a sequence $t_m \in [0,1]$, and a sequence $k_m \in K_{c_0+\delta_m}$ with

$$(c_0+t_m\delta_m,k_m) \notin D_{\epsilon_0} \times G_{\epsilon_0} \text{ for all } m.$$

By the compactness of K we may assume that $k_m \to k$. Since $k_m \in K_{c_0+\delta_m}$ and $\delta_m \to 0$ we conclude from Lemma 1.33 that $\rho(c_0,k) = \rho_1$. Hence $k \in K_{c_0}$ and we conclude that $k_m \in G_{\epsilon_0}$ for sufficiently large m. This contradiction establishes (1.43).

Fix $\delta$ with $|\delta| < \epsilon_1$. Pick $k \in K_{c_0+\delta}$ and define

$$g(t) = \rho(c_0+t\delta,k).$$

Evidently, (1.42) guarantees that

$$g(1) - g(0) \geqslant 0 .$$

Furthermore, (1.41) and (1.43) imply that g is differentiable for all $t \in [0,1]$. The Mean Value Theorem implies that there exists a $t \in [0,1]$ such that $g'(t) \geqslant 0$. So also for such a t we have that

$$D_\delta(c_0+t\delta,k) \geqslant 0.$$

Now apply the construction in the previous paragraph on a sequence $\delta_m = \frac{1}{m}\delta$ to obtain sequences

$$k_m \in \mathrm{K}_{c_0+\frac{1}{m}\delta},$$

and

$$t_m \in [0,1] ,$$

such that

$$D_\delta(c_0+t_m\delta,k_m) \geqslant 0 .$$

As before, we conclude a subsequence of $\{k_m\}$ tends to an element $k \in \mathrm{K}_{c_0}$, and (1.41) then implies that

$$D_\delta(c_0,k) \geqslant 0 .$$

We summarize the result of the previous two paragraphs in the following lemma.

**Lemma 1.43.** *If* $\delta \in \mathbb{C}$, *then there exists* $k \in \mathrm{K}_{c_0}$ *such that*

$$D_\delta(c_0,k) \geqslant 0 .$$

Our next goal will be to improve Lemma 1.43 to Lemma 1.45 below.

For $k \in \mathrm{K}_{c_0}$ define

$$\nabla_k = \begin{pmatrix} D_1(c_0,k) \\ D_i(c_0,k) \end{pmatrix} \in \mathbf{R}^2 .$$

Let

$$\nabla = \left\{ \nabla_k \mid k \in \mathrm{K}_{c_0} \right\} .$$

Observe that Lemma 1.39 implies $\nabla$ is compact and Lemma 1.40 implies that $\nabla$ is convex. If Lemma 1.45 were false, it would mean that $0 \notin \nabla$. The Hahn-Banach Theorem would then imply the existence of a

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \in \mathbf{R}^2$$

such that

$$\delta \cdot \nabla_k < 0 \quad \textit{whenever} \quad k \in \mathrm{K}_{c_0} . \tag{1.44}$$

But

$$\begin{aligned} \delta \cdot \nabla_k &= \delta_1 D_1(c_0,k) + \delta_2 D_i(c_0,k) \\ &= D_{\delta_1 + i\delta_2}(c_0,k) . \end{aligned}$$

Thus, (1.44) contradicts Lemma 1.43, and we have proven the following lemma.

**Lemma 1.45.** *There exists a* $k \in \mathrm{K}_{c_0}$ *such that* $D_\delta(c_0,k) = 0$ *for all* $\delta \in \mathbb{C}$

We are now within striking range of concluding the proof of the theorem. Recall we are operating under the assumption that $\rho_1 > \rho_0$ and wish to derive a contradiction. Let k be fixed with the properties

$$k \in \mathrm{K}_\epsilon \ ,$$

and

$$D_\delta(c_0,k) = 0 \text{ for } all \ \ \delta \in \mathbb{C}$$

From Lemma 1.38 we deduce that

$$<\gamma,\omega> <k_{i_0}, T_k(c)\gamma> = 0$$

where $\omega \in \mathrm{H}_k(I_1) \ominus \mathrm{H}_k(I_0)$ and $<k_{i_0}, \omega> = 1$. (Note that we are in the case where $\omega$ exists: otherwise $\rho_1 = \rho(c_0,k) = ||T_k(c_0)||^2$ and the norm of $T_k(c_0)$ will be attained on $\mathrm{H}_k(I_0)$).

Now, if $<\gamma,\omega> = 0$ we deduce that $\gamma \in \mathrm{H}_k(I_0)$. Hence

$$\rho_1 = ||T_k(c_0)\gamma||^2 = ||T(z,k)||^2 \leqslant \sup_{h \in \mathrm{H}} ||T(z,h)||^2 = \rho_0,$$

a contradiction. Hence $<\gamma,\omega> \neq 0$. Thus,

$$<k_{i_0}, T_k(c_0)\gamma> = 0.$$

Now $\gamma \in ker\Big(\rho_1 - T_k(c_0)^* T_k(c_0)\Big)$. Hence

$$\begin{aligned} <k_{i_0},\gamma> &= \frac{1}{\rho_1} <k_{i_0}, T_k(c_0)^* T_k)\gamma> \\ &= \frac{1}{\rho_1} <T_k(c_0)k_{i_0}, T_k(c_0)\gamma> \\ &= \frac{1}{\rho_1} \overline{c_0} <k_{i_0}, T_k(c_0)\gamma> = 0 \ . \end{aligned}$$

Thus.

$$\gamma,\ T_k(c_0)\gamma \in \mathbf{H}_k(I_0,I_1).$$

Since $\|\gamma\| = 1$ and $\|T_k(c_0)\gamma\|^2 = \rho_1$. We conclude that if $\tilde{z}\colon I_1 \to \mathbb{C}$ is defined by $\tilde{z} \mid I_0 = z$ and $\tilde{z}(i_0) = c_0$, then

$$\|T_{k,z}(I_1)\|^2 \geqslant \|T_{k,z}(I_1)\gamma\|^2 = \|T_{k,\tilde{z}}\gamma\|^2$$
$$= \rho_1 > \rho_0 = \sup_{g \in \mathbf{H}} \|T_{g,z}\|^2 .$$

This contradiction to (1.31) establishes Theorem 1.31.

## § 2. Interpolation in the Dirichlet multiplier norm.

In this section we shall establish the Theorem 2.1 below. Our overall intention is to illustrate the power of Theorem 1.30. Since there is precious little function theory known about multipliers on the Dirichlet it is clear that any of the usual approaches to the classical Nevanlinna-Pick theorem would proceed with great difficulty.

Let D denote the set of analytic functions f on the unit disc $\mathbb{R}$ with

$$\int |f'|^2 dA(z) < \infty .$$

Regard D as a Hilbert space with inner product defined by

$$\begin{aligned} \langle f, g\rangle &= \int_{\partial\mathbb{R}} f\, \overline{g} \frac{d\theta}{2\pi} + \frac{1}{\pi} \iint_{\mathbb{R}} f'\, \overline{g'}\, dA \\ &= \sum_{n=0}^{\infty} (n+1)\, \hat{f}(n)\, \overline{\hat{g}(n)} \end{aligned}$$

where

$$\hat{f}(n) = \int_{\partial\mathbb{R}} f(e^{i\theta})\, e^{-in\theta} \frac{d\theta}{2\pi} .$$

If $\phi$ is holomorphic on $\mathbb{R}$ we say $\phi$ is a multiplier of D if

$$\phi f \in \mathrm{D} \quad \text{whenever} \quad f \in \mathrm{D} .$$

If $\phi$ is a multiplier of $\mathbb{R}$ we define $M_\phi$:D $\to$ D by

$$(M_\phi f)(z) = \phi(z) f(z) .$$

Trivial facts are that if $\phi$ is a multiplier, then $M_\phi$ is bounded and that a bounded operator T commutes with $M_z$ if and only if there exists a multiplier $\phi$ such that $T = M_\phi$.

It is a straight forward computation to see that a reproducing kernel for D is defined by

$$k_\lambda(z) = \frac{1}{\overline{\lambda}z} \ln \frac{1}{1 - \overline{\lambda}z} = \sum_{n=0}^{\infty} \frac{(\overline{\lambda}z)^n}{n+1}$$

Thus, for each $\lambda \in \mathbb{R}$, $k_\lambda$ is the function in D such that

$$<f, k_\lambda> = f(\lambda)$$

for all $f \in$ D .

**Theorem 2.1.** *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and let $z_1, \ldots, z_n \in \mathbb{C}$ There exists a multiplier $\phi$ of* D *such that $\|M_\phi\| \leqslant 1$ and $\phi(\lambda_i) = z_i$ whenever $1 \leqslant i \leqslant n$ if and only if*

$$\left(\left(1 - \overline{z}_i z_j\right) \frac{1}{\overline{\lambda}_i \lambda_j} \ln \frac{1}{1 - \overline{\lambda}_i \lambda_j}\right)_{i,j=1}^{n} \geqslant 0 .$$

To prove Theorem 2.1 we require several lemmas.

**Lemma 2.2.** *$M_z$ is essentially normal.*

**Proof:** Regarding $M_z$ as a weighted shift immediately yields that $M_z$ is a compact perturbation of the ordinary unilateral shift.

**Lemma 2.3.** *If* H *is a Hilbert space and $\pi$: L(D) → L(H) is a unital *–*

*homomorphism, then* $\pi(M_z^*)$ *is unitarily equivalent to an operator of the form* $(M_z^*)^{(v)} \oplus U$, *where* v *is a cardinal,* $(M_z^*)^{(v)}$ *denotes the direct sum of* v *copies of* $M_z^*$ and U is unitary.

Lemma 2.3 follows from Theorem 2.8 in [ ]. Lemma 2.2 guarantees that the W in that theorem is in fact unitary.

**Lemma 2.4.** *Let* T *be an operator with* $\sigma(T) \leqslant \mathbf{D}$. T *has an extension to an operator of the form* $(M_z^*)^{(v)}$ *if and only if* $\frac{1}{k}(T) \geqslant 0$.

Lemma 2.4 follows from Theorem 2.3 in [ ].

**Lemma 2.5.** *Let* T *be an operator with* $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}$. *Let*

$$B = M_z^* \mid \left[k_{\lambda_i} \mid 1 \leqslant i \leqslant n\right] .$$

T *has an extension to an operator of the form* $B^{(v)}$ *if and only if* $\frac{1}{k}(T) \geqslant 0$ *and*

$$\prod_{i=1}^{n} (T-\lambda_i) = 0 .$$

**Proof:** Suppose $T \in \mathbf{L}(\mathrm{H})$, $\frac{1}{k}(T) \geqslant 0$ and $\prod_{i=1}^{n}(T-\lambda_i) = 0$. By Lemma 2.4 there exists an isometry $V: \mathrm{H} \to \mathbf{D}^{(v)}$ with

$$T = V^*(M_z^*)^{(v)} V , \tag{2.6}$$

and ran(V) invariant for $(M_z^*)^{(v)}$. Let $p(z) = \prod_{i=1}^{n}(z-z_i)$ so that p(T) = 0.

Then

$$0 = p(T)^* P(T) = V^* P\left((M_z^*)\right)^* P\left((M_z^*)^{(v)}\right) V$$

and we conclude that

$$ran(V) \subseteq ker\ p((M_z^*)) .$$

But $ker\left(p((M_z^*)^{(v)})\right) = \mathbf{D}_0^{(v)}$ where

$$\mathbf{D}_0 = \left[k_{\lambda_i} \mid 1 \leqslant i \leqslant n\right] .$$

Since $(M_z^*)^{(v)} \mid \mathbf{D}_0^{(v)} = B^{(v)}$ we see that (2.6) implies that T has an extension to an operator of the form $B^{(v)}$. The converse is obvious.

**Lemma 2.7.** $1 - \frac{1}{k}$ *is positive semidefinite.*

**Proof:** Let $1 - \frac{1}{k} = \sum_{n=1}^{\infty} a_n (\overline{\lambda} z)^n$. We need to verify that $a_n \geqslant 0$ if $n \geqslant 1$. Now.

$$\left(1 - \sum_{n=1}^{\infty} a_n (\overline{\lambda} z)^n\right) \left(\sum_{n=0}^{\infty} \frac{1}{n+1} (\overline{\lambda} z)^n\right) = 1 .$$

Hence,

$$\frac{1}{n+1} - \sum_{k=1}^{n} a_k \frac{1}{n+1-k} = 0 \text{ for } all \quad n \geqslant 1 . \tag{2.8}$$

Suppose it has been shown that $a_1, \ldots, a_n > 0$. Now (2.8) implies that

$$\sum_{k=1}^{n} a_k \left( \frac{n+1}{n+2} \cdot \frac{1}{n+1-k} \right) = \frac{1}{n+2} .$$

Further, if $1 \leqslant k \leqslant n$, then

$$\frac{1}{n+2-k} < \frac{n+1}{n+2} \cdot \frac{1}{n+1-k} .$$

Hence, since $a_k > 0$ whenever $1 \leqslant k \leqslant n$,

$$\sum_{n=1}^{n} a_k \frac{1}{n+2-k} < \frac{1}{n+2} ,$$

i.e. $a_{n+1} > 0$. Lemma 2.7 now follows by induction.

**Lemma 2.9.** *Let* $T \in L(H)$ *be an operator with* $\sigma(T) \subseteq \mathbb{R}$. *Assume that* $H_0$ *is semi-invariant for* T *and that* T *has an extension to an operator of the form* $(M_z^*)^{(v)}$. *If* $P$ *denotes the orthogonal projection of* H *onto* $H_0$, *then* $P\, T|\, H_0$ *has an extension to an operator of the form* $(M_z^*)^{(v)}$

**Proof:** By Lemma 2.8

$$\frac{1}{k} = 1 - \sum_{n=1}^{\infty} a_n (\overline{\lambda} z)^n$$

with $a_n > 0$. Because T has an extension to an operator of the form $(M_z^*)^{(v)}$,

$$\frac{1}{k}(T) \geqslant 0.$$

Hence, if is semi-invariant for T, and $r \in$ , then

$$< \sum_{n=1}^{\infty} a_n \Big(P\ T| \Big)^{*n} \Big(P\ T| \Big)^{n} \gamma, \gamma >

$$= \sum_{n=1}^{\infty} a_n \, ||P \ T^n \gamma||^2$$

$$\leqslant \sum_{n=1}^{\infty} a_n \, || \, T\gamma^n \, || \, ^2$$

$$\leqslant <\gamma,\gamma> \ .$$

We conclude that $\frac{1}{k}(P \ T| \, \mathrm{H}_0) \geqslant 0$ and thus via Lemma 2.4 that $P \ T| \, \mathrm{H}_0$ has an extension to an operator of the form $(M_z^*)^{(r)}$.

**Lemma 2.10.** *Let* $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. *If* k *is the kernel defined on* $M = \{1, \ldots, m\}$ *by*

$$k_i(j) = <k_{\lambda_i}, k_{\lambda_j}> \ , \ 1 \leqslant i, j \leqslant m,$$

*then* k *is an* NP *kernel.*

**Proof:** We shall employ Proposition 1.18. Fix $z: I_0 \to \mathbb{C}$ and set $\mathrm{H} = \mathrm{H}_k(I_0; I_1)$. Let $\tilde{z}$ be an extension of $z$ to $I_1$. Evidently we need to show that

$$||T_{k,z}(I_1)|| \leqslant ||T_{k,z}|| \tag{2.11}$$

where recall that $T_{k,z}(I_1) = P \ T_{k,\tilde{z}} \, | \, \mathrm{H}$.

Now observe that if $I \subseteq M$, then the map $V_I: \mathrm{H}_k(I) \to [k_{\lambda_i} \mid i \in I] \subseteq \mathbf{D}$ defined by

$$V_I\left(\sum_{i \in I} c_i k_i\right) = \sum_{i \in I} c_i k_{\lambda_i}$$

is a Hilbert space isormophism. Define $B_0 \in \mathbf{L}(\mathrm{H}_k(I_0))$ by setting

$$B_0 = V_{I_0}^* M_z^* V_{I_0}$$

and define $\tilde{B}_0 \in L(H)$ by setting

$$\tilde{B}_0 = P\ V_{I_1} M_z^* V_{I_1} \mid H .$$

Let p be a polynomial with the property that

$$p(\lambda_i) = \bar{z}(i) \quad whenever \quad i \in I.$$

Calculation reveals that

$$T_{k,z} = p(B_0) \text{ and } T_{k,z}(I_1) = p(\tilde{B}_0) .$$

Since $V_{I_1}^* M_z^* V_{I_1}$ has an extension to $M_z^*$, Lemma 2.9 implies that $\tilde{B}_0$ has an extension to an operator of the form $(M_z^*)^{(\nu)}$. Thus, by Lemma 1.7 and Lemma 2.5 we conclude that $\tilde{B}_0$ has an extension to an operator of the form $B_0^{(\nu)}$. Thus, there exists a $\nu$ and an isometry $V: H \to H_k(I_0)^{(\nu)}$ with ran V invariant for $B_0^{(\nu)}$ and $\tilde{B}_0 = V^* B_0^{(\nu)} V$. Hence

$$\begin{aligned} \|T_{k,z}(I_1)\| &= \|p(\tilde{B}_0)\| = \|p(V^* B_0^{(\nu)} V)\| \\ &\leqslant \|p(B_0)\| = \|T_{k,z}\| , \end{aligned}$$

which concludes the proof of Lemma 2.10.

Finally, we are able to prove Theorem 2.1: it follows in exactly the same way Theorem 1.27 followed from Lemma 1.26.

## § 3. Interpolation on the polydisc.

Fix a positive integer d and set $M = \{1, ..., m\}$. If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is an m-tuple of distinct points in $D^d$, let $K(\lambda)$ denote the set of all kernels k on M with the properties

$$\left(\left(1 - \overline{\lambda}_{ir}\lambda_{jr}\right)k_i(j)\right)_{ij \in M} \geqslant 0 \text{ whenever } 1 \leqslant r \leqslant d. \qquad (3.1)$$

and

$$\text{either } k_i(i) = 0 \text{ or } k_i(i) = 1 \text{ whenever } i \in M. \qquad (3.2)$$

For $k \in K(\lambda)$ and $1 \leqslant r \leqslant d$ define $M_{k,r} \in L(H_k)$ by requiring that

$$M_{k,r} k_i = \overline{\lambda}_{ir} k_i \text{ whenever } i \in M.$$

It is easy to see that $M_{k,r} = T_{k,z}$ if $z: M \to \mathbb{C}$ is defined by $z(i) = \lambda_{ir}$, $i \in M$. If p is a polynomial in d variables we define $\check{p}$ by the formula

$$\check{p}(z_1, \ldots, z_d) = \overline{p(\overline{z}_1, \ldots, \overline{z}_d)} \; .$$

With this convention it should be clear that if $I \subseteq M$, $z: I \to \mathbb{C}$ and p is a polynomial with $p(\lambda_i) = z(i)$ whenever $i \in M$, then

$$T_{k,z} = \check{p}(M_{k,1}, \ldots, M_{k,d}) \mid H_k(I).$$

Our first lemma gives an intrinsic characterization of the d-tuples of operators $(M_{k,1}, \ldots, M_{k,d})$ that can arise from a k in $K(\lambda)$.

**Lemma 3.3.** *Let $T_1, \ldots, T_d$ be a d-tuple of operators acting on a Hilbert space H. The following are equivalent.*

*There exists* $k \in \mathbf{K}(\lambda)$ *and a Hilbert isomorphism* $L : \mathbf{H} \to \mathbf{H}_k$ *such that* $T_r = L^* M_{k,r} L$ *whenever* $1 \leqslant r \leqslant d$. (3.4)

$\|T_r\| \leqslant 1$ *for each* $r$, *and there exist* $g_1, \ldots, g_m \in$ *such that* (3.5)

$$\mathbf{H} = [g_i \mid i \in M]$$

and

$$T_r g_i = \overline{\lambda}_{ir} g_i$$

whenever $1 \leqslant r \leqslant d$ and $i \in M$.

**Proof:** Assume (3.4) holds. Because $k \in \mathbf{K}(\lambda)$, (3.1) guarantees that $\|M_{k,r}\| \leqslant 1$. Hence $\|T_r\| \leqslant 1$. Let $g_i = L^* k_i$. Since $\mathbf{H}_k = [k_i \mid i \in M]$,

$$\mathbf{H} = L^* \mathbf{H}_k = [g_i \mid i \in M].$$

Finally, if $1 \leqslant r \leqslant d$ and $i \in M$, then

$$T_r g_i = T_r L^* k_i = L^* M_{k,r} k_i = L^* (\overline{\lambda}_{ir} k_i) = \overline{\lambda}_{ir} g_i .$$

and we see that (3.5) holds.

Now assume that (3.5) holds. Let k be defined by

$$k_i(j) = \begin{cases} 0 & \text{if } either \ \ g_i = 0 \ \ or \ \ g_j = 0 \\ \dfrac{\langle g_i, g_j \rangle}{\|g_i\| \, \|g_j\|} & otherwise \end{cases} .$$

Let $h_i \in \mathbf{H}$ be defined by $h_i = \dfrac{g_i}{\|g_i\|}$ so that we have the identity

$$\sum_{i,j\in M} k_i(j)c_i\overline{c_j} = \|\sum_{i\in M} c_i h_i\|^2 \ . \tag{3.6}$$

We first show that k is a kernel on M. Since (3.6) implies that $k \geqslant 0$ we need only verify (1.1).

We argue by contradiction. Thus assume that

$$\sum_{i,j\in spt(k)} k_i(j)c_i\overline{c_j} = 0$$

and suppose there exists $i_0 \in spt(k)$ with $c_{i_0} \neq 0$. Choose a polynomial in d variables with $p(\lambda_{i_0}) = 1$ and $p(\lambda_i) = 0$ if $i \in M\setminus\{i_0\}$. Then (3.6) and the fact that $\check{p}(T)h_i = \overline{p(\lambda_i)}h_i$ imply that

$$\begin{aligned} |c_{i_0}|^2 &= \|c_{i_0}h_{i_0}\|^2 \\ &= \|\check{p}(T)\sum_{i\in spt(k)} c_i h_i\|^2 \\ &\leqslant \|\check{p}(T)\|^2 \|\sum_{i\in spt(k)} c_i h_i\|^2 \\ &= 0 \ . \end{aligned}$$

Thus, k is a kernel. We now need to check that in fact $k \in \mathrm{K}(\lambda)$ i.e. that (3.1) and (3.2) hold. (3.2) is immediate from the definition of k and (3.1) follows from (3.6) and the fact that $\|T_r\| \leqslant 1$ whenever $1 \leqslant r \leqslant d$. Thus, $k \in \mathrm{K}(\lambda)$. $L: \mathbf{H} \to \mathbf{H}_k$ is defined by the formula

$$L(\sum c_i h_i) = \sum c_i k_i \ .$$

That L is a Hilbert space isomorphism is equivalent to (3.6). Thus (3.5) implies (3.4) and Lemma 3.3 is established.

**Proposition 3.7.** $\mathrm{K}(\lambda)$ is an NP structure.

Proof: There are a number of points to check. We content ourselves here with the verification of Property 1.3, Property 1.5 and Condition (1.31).

Property 1.3 is straight-forward. Thus, assume $k, k' \in \mathbf{K}(\lambda)$, $c, c' \in \mathbb{C}^m$ and

$$k_i''(j) = \left(\overline{c_i} c_j k_i(j) + \overline{c_i}' c_j' k_i'(j)\right)^{\sim} .$$

Evidently, by the definition of $\sim$, $k''$ satisfies (3.2). We now check Condition (3.1). Choose $d \in \mathbb{C}^m$ with

$$k_i''(j) = \overline{d_i} d_j (\overline{c_i} c_j k_i(j) + \overline{c_i}' c_j' k_i'(j)) .$$

If $1 \leqslant r \leqslant d$, then

$$\left(1 - \overline{\lambda}_{ir} \lambda_{jr}\right) k_i''(j) = \overline{d_i} d_j \overline{c_i} c_j \left[(1-\overline{\lambda}_{ir} \lambda_{jr}) k_i(j)\right]$$
$$+ \overline{d_i} d_j \overline{c_j}' c_j' \left[(1-\overline{\lambda}_{ir} \lambda_{jr}) k_i'(j)\right] \geqslant 0,$$

since k and $k'$ satisfy (3.1).

To see that Property 1.5 holds, by Lemma 1.12 it is sufficient to show that there exists an absolute constant C such that

$$\|z\|_{\mathbf{K}(\lambda)} \leqslant C \max_{i \in I} |z(i)|$$

whenever $I \subseteq M$ and $z: I \to \mathbb{C}$

Observe that (3.1) is equivalent to

$$\|M_{k,r}\| \leqslant 1 .$$

Thus if $w$ is fixed with $w > 1$ then a resolvent estimate yields that there exists a constant $C_0$ such that

$$\|(\eta - M_{k,r})^{-1}\| \leqslant C_0 \tag{3.8}$$

whenever $k \in (\lambda)$ and $|\eta| = w$. Also, there exist a constant $C_1$ such that for any $z: I \to \mathbb{C}$ there exist a polynomial (of d variables) such that

$$\begin{cases} p(\lambda_i) = z(i) & \text{whenever } i \in I \text{ and} \\ \max_{\eta \in w \cdot (\partial D)^d} |p(\eta)| & \leqslant C_1 \max_{i \in I} |z(i)| \, . \end{cases} \tag{3.9}$$

Fix $z: I \to \mathbb{C}$ and choose p so that (3.9) is satisfied. Observe that

$$T_{k,z} = p(M_{k,1}, \ldots, M_{k,d}) \mid \mathbf{H}_k(I).$$

Thus,

$$T_{k,z} = \left( \frac{1}{(2\pi i)^d} \int_{w\partial D} \cdots \int_{w\partial D} p(\eta) \prod_{r=1}^{d} (\eta_r - M_{k,r})^{-1} d\eta \right) \mid \mathbf{H}_k(I)$$

and we conclude from (3.8) and (3.9) that

$$\|T_{k,z}\| \leqslant w^d C_0^d C_1 \max_{i \in I} |z(i)| \quad ,$$

which establishes that Property 1.5 holds.

We now check that Condition (1.31) holds. Fix $I_0 \subseteq I_1 \subseteq M$ and let $z: I_0 \to \mathbb{C}$ Fix $\tilde{z}: I_1 \to \mathbb{C}$ with $\tilde{z} \mid I_0 = z$ and let $\mathbf{G}_k = \mathbf{H}_k(I_0, I_1)$. Recall that

$$T_k(I_1) = P_k T_{k,\tilde{z}} \mid \mathbf{G}_k \, .$$

Thus, we need to check that

$$\sup_{k\in \mathbf{K}(\lambda)} \|P_{\mathbf{G}_k} T_{k,\hat{z}} \mid \mathbf{G}_k\| \leqslant \|z\|_{\mathbf{K}(\lambda)} .$$

Fix $k \in \mathbf{K}(\lambda)$. Define $\tilde{M}_r : \mathbf{G}_k \to \mathbf{G}_k$ by setting

$$\tilde{M}_r = P_k M_{k,r} \mid \mathbf{G}_k \ .$$

By an obvious refinement of Lemma 1.7 and Lemma 3.3, there exists a $g \in \mathbf{K}(\lambda)$ such that

$$\left(\tilde{M}_1, \ldots, \tilde{M}_d\right) \cong \left(M_{g,1}, \ldots, M_{g,d}\right) \ .$$

Also observe that if p is chosen so that

$$p(\lambda_i) = z(i) \quad , \quad i \in I_0 ,$$

then

$$P_{\mathbf{G}_k} T_{k,\hat{z}} \mid \mathbf{G}_k = \check{p}(\tilde{M}_1, \ldots, \tilde{M}_d)$$

and

$$T_{g,z} = \check{p}(M_{g,1}, \ldots, M_{g,d}) \ .$$

Thus,

$$\begin{aligned} \|P_{\mathbf{K}_k} T_{k,\hat{z}} \mid \mathbf{G}_k\| = \|T_{g,z}\| &\leqslant \sup_{h\in \mathbf{K}(\lambda)} \|T_{h,z}\| \\ &= \|z\|_{(\lambda)} , \end{aligned}$$

which concludes the proof that $\mathbf{K}(\lambda)$ is an NP structure.

For d a positive integer let us agree to let $\mathbf{F}_d$ denote the collection of all d-tuples $T = (T_1, \ldots, T_d)$, such that $\|T_r\| \leqslant 1$ whenever $1 \leqslant r \leqslant d$ and such that $T_1, \ldots, T_d$ pairwise commute. If $\phi$ is holomorphic on a neighborhood of $(\mathbf{D}^-)^d$ define

$$\|\phi\|_{d,\infty} = \sup\left\{\|\phi(T)\| \mid T \in \mathbf{F}_d\right\}. \tag{3.10}$$

In (3.10), $\phi(T)$ can be defined either through the Riesz-Dunford functional calculus or by power series methods. If $\rho > 1$ and $\phi$ is holomorphic on $\mathbf{D}^d$ define $\phi_\rho$ by $\phi_\rho(z) = \phi(\rho z)$ and if $T \in \mathbf{F}_d$ define $\rho T$ by $\rho T = (\rho T_1, \ldots, \rho T_d)$. Obviously $\phi_\rho(T) = \phi(\rho T)$ whenever $\phi$ is analytic on $\mathbf{D}^d$ and $T \in \mathbf{F}_d$. Let $H^\infty(\mathbf{F}_d)$ denote the collection of all holomorphic functions $\phi$ on $\mathbf{D}^d$ such that

$$\|\phi\|_{d,\infty} \equiv \sup_{\rho<1} \|\phi_\rho\|_{d,\infty} < \infty.$$

$H_\infty(\mathbf{F}_d)$ is a Banach space when equipped with the norm $\|\cdot\|_{d,\infty}$. Furthermore, $H^\infty(\mathbf{F}_d) \subseteq H^\infty(\mathbf{D}^d)$ and if $\phi \in H^\infty(\mathbf{F}_d)$ then

$$\|\phi\|_\infty \equiv \sup_{\lambda \in \mathbf{D}^d} |\phi(\lambda)| \leqslant \|\phi\|_{d,\infty}.$$

**Lemma 3.11.** *If* d = 1 or 2, *then* $H^\infty(\mathbf{F}_d) = H^\infty(\mathbf{D}^d)$. *Furthermore* $\|\phi\|_\infty = \|\phi\|_{d,\infty}$ *whenever* $\phi \in H^\infty(\mathbf{F}_d)$.

**Proof:** d = 1 is von Neumann's inequality. d = 2 is the fact that the closed bidisc is a spectral set for any commuting pair of contractions (a consequence of Ando's theorem or the commutant lifting theorem).

**Lemma 3.12.** *Let* $\lambda = (z_i)_{i=1}^\infty$ *be a sequence of distinct points in* $\mathbf{D}^d$ *that forms a set of uniqueness for* $H^2(\mathbf{D}^d)$. *If* $\phi$ *is analytic on* $\mathbf{D}^d$, *then*

*$\phi \in H^\infty(\mathbf{F}_d)$ if and only if*

$$||\phi||_\lambda \equiv \sup_{m \geqslant 1} \sup_{k \in \mathbf{K}((\lambda_1, \ldots, \lambda_m))} ||\phi(M_{k,1}, \ldots, M_{k,d})|| < \infty.$$

*Furthermore, if $\phi \in H^\infty(\mathbf{F}_d)$, then*

$$||\phi||_{d,\infty} = ||\phi||_\lambda \ .$$

**Proof:** It follows immediately from Lemma 3.3 that if $\phi \in H^\infty(\mathbf{F}_d)$, then $||\phi|| \leqslant ||\phi||_{d,\infty}$. Conversely, assume that $\phi$ is analytic on $\mathbf{D}$ and

$$||\phi||_\lambda < \infty \ .$$

We need to show that $||\phi||_{d,\infty} < \infty$ and that $||\phi||_{d,\infty} \leqslant ||\phi||_\lambda$. Fix $\rho < 1$ and fix $T \in \mathbf{F}_d$. From the definition of $||\cdot||_{d,\infty}$ it is enough to establish that

$$||\phi(\rho T)|| \leqslant ||\phi||_\lambda \, .$$

Assume $T$ acts on the Hilbert space $G$, and set $R = \rho T$. Fix $\epsilon > 0$ and choose a vector $\gamma \in G$ such that $||\gamma|| = 1$ and

$$||\phi(R)\gamma||^2 > ||\phi(R)||^2 - \epsilon \ .$$

Fix $\eta > 0$ and consider the following sesquilinear form on $H^2(\mathbf{D}^d)$.

$$[f,g]_\eta = \eta\langle f,g\rangle_2 + \langle f(R)\gamma, g(R)\gamma\rangle$$

where $\langle\cdot,\cdot\rangle_2$ denotes the usual inner product on $H^2(\mathbf{D}^d)$, $\langle f, g\rangle_2 = \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}$, and $f(T)$ is defined by

$$f(T) = \frac{1}{(2\pi i)^d} \int_{\partial D} \cdots \int_{\partial D} f(\eta) \prod_{r=1}^{d} (\eta_r - R_r)^{-1} d\eta_1 \ldots d\eta_d .$$

We observe that this last integral makes sense since f is holomorphic on $D^d$ and $R = \rho T$ has spectrum in $D^d$. It is elementary to ascertain that $H^2(D^d)$ is complete with respect to the metric induced by $[\cdot,\cdot]_\eta$, and also that $[\cdot,\cdot]_\eta$ is positive definite. Let $H_\eta^2$ denote the corresponding Hilbert space. Define $M_\eta$ on $H_\eta^2$ by

$$M_\eta = (M_{\eta,1}, \ldots, M_{\eta,d})$$

where

$$(M_{\eta,r} f)(z) = z_r f(z) .$$

We observe that $M_\eta$ is a bounded n-tuple of commuting operators with spectrum in $(D^-)^d$ . Also, since

$$\|\phi(\rho M_\eta)1\|_{H_\eta^2}^2 = \eta\|\phi_\rho\|_{H^2(D^d)}^2 + \|\phi(R)\gamma\|^2 ,$$

$$\|\phi(R)\gamma\|^2 > \|\phi(R)\|^2 - \epsilon ,$$

and

$$\|1\|_{H_\eta^2}^2 = \eta + 1 ,$$

we deduce that

$$\frac{\eta}{\eta+1} \|\phi_\rho\|^2 + \frac{1}{\eta+1} \left(\|\rho(R)\|^2 - \epsilon\right) < \|\phi(\rho M_\eta)\|.$$

Hence

$$\|\phi(R)\|^2 - \epsilon \leqslant \lim_{\eta \to 0} \|\phi(\rho M_\eta)\|. \tag{3.12}$$

Now define a map $L_\eta: H^2(\mathrm{D}^d) \to \mathrm{H}_\eta^2$ by

$$L_\eta(f) = f \quad .$$

Evidently, $L_\eta$ is a bounded invertible operator and

$$M_{\eta,r} L_\eta = L_\eta M_r$$

whenever $1 \leqslant r \leqslant d$. Here, $M_r$ denotes the operator multiplication by $z_r$ on $H^2(\mathrm{D}^d)$. Consequently, if we define

$$g_\lambda = L_\eta^{-1*} k_\lambda$$

where $k_\lambda$ is the Szego kernel for $\lambda \in \mathrm{D}^d$, then

$$M_{\eta,r}^* g_\lambda = \overline{\lambda}_r g_\lambda \tag{3.13}$$

whenever $1 \leqslant r \leqslant d$ and $\lambda \in \mathrm{D}^d$. Now since $\{\lambda_i\}$ is a set of uniqueness for $H^\infty(\mathrm{D}^d)$, $\{k_{\lambda_i}\}$ has dense linear span in $H^2(\mathrm{D}^2)$, and hence $\{g_{\lambda_i}\}$ has dense linear span in $H_\nu^2$. Consequently,

$$\begin{aligned} \|\phi(\rho M_\eta)\| &= \|\phi(\rho M_\eta)^*\| \\ &= \sup_{m \geqslant 1} \|\breve{\phi}(\rho M_\nu^*) \mid \left[g_{\lambda_i} \mid 1 \leqslant i \leqslant m\right]\|. \end{aligned} \tag{3.14}$$

Now calculating with the definition of $M_\nu$ it is easy to conclude that $\rho M_\nu^* \in \mathbf{F}_d$. Hence (3.13) and Lemma 3.3 imply that $\left(g_{\lambda_i}(x_j)\right)_{1 \leqslant i,\, j \leqslant m} \in \mathbf{K}\left((\lambda_1, \ldots, \lambda_m)\right)$. Hence

$$\|\breve{\phi}(\rho M_\nu^*) \mid \left[g_{\lambda_i} \mid 1 \leqslant i \leqslant m\right]\| \tag{3.15}$$

$$\leqslant \|\tilde{\phi}\|_{\{\lambda_i\}_{i=1}^{\infty}} = \|\phi\|_{\lambda} .$$

Putting together (3.12) (3.14) and (3.15) yields that

$$\|\phi(R)\|^2 - \epsilon \leqslant \|\phi\|_{\lambda}^2 ,$$

which concludes the proof of Lemma 3.11.

**Theorem 3.16.** *Fix positive integers* d *and* n, *let* $\lambda_1, \ldots, \lambda_n$ *be distinct points in* $\mathrm{D}^d$, *and let* $z_1, \ldots, z_n \in \mathbb{C}$ *There exists* $\phi \in H^\infty(\mathrm{F}_d)$ *with* $\|\phi\|_{d,\infty} \leqslant 1$ *and* $\phi(\lambda_i) = z_i$ *for all* i *if and only if there exist* d *positive semidefinite* $n \times n$ *matrices* $a^1, \ldots, a^d$ *such that*

$$1 - \overline{z_i} z_j = \sum_{r=1}^{d} \left(1 - \overline{\lambda}_{ir} \lambda_{jr}\right) a_{ij}^r$$

*whenever* $1 \leqslant i, j \leqslant n$ .

**Proof:** First assume that $\phi \in H^\infty(\mathrm{F}_d)$, $\|\phi\|_{d,\infty} \leqslant 1$ and $\phi(\lambda_i) = z_i$. Let P be the cone in the $n \times n$ matrices generated by the matrices of the form

$$\left(\left(1 - \overline{\lambda}_{ir} \lambda_{jr}\right) a_{ij}\right)_{i,j=1}^{n}$$

where $1 \leqslant r \leqslant n$ and $(a_{ij})$ is positive semidefinite. We wish to show that $(1 - \overline{z}_i z_j) \in \mathrm{P}$. But by the Hahn-Banach theorem this will follow if we can show that

$$tr(1 - \overline{z}_i z_j) W \geqslant 0 \tag{3.17}$$

whenever W is an $n \times n$ matrix with

$$tr\, CW \geqslant 0 \text{ for } all \quad C \in \mathrm{P}. \tag{3.18}$$

Accordingly, fix W satisfying (3.18).

Note that since $\left(\dfrac{1}{1 - \overline{\lambda}_{ir} \lambda_{jr}}\right) \geqslant 0$, if $(a_{ij}) \geqslant 0$, then $\left(\dfrac{a_{ij}}{1 - \overline{\lambda}_{ir} \lambda_{jr}}\right) = (a_{ij}) * \left(\dfrac{1}{1 - \overline{\lambda}_{ir} \lambda_{jr}}\right) \geqslant 0$. This is because the Shur product $\left((a_{ij}) * (b_{ij}) = (a_{ij} b_{ij})\right)$ of positive semidefinite matrices is positive semidefinite. In particular, we find that if $(a_{ij}) \geqslant 0$ then $(a_{ij}) \in \mathrm{P}$. Consequently we deduce from (3.18) that $W \geqslant 0$.

Define a sesquilinear form $<\cdot,\cdot>$ on $\mathbb{C}$ by

$$<c,d> = (Wc,d) \ . \tag{3.19}$$

Since $W \geqslant 0$, $<\cdot,\cdot>$ is positive semidefinite. Let $\mathrm{N} = \left\{c \in \mathbb{C}^n \mid <c<c> = 0\right\}$.

Cauchy's inequality guarantees that N is a subspace of $\mathbb{C}^n$ and that $<\cdot,\cdot>$ is well defined $\frac{\mathbb{C}^n}{N}$.

Let H denote this Hilbert space. Define operators $T_1, \ldots, T_d$ on H by the formulas

$$T_r e_i = \overline{\lambda}_{ir} e_i \qquad 1 \leq i \leq n, \ 1 \leq r \leq d. \tag{3.20}$$

It is not obvious that $T_1, \ldots, T_d$ are actually well defined on H (i.e. $c \in N$ implies $Tc \in N$). That $T_r$ is well defined and that $\|T_r\| \leq 1$ whenever $1 \leq r \leq d$ both will follow from the following formula

$$\|c\|^2 - \|T_r c\|^2 = <\left(W^T * (1 - \overline{\lambda}_{ir}\lambda_{jr})\right)\overline{c},\overline{c}>. \tag{3.21}$$

In (3.21) r and $c \in \mathbb{C}^n$ are arbitrary. $\overline{c} \in \mathbb{C}^n$ is defined by $\overline{c} = (\overline{c_i})$. The derivation of (3.21) is a routine matter based on formulas (3.19) and (3.20).

Now observe that $T_r$ will be well defined and $\|T_r\| \leq 1$ provided that the expression in (3.21) is nonnegative. From (3.18) we know that

$$\sum_{i,j} w_{ji} a_{ij} (1 - \overline{\lambda}_{ir}\lambda_{jr}) \geq 0$$

whenever $(a_{ij}) \geq 0$. Hence

$$W^T * (1 - \overline{\lambda}_{ir}\lambda_{jr}) \geq 0$$

and we conclude from (3.21) that $T_r$ is well defined and $\|T_r\| \leq 1$ whenever $1 \leq r \leq d$. Since tautologically $H = [e_i \mid 1 \leq i \leq n]$, we conclude from the relations in (3.20) and Lemma 3.3 that there exists $k \in K(\lambda_1, \ldots, \lambda_n)$ and a Hilbert space isomorphism $L : H \to H_k$ such that

$$T_r = L' M_{k,r} L \quad whenever \quad 1 \leqslant r \leqslant d. \tag{3.22}$$

Now, if $\breve{\phi}$ is defined by $\breve{\phi}(z) = \overline{\phi(\bar{z})}$ it is easy to deduce from the fact that $T \in \mathbf{F}_d$ if and only if $T' \in \mathbf{F}_d$ that $\|\breve{\phi}\|_{d,\infty} = \|\phi\|_{d,\infty}$. Since $\|\phi\|_{d,\infty} \leqslant 1$ we conclude from (3.22) and Lemma 3.12 that

$$\begin{aligned}\|\breve{\phi}(T_1, \ldots, T_d)\| &= \|\breve{\phi}(M_{k,1}, \ldots, M_{k,d})\| \\ &\leqslant \|\breve{\phi}\|_{d,\infty} = \|\phi\|_{d,\infty} \leqslant 1 .\end{aligned}$$

Hence

$$\|c\|^2 - \|\breve{\phi}(T)c\|^2 \geqslant 0 .$$

From the same calculation that proves (3.21) we conclude that

$$W^T * \left(1 - \overline{\phi(\lambda_i)}\,\phi(\lambda_j)\right) \geqslant 0 .$$

Hence since $\phi(\lambda_i) = z_i$ we see that

$$tr\,(1 - \bar{z}_i z_j)\, W = \sum \left(1 - \overline{\phi(\lambda_i)}\,\phi(\lambda_j)\right) W_{ji} \geqslant 0$$

which establishes (3.17) and concludes the proof of the necessity of the interpolation condition.

Now assume that

$$(1 - \bar{z}_i z_j) = \sum_{r=1}^{d} (1 - \bar{\lambda}_{ir} \lambda_{ir}) a_{ij}^r \qquad 1 \leqslant i, j \leqslant n$$

where $(a_{ij}^r) \geqslant 0$ for each r. Extend $\lambda_1, \ldots, \lambda_n$ to an infinite sequence of distinct points $\{\lambda_i\}_{i=1}^{\infty} \subseteq \mathbb{D}^d$ such that $\{\lambda_i\}_{i=1}^{\infty}$ is a set of uniqueness for $H^2(\mathbb{D}^d)$. For $m \geqslant 1$ let

$$\mathrm{K}_m = \mathrm{K}((\lambda_1, \ldots, \lambda_m)) .$$

By Proposition 3.7 and Theorem 1.31 there exists a sequence $\{z_i\}_{i=1}^{\infty}$ such that if $z_m : \{1, \ldots, m\} \to \mathbb{C}$ is defined by

$$z_m(i) = z_i \quad , \ 1 \leqslant i \leqslant m,$$

then

$$\sup_{m \geqslant 1} \sup_{k \in \mathrm{K}_m} \|T_{k,z_m}\| \leqslant 1 . \tag{3.23}$$

Defined a sequence $g_m \in \mathrm{K}_m$ by setting

$$(g_m)_i(j) = \prod_{r=1}^{d} \frac{1}{1 - \overline{\lambda}_{ir} \lambda_{jr}} \quad , \ 1 \leqslant i, j \leqslant m.$$

Evidently, if G is the Szego kernel for $H^2(\mathbb{D}^d)$, then

$$(g_m)_i(j) = G_{\lambda_i}(\lambda_j) \quad , \ 1 \leqslant i, j \leqslant m,$$

and so it is clear that in fact $g_m \in \mathrm{K}_m$. Also observe that if we define $R_m$ on $H_m^2 = \left[ G_{\lambda_i} \mid 1 \leqslant i \leqslant m \right]$ by

$$R_m G_{\lambda_i} = \overline{z_i} G_{\lambda_i} \quad , \quad 1 \leqslant i \leqslant m,$$

then

$$R_m \cong T_{g_m, z_m} .$$

Thus, (3.23) implies that

$$\|R_m\| \leqslant 1 \text{ for } all \quad m.$$

Also, we have that

$$R_{m_1} \mid H^2_{m_0} = R_{m_0} \text{ if } m_0 \leqslant m_1 .$$

Define R on $H^2(\mathrm{D}^{\mathrm{d}})$ by requiring that

$$R \mid H^2_m = R_m \text{ for all } m \geqslant 1.$$

Since $RM^*_{z_i} = M^*_{z_i}R$ and $\|R\| \leqslant 1$, there exists $\phi \in H^\infty(\mathrm{D}^{\mathrm{d}})$ with

$$\|\phi\|_\infty \leqslant 1$$

and

$$\phi(\lambda_i) = z_i \quad i \geqslant 1.$$

In particular $\phi(\lambda_i) = z_i$ when $1 \leqslant i \leqslant n$. We claim that in fact,

$$\|\phi\|_{d,\infty} \leqslant 1 \;.$$

Observe that if $m \geqslant 1$, then since

$$\phi(\lambda_i) = z_i,$$

when $1 \leqslant i \leqslant m$ we have that

$$T_{k,z_m} = \check{\phi}(M_{k,1}, \; ... \; , M_{k,d}) \;.$$

Thus, by Lemma 3.12 and (3.23),

$$\|\phi\|_{d,\infty} = \|\phi\|_{d,\infty} = \sup_{m \geq 1} \sup_{k \in K_m} \|\check{\phi}(M_{k,1}, \ldots, M_{k,d})\|$$

$$= \sup_{m \geq 1} \sup_{k \in K_m} \|T_{k,z_m}\| \leq 1 .$$

This concludes the proof of Theorem 3.16.

**Corollary 1.** We observe from Theorem 3.16 and Lemma 3.11 that if d = 2 then there exists $\phi \in H^\infty(\mathbf{D}^2)$ with $\|\phi\|_\infty \leq 1$ and $\phi(\lambda_i) = z_i$ if and only if there exist positive semi-definite matrices $(a_{ij}^1)$ and $(a_{ij}^2)$ such that

$$(1 - \overline{z}_i z_j) = \left(1 - \overline{\lambda}_{i1}\lambda_{j1}\right) a_{ij}^1 + \left(1 - \overline{\lambda}_{i2}\lambda_{j2}\right) a_{ij}^2 .$$

**Corollary 2.** If d is arbitrary and

$$1 - \overline{z}_i z_j = \sum_r \left(1 - \overline{\lambda}_{ir} \lambda_{jr}\right) a_{ij}^r$$

with $(a_{ij}^r) \geq 0$ then there exists $\phi \in H^\infty(\mathbf{D}^d)$ with $\|\phi\|_\infty \leq 1$ and $\phi(\lambda_i) = z_i$.

**Proof:** Theorem 3.16 together with the fact that

$$\|\phi\|_\infty \leq \|\phi\|_{d,\infty} .$$